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Pieri Rule and Pieri Algebras

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1 Introduction

Let G be a complex classical group, and U, V be finite dimensional irreducible representations of G. The tensor product $U\otimes V$ is also a representation of G, but it is not irreducible in general. It is an important problem to describe the decomposition of $U\otimes V$ into a sum of irreducible representations of G.

In the case of complex general linear groups, the finite dimensional irreducible rational representations of $\mathrm{GL}_n := \mathrm{GL}_n(\mathbb{C})$ are indexed by non-increasing sequences $\lambda=(\lambda_1,\ldots,\lambda_n)$ of integers. We denote the representation corresponding to λ by ρ_n^{λ}. Specifically, the irreducible polynomial representations are indexed by sequences of non-negative integers. These sequences are denoted by capital characters D, E etc. There is a combinatorial description of how a tensor product of the form $\rho_n^D\otimes\rho_n^F$ decomposes. It is called the **Littlewood-Richardson rule** ([16], [8]).

In the case when $F=(\alpha)$ is a sequence with only one nonzero entry α, the description of how $\rho_n^D\otimes\rho_n^{(\alpha)}$ decomposes is called the **Pieri rule** ([15], [4], [10]). Although the Pieri rule is a very special case of tensor product, it is of particular interest because it is connected with the branching rule from GL_n to GL_{n-1} ([10], [17]). It is a natural question to consider a more general version of the Pieri rule, that is, a description of how tensor products of the form

$$\rho_n^{\lambda}\otimes\left(\bigotimes_{s=1}^{h}\rho_n^{(\alpha_s)}\right)\otimes\left(\bigotimes_{t=1}^{l}\rho_n^{(\beta_t)^*}\right),\qquad \alpha_s,\beta_t\in\mathbb{Z}_{\geq 0} \tag{1.1}$$

decomposes. Here the representation $\rho_n^{(\beta_t)^*}$ is dual to $\rho_n^{(\beta_t)}$.

Let k, p, h and l be positive integers. Assume that there are at most k positive entries and p negative entries of λ. In [7], Roger Howe, Sangjib Kim and Soo Teck Lee construct an algebra $\mathcal{A}_{n,k,p,h,l}$ which encodes information on the decomposition of (1.1). The algebra is called a $((k,p),h,l)$-**Pieri algebra** for GL_n. Specifically, when $p=l=0$ and $h=1$, the algebra encodes the Pieri rule for $\rho_n^D\otimes\rho_n^{(\alpha)}$. There are also analogues of Pieri algebras for $\mathrm{O}_n=\mathrm{O}_n(\mathbb{C})$ and $\mathrm{Sp}_{2n}=\mathrm{Sp}_{2n}(\mathbb{C})$, which are discussed in [13].

In [7], the authors reveal the structure of two kinds of $((k,p),h,l)$-**Pieri algebras**, $p=l=0$ and $k+p+h+l\leq n$. For the algebras discussed in [7], the structure is controlled by a semigroup, called the **Hibi cone** ([11]). The Hibi cone is constructed from a finite poset Γ: it is the set $\mathbb{Z}_{\geq 0}^{\Gamma,\succeq}$ of all order preserving functions $f:\Gamma\to\mathbb{Z}_{\geq 0}$ with semigroup operations [12] given by the addition of functions. The Hibi cone $\mathbb{Z}_{\geq 0}^{\Gamma,\succeq}$ has a very nice and simple structure:

1. It has a finite set $\mathcal{G}$ of generators.

2. One can define a partial ordering $\succeq$ on $\mathcal{G}$, such that, each nonzero element f of $\mathbb{Z}_{\geq 0}^{\Gamma,\succeq}$ has a unique standard expression as a sum $f = \sum_{i=1}^{u} g_i^{a_i}$ where $g_i \in \mathcal{G}$ and $a_i \in \mathbb{Z}_{>0}$ for $1 \leq i \leq u$ and $g_1 \preceq g_2 \preceq \cdots \preceq g_u$ with respect to the partial ordering in $\mathcal{G}$.

In [7], the authors define an element v_g in the algebra $\mathcal{A}_{n,k,p,h,l}$ for each g in $\mathcal{G}$. Then the partial ordering on $\mathcal{G}$ induces a partial ordering on $\mathcal{S} := \{v_g : g \in \mathcal{G}\}$. A monomial on $\mathcal{S}$ of the form $v_{g_1}^{a_1} v_{g_2}^{a_2} \cdots v_{g_u}^{a_u}$ is called **standard** if $v_{g_1} \leq v_{g_2} \leq \ldots \leq v_{g_u}$ and $a_i \in \mathbb{Z}_{>0}$ for $1 \leq i \leq u$. The authors prove that the set of standard monomials on $\mathcal{S}$ form a vector space basis for $\mathcal{A}_{n,k,p,h,l}$ ([5]). Furthermore, $\mathcal{A}_{n,k,p,h,l}$ has a flat deformation to the semigroup algebra $\mathbb{C}[\mathbb{Z}_{\geq 0}^{\Gamma,\succeq}]$ on the Hibi cone. Similar results for Sp_{2n} and O_n are obtained in the paper [13].

We shall study another algebra, the structure of the *anti-row iterated Pieri algebra* $\mathfrak{A}_{n,k,l} = \mathcal{A}_{n,k,0,0,l}$. But Hibi cone is not enough for this case. So first we need to define another semigroup and call it **sign Hibi cone**. It retains many nice properties of Hibi cones. In fact, it is generated by two subsemigroups which are both Hibi cones. Then we describe the structure of $\mathfrak{A}_{n,k,l}$ with sign Hibi cones. The results on the anti-row iterated Pieri algebras also have applications in the study of lowest weight modules appearing in Howe duality.

2 Preliminaries

In this section, we review several necessary definitions, notations and theorems.

2.1 Pieri Rule for GL_n

Let A_n be the subgroup of all the diagonal matrices and U_n be the collection of upper triangular matrices with 1's on the diagonal. So A_n is the **maximal torus** and U_n is the **unipotent subgroup**.

Let

- $\Lambda_n^+ = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n : \lambda_1 \geq \ldots \geq \lambda_n\}$ and
- $\Lambda_n^{++} = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda_n^+ : \lambda_n \geq 0\}$.

For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda_n^+$, define $|\lambda| := \sum_{h=1}^{n} \lambda_h$. For $D \in \Lambda_n^{++}$, depth(D) is the number of positive entries in D.

Let ρ_n^λ be the GL_n irreducible **rational representation** with highest weight ψ_n^λ, where

$$\psi_n^\lambda(a) = a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n} \tag{2.1}$$

with $a = \mathrm{Diag}(a_1, \ldots, a_n) \in A_n$. An irreducible **polynomial representation** can be written as ρ_n^D with $D \in \Lambda_n^{++}$.

The following are two important examples. For a positive integer α, $(\alpha, 0, \ldots, 0) \in \Lambda_n^{++}$ is denoted by (α). Then $\rho_n^{(\alpha)} \cong S^\alpha(\mathbb{C}^n)$. In particular, $\rho_n^{(1)} \cong \mathbb{C}^n$ is the standard representation of GL_n. For a positive integer $\beta \leq n$, let

$$1_\beta = \overbrace{(1, 1, \ldots, 1)}^{\beta} \in \Lambda_n^{++}.$$

Then $\rho_n^{1_\beta} \cong \bigwedge^\beta \mathbb{C}^n$. Specifically, $\rho_n^{1_n} \cong \det_n$.

Definition 2.1.1. If $\lambda = (\lambda_1, ..., \lambda_n)$ and $\mu = (\mu_1, ..., \mu_n) \in \Lambda_n^+$ satisfy

$$\mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \ldots \geq \mu_n \geq \lambda_n,$$

then we say μ **interlaces** λ and write $\lambda \sqsubseteq \mu$.

Theorem 2.1.1 (Pieri Rule [4], [10]**).** *Let $D \in \Lambda_n^{++}$ and $\alpha \in \mathbb{Z}_{\geq 0}$. Then*

$$\rho_n^D \otimes \rho_n^{(\alpha)} = \bigoplus_{\substack{F \in \Lambda_n^{++}, D \sqsubseteq F \\ |D|+\alpha = |F|}} \rho_n^F. \tag{2.2}$$

By iterating the Pieri rule, we obtain the following result.

Theorem 2.1.2 ([7])**.** *Let $D \in \Lambda_n^{++}$, $\alpha = (\alpha_1, \ldots, \alpha_h) \in \mathbb{Z}_{\geq 0}^h$. We have*

$$\rho_n^D \otimes \left(\bigotimes_{s=1}^{h} \rho_n^{(\alpha_s)} \right) = \bigoplus K_{F/D,\alpha} \rho_n^F,$$

where the multiplicity $K_{F/D,\alpha}$ equals to the number of sequences

$$D = D^{(0)} \sqsubseteq D^{(1)} \sqsubseteq D^{(2)} \sqsubseteq \ldots \sqsubseteq D^{(h)} = F$$

satisfying $|D^{(s-1)}| + \alpha_s = |D^{(s)}|$ for $1 \leq s \leq h$.

This iterated Pieri rule is called **polynomial iterated Pieri rule.** An algebra $\mathfrak{R}$ is called a **polynomial iterated Pieri algebra** if

1. $\mathfrak{R}$ is graded, $\mathfrak{R} = \oplus_{D,\alpha,F} \mathfrak{R}_{D,\alpha,F}$;
2. $\dim(\mathfrak{R}_{D,\alpha,F}) = K_{F/D,\alpha}$.

In [7], the authors described the structure of polynomial iterated Pieri algebra very carefully. The multiplicity $K_{F/D,\alpha}$ is the key part. We shall review a combinatorial way to describe it in next subsection.

2.2 Gelfand-Tsetlin Patterns

The following array of integers

$$\begin{array}{ccccccc} & & & \mu_1^{(0)} & & & \\ & & \mu_1^{(1)} & & \mu_2^{(1)} & & \\ & \cdot^{\cdot^{\cdot}} & & \cdot^{\cdot^{\cdot}} & & \ddots & \\ \mu_1^{(l)} & & \mu_2^{(l)} & & \cdots & & \mu_n^{(l)} \end{array}$$

is called a **Gelfand-Tsetlin (GT) pattern** if

$$\mu_t^{(s+1)} \geq \mu_t^{(s)} \geq \mu_{t+1}^{(s+1)} \tag{2.3}$$

for all applicable s and t. This is the original GT pattern. We may generalize this concept to all patterns satisfying the condition that **each entry is not greater than the one on the left bottom and not less than the one on the right bottom.** A sequence of Λ_n^{++}

$$D = D^{(0)} \sqsubseteq D^{(1)} \sqsubseteq D^{(2)} \sqsubseteq \ldots \sqsubseteq D^{(h)} = F$$

corresponds to a GT pattern of the form

$$\begin{array}{ccccccccc} & & & d_1^{(0)} & & d_2^{(0)} & & \cdots & & d_n^{(0)} \\ & & d_1^{(1)} & & d_2^{(1)} & & \cdots & & d_n^{(1)} & \\ & \cdot^{\cdot^{\cdot}} & & \cdot^{\cdot^{\cdot}} & & & & \cdot^{\cdot^{\cdot}} & & \\ d_1^{(h)} & & d_2^{(h)} & & \cdots & & d_n^{(h)} & & & \end{array} \tag{2.4}$$

where $D^{(s)} = (d_1^{(s)}, d_2^{(s)}, \cdots, d_n^{(s)})$ for $1 \leq s \leq h$. In fact, there is a bijection between the set of sequences

$$D = D^{(0)} \sqsubseteq D^{(1)} \sqsubseteq D^{(2)} \sqsubseteq \ldots \sqsubseteq D^{(h)} = F$$

satisfying $|D^{(s-1)}| + \alpha_s = |D^{(s)}|$ for $1 \leq s \leq h$ and the set of all the GT patterns of the form (2.4) with nonnegative integer entries satisfying

1). $D = (d_1^{(0)}, d_2^{(0)}, \ldots, d_n^{(0)})$, $F = (d_1^{(h)}, d_2^{(h)}, \ldots, d_n^{(h)})$ and

2). $\alpha_s = \sum_{t=1}^{n} d_t^{(s)} - \sum_{t=1}^{n} d_t^{(s-1)}$ for $1 \leq s \leq h$.

Therefore, the number of these GT patterns equals to $K_{F/D,\alpha}$.

2.3 Hibi Cones

We now review the definition and structure of Hibi cones. The results of this and the next part are due to Howe ([11]).

Definition 2.3.1. Let $(\Gamma, \succeq)$ be a **poset** (partially ordered set) and B be a nonempty subset of $\mathbb{R}$. A map $f : \Gamma \to B$ is called **order preserving** if $f(x) \geq f(y)$ for $x \succeq y$.

We denote the set of all order preserving maps from Γ to B by $B^{\Gamma,\succeq}$. When $B = \mathbb{Z}_{\geq 0}$, the semigroup $\mathbb{Z}_{\geq 0}^{\Gamma,\succeq}$ is called a **Hibi cone**. We are interested in the case of $B = \mathbb{Z}_{\geq 0}$ because GT-patterns with nonnegative integer entries can be identified with the elements of $\mathbb{Z}_{\geq 0}^{\Gamma,\succeq}$ for a suitable finite poset. Here the poset plays the role of a placeholder.

Definition 2.3.2. Define a poset $(\Gamma_{n,h}, \succeq)$ where the underlying set

$$\Gamma_{n,h} = \{\eta_t^{(s)} : 1 \leq t \leq n, 0 \leq s \leq h\} \tag{2.5}$$

and the partial ordering on it is defined by the **interlacing conditions**

$$\eta_t^{(s+1)} \succeq \eta_t^{(s)} \succeq \eta_{t+1}^{(s+1)} \tag{2.6}$$

for every s and t.

The poset $(\Gamma_{n,h}, \succeq)$ can be illustrated as

$$\begin{array}{ccccccccc} & & & \eta_1^{(0)} & & \eta_2^{(0)} & & \cdots & & \eta_n^{(0)} \\ & & \eta_1^{(1)} & & \eta_2^{(1)} & & \cdots & & \eta_n^{(1)} & \\ & \cdot^{\cdot^{\cdot}} & & \cdot^{\cdot^{\cdot}} & & & & \cdot^{\cdot^{\cdot}} & & \\ \eta_1^{(h)} & & \eta_2^{(h)} & & \cdots & & \eta_n^{(h)}. & & & \end{array}$$

Then an element $f \in \mathbb{Z}_{\geq 0}^{\Gamma_{n,h},\succeq}$ can be illustrated as

$$\begin{array}{ccccccccc} & & f(\eta_1^{(0)}) & & f(\eta_2^{(0)}) & & \cdots & & f(\eta_n^{(0)}) \\ & f(\eta_1^{(1)}) & & f(\eta_2^{(1)}) & & \cdots & & f(\eta_n^{(1)}) & \\ \iddots & & \iddots & & & & \iddots & & \\ f(\eta_1^{(h)}) & & f(\eta_2^{(h)}) & & \cdots & & f(\eta_n^{(h)}). & & \end{array}$$

This is a GT pattern of the form (2.4).

Let

$$f^{(s)} := (f(\eta_1^{(s)}), f(\eta_2^{(s)}), \cdots, f(\eta_n^{(s)})).$$

Then $f^{(s)} \in \Lambda_n^{++}$. We define the **weight** of $f \in \mathbb{Z}_{\geq 0}^{\Gamma_{n,h},\succeq}$ by

$$\mathrm{wt}(f) := (|f^{(1)}| - |f^{(0)}|, |f^{(2)}| - |f^{(1)}|, \dots, |f^{(h)}| - |f^{(h-1)}|).$$

Define

$$(\mathbb{Z}_{\geq 0}^{\Gamma_{n,h},\succeq})_{F,D,\alpha} := \{f \in \mathbb{Z}_{\geq 0}^{\Gamma_{n,h},\succeq} : f^{(0)} = D, f^{(h)} = F, \mathrm{wt}(f) = \alpha\}.$$

Lemma 2.3.1. *There is a bijection between the set of all the GT patterns of the form (2.4) with nonnegative integer entries satisfying*

1). $D = (d_1^{(0)}, d_2^{(0)}, \dots, d_n^{(0)})$, $F = (d_1^{(h)}, d_2^{(h)}, \dots, d_n^{(h)})$ and

2). $\alpha_s = \sum_{t=1}^n d_t^{(s)} - \sum_{t=1}^n d_t^{(s-1)}$ for $1 \leq s \leq h$

and the set $(\mathbb{Z}_{\geq 0}^{\Gamma_{n,h},\succeq})_{F,D,\alpha}$.

Therefore, $K_{F/D,\alpha}$ equals the cardinality of $(\mathbb{Z}_{\geq 0}^{\Gamma_{n,h},\succeq})_{F,D,\alpha}$, denoted by $\#\left((\mathbb{Z}_{\geq 0}^{\Gamma_{n,h},\succeq})_{F,D,\alpha}\right)$.

2.4 The Structure of Hibi Cones

To describe the structure of Hibi cones, we introduce several concepts of poset.

Definition 2.4.1. [19] Let Γ be a finite poset.

- A subset S of Γ is called **increasing** if for any $x \in S$ and any $y \in \Gamma$,

$$y \succeq x \Rightarrow y \in S.$$

  The collection of all increasing subsets of Γ is denoted by $J^*(\Gamma, \succeq)$. Similarly, we can define **decreasing** sets.

- For any subset S of Γ, the **indicator function** of S is the map $\chi_S : \Gamma \to \{0, 1\}$ defined by

$$\chi_S(x) = \begin{cases} 1 & x \in S \\ 0 & x \notin S. \end{cases} \tag{2.7}$$

- The **dual** of a poset Γ is the poset Γ^* with the same underlying set Γ such that $x \preceq y$ in Γ^* if and only if $y \preceq x$ in Γ.

One important property of Hibi cone is that each nonzero element of it has a unique "standard" expression.

Theorem 2.4.1 ([11]). *The semigroup $\mathbb{Z}_{\geq 0}^{\Gamma,\succeq}$ is generated by $\{\chi_S : S \in J^*(\Gamma,\succeq)\}$. More precisely, every nonzero element f of $\mathbb{Z}_{\geq 0}^{\Gamma,\succeq}$ has a unique expression*

$$f = \sum_{j=1}^{h} a_j \chi_{S_j}$$

where a_j are positive integers for $1 \leq j \leq h$ and

$$\emptyset \subsetneq S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_h$$

is a chain in the poset $J^*(\Gamma,\succeq)$.

2.5 Semigroup Algebras on Hibi Cones

Definition 2.5.1 ([1]). For a semigroup S, let $\mathbb{C}[S]$ be the vector space with basis

$$\mathcal{B} = \{X^f : f \in S\}.$$

For f, $g \in S$, define the multiplication

$$X^f X^g = X^{f+g}.$$

Then the vector space $\mathbb{C}[S]$ together with the multiplication operation forms a complex algebra, called the **semigroup algebra** on S.

When $S = \mathbb{Z}_{\geq 0}^{\Gamma,\succeq}$, $\mathbb{C}[\mathbb{Z}_{\geq 0}^{\Gamma,\succeq}]$ is a **Hibi algebra** [6]. The Hibi cone is named after this property.

Definition 2.5.2 ([18]). Let R be a complex algebra and let $\mathcal{G}$ be a finite set of elements of R with a partial ordering $\preceq$.

(a). If $g_1 \preceq g_2 \preceq \ldots \preceq g_s$ is a multichain in $\mathcal{G}$, then we call the product $g_1 g_2 \cdots g_s$ a **standard monomial** on $\mathcal{G}$.

(b). Let $\mathcal{B}$ be the set of all standard monomials on $\mathcal{G}$. If $\mathcal{B}$ forms a basis for R, then we call $\mathcal{B}$ a **standard monomial basis** and say that R has a **standard monomial theory** for $\mathcal{G}$.

The semigroup algebra $\mathbb{C}[\mathbb{Z}_{\geq 0}^{\Gamma,\succeq}]$ has a standard monomial theory for $\{\chi_S : S \in J^*(\Gamma,\succeq)\}$.

2.6 Flat Deformation

In this part, we briefly review the concepts of **flat deformation** and **Sagbi basis**.

Definition 2.6.1. Let R be a subalgebra of the polynomial algebra $\mathbb{C}[x_1,\ldots,x_m]$, with well-defined monomial order.

(a). For $f \in R$, denote $\mathrm{LM}(f)$ the **leading monomial** of f. Let $\mathrm{LM}(R) := \{\mathrm{LM}(f) : f \in R\}$.

(b). The subalgebra of $\mathbb{C}[x_1,\ldots,x_m]$ generated by $\mathrm{LM}(R)$ is called the **initial algebra** of R. It is denoted by $\mathbb{C}[\mathrm{LM}(R)]$.

(c). A set S of nonzero polynomials in R is called a **Sagbi basis** for R if the set

$$\mathrm{LM}(S) = \{\mathrm{LM}(f) : f \in S\}$$

generates the initial algebra $\mathbb{C}[\mathrm{LM}(R)]$ of R.

The initial algebra $\mathbb{C}[\mathrm{LM}(R)]$ is the semigroup algebra on $\mathrm{LM}(R)$. If the initial algebra $\mathbb{C}[\mathrm{LM}(R)]$ of R is finitely generated, then a general result says that $\mathbb{C}[\mathrm{LM}(R)]$ is a good approximation to R in the following sense.

Theorem 2.6.1 ([2]). *Let $\mathbb{C}[x_1, \ldots, x_m]$ be given a monomial ordering and let R be a subalgebra of $\mathbb{C}[x_1, \ldots, x_m]$. If the initial algebra $\mathbb{C}[\mathrm{LM}(R)]$ is finitely generated, then there exists a flat one-parameter family of $\mathbb{C}$-algebras with general fibre R and special fibre $\mathbb{C}[\mathrm{LM}(R)]$.*

3 Anti-row Iterated Pieri Rule for GL_n

In this section, we discuss the specific Pieri rule studied in this paper.

3.1 Generalized Pieri Rules

There is a more general version of the Pieri rule. It can be considered as folklore.

Theorem 3.1.1 (Generalized Pieri Rules). *Let $\lambda \in \Lambda_n^+$ and $\alpha \in \mathbb{Z}_{\geq 0}$. Then*

$$(a). \qquad \rho_n^\lambda \otimes \rho_n^{(\alpha)} = \bigoplus_{\substack{\lambda \sqsubseteq \mu \\ |\lambda|+\alpha=|\mu|}} \rho_n^\mu$$

and

$$(b). \qquad \rho_n^\lambda \otimes \rho_n^{(\alpha)^*} = \bigoplus_{\substack{\mu \sqsubseteq \lambda \\ |\lambda|-\alpha=|\mu|}} \rho_n^\mu.$$

Here $\rho_n^{(\alpha)^*}$ is contragrediant to $\rho_n^{(\alpha)}$.

3.2 Anti-row Iterated Pieri Rule for GL_n

Let $D \in \Lambda_n^{++}$, $\alpha = (\alpha_1, \ldots, \alpha_l) \in \mathbb{Z}_{\geq 0}^l$. By iterating the formula in Theorem 3.1.1 (b), we have

$$\rho_n^D \otimes \left(\bigotimes_{s=1}^{l} \rho_n^{(\alpha_s)^*} \right) = \bigoplus K_{\lambda/D,-\alpha} \rho_n^\lambda,$$

where the multiplicity $K_{\lambda/D,-\alpha}$ is equal to the number of sequences

$$D = \lambda^{(0)} \sqsupseteq \lambda^{(1)} \sqsupseteq \lambda^{(2)} \sqsupseteq \ldots \sqsupseteq \lambda^{(l)} = \lambda$$

satisfying $|\lambda^{(s-1)}| - \alpha_s = |\lambda^{(s)}|$ for $1 \leq s \leq l$.

Follow previous idea, each sequence

$$\lambda^{(0)} \sqsupseteq \lambda^{(1)} \sqsupseteq \lambda^{(2)} \sqsupseteq \ldots \sqsupseteq \lambda^{(l)}$$

corresponds to a GT pattern

$$\begin{array}{ccccccccc}
\lambda_1^{(0)} & & \lambda_2^{(0)} & & \cdots & & \lambda_n^{(0)} & & \\
& \lambda_1^{(1)} & & \lambda_2^{(1)} & & \cdots & & \lambda_n^{(1)} & \\
& & \ddots & & \ddots & & & & \ddots \\
& & & \lambda_1^{(l)} & & \lambda_2^{(l)} & & \cdots & & \lambda_n^{(l)}
\end{array}$$

and vice versa. Therefore, the multiplicity $K_{\lambda/D,-\alpha}$ equals the number of the GT patterns of the form

$$\begin{array}{ccccccccc} \lambda_1^{(0)} & & \lambda_2^{(0)} & & \cdots & & \lambda_n^{(0)} & & \\ & \lambda_1^{(1)} & & \lambda_2^{(1)} & & \cdots & & \lambda_n^{(1)} & \\ & & \ddots & & \ddots & & & & \ddots \\ & & & \lambda_1^{(l)} & & \lambda_2^{(l)} & & \cdots & & \lambda_n^{(l)} \end{array}$$

where

1). $D = (\lambda_1^{(0)}, \ldots, \lambda_n^{(0)})$, $\lambda = (\lambda_1^{(l)}, \ldots, \lambda_n^{(l)})$ and

2). $-\alpha_s = \sum_{t=1}^n \lambda_t^{(s)} - \sum_{t=1}^n \lambda_t^{(s-1)}$ for $1 \le s \le l$.

Here the GT pattern cannot be identified with an element of $\mathbb{Z}_{\ge 0}^{\Gamma,\succeq}$ for certain poset Γ because some $\lambda_j^{(i)}$s can be negative. We shall generalize the concept of Hibi cones to sign Hibi cones.

4 Sign Hibi Cones

All the entries of a Hibi cone $\mathbb{Z}_{\ge 0}^{\Gamma,\succeq}$ are nonnegative. To obtain negative entries, it is natural to consider $\mathbb{Z}^{\Gamma,\succeq}$. It is still a semigroup. So we consider a specific subsemigroup of $\mathbb{Z}^{\Gamma,\succeq}$.

4.1 Sign Hibi Cones

Definition 4.1.1. Let A and B be two subsets of a poset Γ. Define

$$\Omega_{A,B}(\Gamma) := \{f \in \mathbb{Z}^{\Gamma,\succeq} : f(A) \ge 0, f(B) \le 0\}. \tag{4.1}$$

Here $f(A) \ge 0$ means that $f(x) \ge 0$ for all $x \in A$. We call $\Omega_{A,B}(\Gamma)$ a **sign Hibi Cone** if $\Omega_{A,B}(\Gamma) \ne \{0\}$.

Clearly, it forms a subsemigroup of $\mathbb{Z}^{\Gamma,\succeq}$. If $A = B = \emptyset$, then $\Omega_{A,B}(\Gamma) = \mathbb{Z}^{\Gamma,\succeq}$, and if $A = \Gamma$, $B = \emptyset$, then $\Omega_{A,B}(\Gamma) = \mathbb{Z}_{\ge 0}^{\Gamma,\succeq}$. Therefore, sign Hibi cone is a more general construction than both Hibi cones and $\mathbb{Z}^{\Gamma,\succeq}$. In the absence of ambiguity, we shall write $\Omega_{A,B}$ instead of $\Omega_{A,B}(\Gamma)$.

4.2 Structure of Sign Hibi Cones

First let us connect sign Hibi cones with Hibi cones.

For A, $B \subseteq \Gamma$, let P_A be **the smallest increasing subset** of Γ containing A and N_B **the smallest decreasing subset** of Γ containing B. Define

$$\Gamma_{A,B}^{+} := \Gamma \setminus N_B \quad \text{and} \quad \Gamma_{A,B}^{-} := \Gamma \setminus P_A. \tag{4.2}$$

Then $\Gamma_{A,B}^{+}$ is an increasing subset of Γ and $\Gamma_{A,B}^{-}$ is decreasing.

Theorem 4.2.1 ([21]). *Let A, $B \subseteq \Gamma$.*

(a). Ω_{Γ,N_B} and $\Omega_{P_A,\Gamma}$ are subsemigroups of $\Omega_{A,B}$. Moreover, $\Omega_{\Gamma,N_B} \cong \mathbb{Z}_{\ge 0}^{\Gamma_{A,B}^{+},\succeq}$ and $\Omega_{P_A,\Gamma} \cong \mathbb{Z}_{\ge 0}^{\Gamma_{A,B}^{-*},\succeq}$. Here $\Gamma_{A,B}^{-*}$ is the dual poset of $\Gamma_{A,B}^{-}$.

(b). The semigroup $\Omega_{A,B}$ is generated by Ω_{Γ,N_B} and $\Omega_{P_A,\Gamma}$. That is, it is the smallest subsemigroup of $\mathbb{Z}^{\Gamma,\succeq}$ which contains Ω_{Γ,N_B} and $\Omega_{P_A,\Gamma}$.

(c). Specifically,

$$\Omega_{A,B} \cong \mathbb{Z}_{\geq 0}^{\Gamma^{+}_{A,B},\succeq} \times \mathbb{Z}_{\geq 0}^{\Gamma^{-*}_{A,B},\succeq} \tag{4.3}$$

if $\Gamma^{+}_{A,B} \cap \Gamma^{-}_{A,B} = \emptyset$.

Remark. The cross product of two Hibi cones is still a Hibi cone.

For Hibi cones, each nonzero element has a unique "standard" expression (Theorem 2.4.1). The second part is to establish a parallel result for sign Hibi cones.

Corollary 4.2.2. *Let*

$$\mathcal{G}^{+}_{A,B} = \{\chi_P : P \in J^*(\Gamma^{+}_{A,B}, \succeq)\} \tag{4.4}$$

and

$$\mathcal{G}^{-}_{A,B} = \{-\chi_Q : Q \in J^*(\Gamma^{-*}_{A,B}, \succeq)\}. \tag{4.5}$$

Then the semigroup $\Omega_{A,B}$ *is generated by* $\mathcal{G}^{+}_{A,B}$ *and* $\mathcal{G}^{-}_{A,B}$.

Definition 4.2.1. Let

$$\mathcal{G}_{A,B} = \mathcal{G}^{+}_{A,B} \cup \mathcal{G}^{-}_{A,B}.$$

Define the partial ordering on $\mathcal{G}_{A,B}$ as follows: For P_1 and $P_2 \in J^*(\Gamma^{+}_{A,B}, \succeq)$, Q_1 and $Q_2 \in J^*(\Gamma^{-*}_{A,B}, \succeq)$,

(a). $\chi_{P_1} \preceq \chi_{P_2}$ if and only if $P_1 \subseteq P_2$;

(b). $-\chi_{Q_1} \preceq -\chi_{Q_2}$ if and only if $Q_1 \supseteq Q_2$; and

(c). $\chi_{P_1} \preceq -\chi_{Q_1}$ if and only if $P_1 \cap Q_1 = \emptyset$.

Now we can state the main theorem.

Theorem 4.2.3 ([21]). *Each nonzero element* f *of* $\Omega_{A,B}$ *can be expressed uniquely as*

$$f = \sum_{i=1}^{s} a_i \chi_{P_i} + \sum_{j=1}^{t} b_j(-\chi_{Q_j}),$$

where

$$\chi_{P_1} \preceq \cdots \preceq \chi_{P_s} \preceq -\chi_{Q_1} \preceq \cdots \preceq -\chi_{Q_t}$$

is a chain in $\mathcal{G}_{A,B}$ *and* $a_1, \ldots, a_s, b_1, \ldots, b_t$ *are positive integers.*

4.3 Semigroup Algebras on Sign Hibi Cones

Finally, we shall study the semigroup algebra $\mathbb{C}[\Omega_{A,B}]$. Define

$$\mathfrak{B}_{A,B} = \{X^f : f \in \Omega_{A,B}\}. \tag{4.6}$$

Then $\mathfrak{B}_{A,B}$ is a basis for $\mathbb{C}[\Omega_{A,B}]$. Let

$$\mathfrak{G}_{A,B} = \{X^f : f \in \mathcal{G}_{A,B}\}, \tag{4.7}$$

and define a partial ordering on $\mathfrak{G}_{A,B}$ by

$$X^{f_1} \preceq X^{f_2} \text{ if and only if } f_1 \preceq f_2 \text{ in } \mathcal{G}_{A,B}.$$

By Theorem 4.2.3, we have the following theorem.

Theorem 4.3.1. *The set* $\mathfrak{B}_{A,B}$ *is a standard monomial basis for* $\mathbb{C}[\Omega_{A,B}]$ *and* $\mathbb{C}[\Omega_{A,B}]$ *has a standard monomial theory for* $\mathfrak{G}_{A,B}$.

4.4 Sign Hibi Cone $\Omega_{n,k,l}$

In this part, we look at a concrete example of sign Hibi cone, which is also necessary for the next section. The first step is to define the poset.

Definition 4.4.1.

1. Define a poset $(\Gamma_{n,l}, \succeq)$

$$\begin{array}{ccccccccccc} \gamma_1^{(0)} & & \gamma_2^{(0)} & & \cdots & & \gamma_n^{(0)} & & & & \\ & \gamma_1^{(1)} & & \gamma_2^{(1)} & & \cdots & & \gamma_n^{(1)} & & & \\ & & \ddots & & \ddots & & & & \ddots & & \\ & & & \gamma_1^{(l)} & & \gamma_2^{(l)} & & \cdots & & \gamma_n^{(l)}, & \end{array}$$

   where the elements satisfy the **interlacing conditions**

$$\gamma_t^{(s)} \succeq \gamma_t^{(s+1)} \succeq \gamma_{t+1}^{(s)} \tag{4.8}$$

   for every s and t.

2. Define $\Omega_{n,k,l} := \Omega_{A,B}(\Gamma_{n,l})$ where

$$A = \{\gamma_n^{(0)}\} \quad \text{and} \quad B = \begin{cases} \{\gamma_{k+1}^{(0)}\} & k < n \\ \emptyset & k = n. \end{cases} \tag{4.9}$$

Remark. The poset $\Gamma_{n,l}$ is the same as the one in Definition 2.3.2 when we identify $\gamma_t^{(s)}$ with $\eta_t^{(l-s)}$.

By Theorem 4.2.1, to describe the structure of $\Omega_{A,B}$, there are two important sets, $\Gamma_{A,B}^+$ and $\Gamma_{A,B}^-$. In this case, denote $\Gamma_{n,k,l}^+ := \Gamma_{A,B}^+$ and $\Gamma_{n,k,l}^- := \Gamma_{A,B}^-$. Then we have

$$\Gamma_{n,k,l}^+ = \begin{cases} \left\{\gamma_t^{(s)} : 0 \le s \le l,\ 1 \le t \le k\right\} & \text{when} \quad k < n \\ \Gamma_{n,l} & \text{when} \quad k = n. \end{cases}$$

and

$$\Gamma_{n,k,l}^- = \left\{\gamma_t^{(s)} : 1 \le s \le l,\ \max\{n-s+1, 1\} \le t \le n\right\}.$$

Definition 4.4.2. Let c be an integer such that $1 \le c \le k$. Let I and J be two subsets of $\{1, 2, \ldots, l\}$ such that $\#(I) \le c$ and $1 \le \#(J) \le n$. Define

(a).

$$A(c, I) = \{\gamma_t^{(s)} \in \Gamma_{n,l} : 1 \le t \le a_s, \quad 0 \le s \le l\},$$

where $a_0 = c$, $a_i = \begin{cases} a_{i-1} & \text{if} \quad i \notin I \\ a_{i-1} - 1 & \text{if} \quad i \in I. \end{cases}$

(b).

$$B(J) = \{\gamma_t^{(s)} \in \Gamma_{n,l} : 0 \le s \le l, \quad b_s \le t \le n\},$$

where $b_0 = n+1$, $b_j = \begin{cases} b_{j-1} & \text{if} \quad j \notin J \\ b_{j-1} - 1 & \text{if} \quad j \in J. \end{cases}$

It is easy to check that $A(c, I) \in J^*(\Gamma_{n,k,l}^+, \succeq)$ and $B(J) \in J^*(\Gamma_{n,k,l}^{-*}, \succeq)$.

Proposition 4.4.1. *We have*

$$J^*(\Gamma^+_{n,k,l}, \succeq) = \{A(c,I) : 1 \le c \le k,\ I \subseteq \{1,2,\ldots,l\},\ \#(I) \le c\}$$

and

$$J^*(\Gamma^{-*}_{n,k,l}, \succeq) = \{B(J) : J \subseteq \{1,2,\ldots,l\},\ 1 \le \#(J) \le n\}.$$

By Corollary 4.2.2, $\Omega_{n,k,l}$ is generated by $\mathcal{G}^+_{n,k,l} = \{\chi_{A(c,I)}\}$ and $\mathcal{G}^-_{n,k,l} = \{-\chi_{B(J)}\}$.

Corollary 4.4.2. *Let $\mathcal{G}_{n,k,l} := \mathcal{G}^+_{n,k,l} \cup \mathcal{G}^-_{n,k,l}$. Then $\Omega_{n,k,l}$ is generated by $\mathcal{G}_{n,k,l}$. More precisely, each nonzero function $f \in \Omega_{n,k,l}$ can be uniquely written as*

$$f = \sum_{s=1}^{p} a_s \chi_{A(c_s,I_s)} + \sum_{t=1}^{q} b_t(-\chi_{B(J_t)}),$$

where a_s and b_t are positive integers for $1 \le s \le p$, $1 \le t \le q$ and

$$\chi_{A(c_1,I_1)} \prec \cdots \prec \chi_{A(c_p,I_p)} \prec -\chi_{B(J_1)} \prec \cdots \prec -\chi_{B(J_q)}$$

is a chain in $(\mathcal{G}_{n,k,l}, \succeq)$.

Now we show the relation between sign Hibi cone $\Omega_{n,k,l}$ and anti-row iterated Pieri rule. As in subsection 2.3, for each $f \in \mathbb{Z}^{\Gamma_{n,l},\succeq}$, define

$$f^{(s)} = (f(\gamma_1^{(s)}), f(\gamma_2^{(s)}), \cdots, f(\gamma_n^{(s)})) \qquad (0 \le s \le l)$$

and define the **weight** of f by

$$\mathrm{wt}(f) := (|f^{(1)}| - |f^{(0)}|, |f^{(2)}| - |f^{(1)}|, \ldots, |f^{(h)}| - |f^{(h-1)}|).$$

For $D \in \Lambda_n^{++}$ with $\mathrm{depth}(D) \le k$, $\lambda \in \Lambda_n^+$ and $\alpha \in \mathbb{Z}^l_{\ge 0}$, let

$$\Omega_{\lambda,D,\alpha} = \{f \in \mathbb{Z}^{\Gamma_{n,l},\succeq} : f^{(0)} = D,\ f^{(l)} = \lambda,\ \mathrm{wt}(f) = -\alpha\}.$$

Theorem 4.4.3. *(a). We have*

$$\Omega_{n,k,l} = \bigcup_{\lambda,D,\alpha} \Omega_{\lambda,D,\alpha}$$

where the union is taken over all $D \in \Lambda_n^{++}$ with $\mathrm{depth}(D) \le k$, $\lambda \in \Lambda_n^+$ and $\alpha \in \mathbb{Z}^l_{\ge 0}$.

(b). The number of elements in $\Omega_{\lambda,D,\alpha}$ is equal to $K_{\lambda/D,-\alpha}$.

5 Anti-row Iterated Pieri Algebras

Let n, k and l be positive integers such that $k \le n$. In this section, we provide results about the structure of an algebra $\mathfrak{A}_{n,k,l}$ called the **anti-row iterated Pieri algebra**. It is named after the property that it encodes the anti-row iterated Pieri rule.

5.1 $(\mathrm{GL}_n,\ \mathrm{GL}_k)$-Duality

First, we state the key theorem for the realization of the representations.

Let M_{nk} be the space of all complex $n \times k$ matrices and $\mathcal{P}(\mathrm{M}_{nk})$ be the algebra of polynomial functions on M_{nk}. Define

$$\left(\tau^*_{n,k}(g,h)(f)\right)(T) = f(g^tTh) \tag{5.1}$$

and

$$\left(\tau'^*_{n,k}(g,h)(f)\right)(T) = f(g^{-1}Th), \tag{5.2}$$

where $(g,h) \in \mathrm{GL}_n \times \mathrm{GL}_k$, $f \in \mathcal{P}(\mathrm{M}_{nk})$ and $T \in \mathrm{M}_{nk}$.

Theorem 5.1.1 ($(\mathrm{GL}_n,\ \mathrm{GL}_k)$-duality, [10]).

(a). Under the action $\tau^*_{n,k}$, $\mathcal{P}(\mathrm{M}_{nk})$ is decomposed into a direct sum of irreducible $\mathrm{GL}_n \times \mathrm{GL}_k$ representations as

$$\mathcal{P}(\mathrm{M}_{nk}) \cong \sum_{\mathrm{depth}(D)\leq \min(n,k)} \rho_n^D \otimes \rho_k^D.$$

(b). Under the action $\tau'^*_{n,k}$, $\mathcal{P}(\mathrm{M}_{nk})$ is decomposed as

$$\mathcal{P}(\mathrm{M}_{nk}) \cong \sum_{\mathrm{depth}(D)\leq \min(n,k)} \rho_n^{D^*} \otimes \rho_k^D.$$

5.2 Anti-row Iterated Pieri Algebras

For the algebra of polynomial functions $\mathcal{P}(\mathrm{M}_{n,k+l})$, we have

$$\mathcal{P}(\mathrm{M}_{n,k+l}) \cong \mathcal{P}\left(\mathrm{M}_{n,k} \oplus \left(\bigoplus_{j=1}^{l} \mathbb{C}_j^n\right)\right) \cong \mathcal{P}(\mathrm{M}_{n,k}) \otimes \left(\bigotimes_{j=1}^{l} \mathcal{P}(\mathbb{C}_j^n)\right).$$

Let $\mathrm{GL}_n \times \mathrm{GL}_k$ act on $\mathcal{P}(\mathrm{M}_{n,k})$ by $\tau^*_{n,k}$ and $\mathrm{GL}_n \times \mathrm{GL}_1$ act on $\mathcal{P}(\mathbb{C}_j^n)$ for $1 \leq j \leq l$ by $\tau'^*_{n,1}$. Then $\mathcal{P}(\mathrm{M}_{n,k+l})$ becomes a representation of

$$(\mathrm{GL}_n \times \mathrm{GL}_k) \times (\mathrm{GL}_n \times \mathrm{GL}_1)^l \cong (\mathrm{GL}_n \times \mathrm{GL}_n^l) \times \mathrm{GL}_k \times (\mathrm{GL}_1)^l \cong \mathrm{GL}_n^{l+1} \times \mathrm{GL}_k \times A_l.$$

We denote it by $(\rho, \mathcal{P}(\mathrm{M}_{n,k+l}))$. By the $(\mathrm{GL}_n, \mathrm{GL}_k)$-duality, we have

$$\begin{aligned}\mathcal{P}(\mathrm{M}_{n,k+l}) &\cong \left(\bigoplus_{\mathrm{depth}(D)\leq k} \rho_n^D \otimes \rho_k^D\right) \otimes \left\{\bigotimes_{j=1}^{l}\left(\bigoplus_{\alpha_j \in \mathbb{Z}_{\geq 0}} \rho_n^{(\alpha_j)^*} \otimes \rho_1^{(\alpha_j)}\right)\right\} \\ &\cong \bigoplus_{\substack{\mathrm{depth}(D)\leq k \\ (\alpha_1,\dots,\alpha_l)\in \mathbb{Z}^l_{\geq 0}}} \left(\rho_n^D \otimes \rho_n^{(\alpha_1)^*} \otimes \cdots \otimes \rho_n^{(\alpha_l)^*}\right) \otimes \rho_k^D \otimes \rho_1^{(\alpha_1)} \otimes \cdots \otimes \rho_1^{(\alpha_l)}.\end{aligned}$$

By extracting the U_k invariants in $\mathcal{P}(\mathrm{M}_{n,k+l})$, we obtain

$$\mathcal{P}(\mathrm{M}_{n,k+l})^{U_k} \cong \bigoplus_{\substack{\mathrm{depth}(D)\leq k \\ (\alpha_1,\dots,\alpha_l)\in \mathbb{Z}^l_{\geq 0}}} \left(\rho_n^D \otimes \rho_n^{(\alpha_1)^*} \otimes \cdots \otimes \rho_n^{(\alpha_l)^*}\right) \otimes \left(\rho_k^D\right)^{U_k} \otimes \psi_l^\alpha.$$

The $\psi_k^D \times \psi_l^\alpha$ eigenspace of $A_k \times A_l$ in $\mathcal{P}(\mathrm{M}_{n,k+l})^{U_k}$ is the realization of the tensor product $\rho_n^D \otimes \rho_n^{(\alpha_1)^*} \otimes \cdots \otimes \rho_n^{(\alpha_l)^*}$.

Now we restrict the representation ρ to $\mathrm{GL}_n \times \mathrm{GL}_k \times A_l$ where $\mathrm{GL}_n \cong \triangle(\mathrm{GL}_n^{l+1})$ is the diagonal subgroup of GL_n^{l+1}. Apply the anti-row iterated Pieri rule,

$$\mathcal{P}(\mathrm{M}_{n,k+l})^{U_k} \cong \bigoplus_{\substack{\mathrm{depth}(D)\leq k \\ (\alpha_1,\ldots,\alpha_l)\in\mathbb{Z}^l_{\geq 0}}} \left(\bigoplus_{\lambda\in\Lambda_n^+} K_{\lambda/D,-\alpha}\rho_n^\lambda \right) \otimes \left(\rho_k^D\right)^{U_k} \otimes \psi_l^\alpha.$$

Define

$$\mathfrak{A}_{n,k,l} := \mathcal{P}(\mathrm{M}_{n,k+l})^{U_n\times U_k}. \tag{5.3}$$

Then $\mathfrak{A}_{n,k,l}$ is a module for $A_n \times A_k \times A_l$. Let $\mathfrak{A}_{\lambda,D,\alpha}$ be the $\psi_n^\lambda \times \psi_k^D \times \psi_l^\alpha$ eigenspace of $A_n \times A_k \times A_l$ in $\mathfrak{A}_{n,k,l}$, then

- $\mathfrak{A}_{n,k,l} = \bigoplus_{\lambda,D,\alpha} \mathfrak{A}_{\lambda,D,\alpha}$ and
- $\dim \mathfrak{A}_{\lambda,D,\alpha} = K_{\lambda/D,-\alpha}$.

Thus, we call $\mathfrak{A}_{n,k,l}$ an **anti-row iterated Pieri algebra**. One of the main goals is to determine the structure of this algebra.

5.3 Standard Monomial Basis of Anti-row Iterated Pieri Algebras

In this part, we only summarize the results about the structure of the anti-row iterated Pieri algebra $\mathfrak{A}_{n,k,l}$ without detail. First, we need to state several definitions and notations.

1. For $f \in \mathcal{G}_{n,k,l}$, define $v_f \in \mathcal{P}(\mathrm{M}_{n,k+l})$ **explicitly** [21].
2. By Corollary 4.4.2, for each $f \in \Omega_{n,k,l}$, there is a unique standard expression $f = \sum_{s=1}^p a_s f_s$ such that a_s are all positive and $f_1 \prec f_2 \prec \cdots \prec f_p$ in $\mathcal{G}_{n,k,l}$. Define

$$v_f := \Pi_{s=1}^p \left(v_{f_s}\right)^{a_s} \tag{5.4}$$

   and

$$\mathfrak{B}_{n,k,l} := \{v_f : f \in \Omega_{n,k,l}\}. \tag{5.5}$$

3. Let

$$\mathfrak{G}_{n,k,l} := \{v_f : f \in \mathcal{G}_{n,k,l}\} \tag{5.6}$$

   and define a **partial ordering** $\succeq$ on $\mathfrak{G}_{n,k,l}$ as $v_f \succeq v_g$ if and only if $f \succeq g$ in $\mathcal{G}_{n,k,l}$.
4. With a graded lexicographic order, define **leading monomial** for each element of $\mathcal{P}(\mathrm{M}_{n,k+l})$.

The following is the main structure theorem.

Theorem 5.3.1. *Let n, k, l be positive integers such that $k \leq n$.*

(a). $\mathfrak{A}_{n,k,l}$ has a standard monomial theory on $\mathfrak{G}_{n,k,l}$ and $\mathfrak{B}_{n,k,l}$ is a standard monomial basis for $\mathfrak{A}_{n,k,l}$.

(b). We have

$$\mathrm{LM}(\mathfrak{A}_{n,k,l}) \cong \Omega_{n,k,l},$$

so that the initial algebra of $\mathfrak{A}_{n,k,l}$

$$\mathbb{C}[\mathrm{LM}(\mathfrak{A}_{n,k,l})] \cong \mathbb{C}[\Omega_{n,k,l}].$$

(c). $\mathbb{C}[\mathrm{LM}(\mathfrak{A}_{n,k,l})]$ *has a standard monomial theory on* $\mathrm{LM}(\mathfrak{G}_{n,k,l})$ *and* $\mathrm{LM}(\mathfrak{B}_{n,k,l})$ *is a standard monomial basis for* $\mathbb{C}[\mathrm{LM}(\mathfrak{A}_{n,k,l})]$.

(d). $\mathfrak{G}_{n,k,l}$ *is a finite Sagbi basis for* $\mathfrak{A}_{n,k,l}$.

(e). There exists a flat one-parameter family of $\mathbb{C}$*-algebras with general fibre* $\mathfrak{A}_{n,k,l}$ *and special fibre* $\mathbb{C}[\Omega_{n,k,l}]$.

Sketch of Proof.

(a). Let $D \in \Lambda_n^{++}$ with $\mathrm{depth}(D) \leq k$, $\lambda \in \Lambda_n^+$ and $\alpha \in \mathbb{Z}_{\geq 0}^l$. For each $f \in \Omega_{\lambda,D,\alpha}$, prove that $v_f \in \mathfrak{A}_{\lambda,D,\alpha}$. It can be proved that $\mathrm{LM}(v_f)$ is uniquely determined by f. Then all the v_f have distinct leading monomials. Because the cardinality of $\mathfrak{B}_{\lambda,D,\alpha} := \{v_f : f \in \Omega_{\lambda,D,\alpha}\}$ is correct for a basis of $\mathfrak{A}_{\lambda,D,\alpha}$. By equations 5.4 and 5.6, $\mathfrak{B}_{n,k,l}$ is a standard monomial basis.

(b). It suffices to prove that $\mathrm{LM}(\mathfrak{B}_{n,k,l}) \cong \Omega_{n,k,l}$ as semigroups and $\mathrm{LM}(\mathfrak{A}_{n,k,l}) = \mathrm{LM}(\mathfrak{B}_{n,k,l})$.

(c). For f and $g \in \mathcal{P}(\mathrm{M}_{n,k+l})$, $\mathrm{LM}(fg) = \mathrm{LM}(f)\,\mathrm{LM}(g)$. Then it is clear by (a).

(d). By (c).

(e). $\mathrm{LM}(\mathfrak{G}_{n,k,l}) \cong \mathcal{G}_{n,k,l}$ is finite.

Remarks. To understand the structure of an algebra, the classical method is to figure out the generators and relators. For $\mathfrak{A}_{n,k,l}$, the relators among the generators $\mathfrak{G}_{n,k,l}$ are very complicated. It is meaningless for us to understand the structure. So I choose another way: determine a basis of the algebra. And the basis has good properties. I borrowed the idea from [7], [13].

5.4 Applications to Howe Duality

For each positive integer m, let $\mathfrak{gl}_m = \mathfrak{gl}_m(\mathbb{C})$ be the general Lie algebra of all $m \times m$ complex matrices. In this subsection, we consider the lowest weight modules of $\mathfrak{gl}_{k+l}$ which occur in $\mathcal{P}(\mathrm{M}_{n,k+l})$.

Theorem 5.4.1 ([9],[3]). *There is a multiplicity free decomposition of* $\mathrm{GL}_n \times \mathfrak{gl}_{k+l}$*-modules given by*

$$\mathcal{P}(\mathrm{M}_{n,k+l}) \cong \sum_{\lambda \in \Lambda_n^+} \rho_n^\lambda \otimes \mathcal{L}_{k,l}^\lambda, \tag{5.7}$$

where λ *has at most* k *positive entries and at most* l *negative entries and* $\mathcal{L}_{k,l}^\lambda$ *is an irreducible lowest weight module of* $\mathfrak{gl}_{k+l}$ *with its lowest weight uniquely determined by* λ.

By previous discussion,

$$\mathfrak{A}_{n,k,l} = \mathcal{P}(\mathrm{M}_{n,k+l})^{U_n \times U_k} \cong \sum_\lambda \left(\rho_n^\lambda\right)^{U_n} \otimes \left(\mathcal{L}_{k,l}^\lambda\right)^{\mathfrak{n}_k^+},$$

where $\left(\mathcal{L}_{k,l}^\lambda\right)^{\mathfrak{n}_k^+}$ is spanned by all $\mathfrak{gl}_k$ highest weight vectors in $\mathcal{L}_{k,l}^\lambda$. In particular, $\left(\mathcal{L}_{k,l}^\lambda\right)^{\mathfrak{n}_k^+}$ can be identified with the ψ_n^λ eigenspace of A_n in $\mathfrak{A}_{n,k,l}$.

Corollary 5.4.2. *For* $\lambda \in \Lambda_n^+$ *with at most* k *positive entries and at most* l *negative entries, define*

$$\mathfrak{B}_\lambda = \{v_f \in \mathfrak{B}_{n,k,l} : f^{(l)} = \lambda\}. \tag{5.8}$$

Then $\mathfrak{B}_\lambda$ *forms a basis of* $\left(\mathcal{L}_{k,l}^\lambda\right)^{\mathfrak{n}_k^+}$.

6 Further Problems

In the last section, we describe two related problems.

6.1 Generalized Iterated Pieri Algebras for GL_n

Let k, p, h, l be nonnegative integers. In Section 1, we introduced the $((k,p),h,l)$-Pieri rule for GL_n. The anti-row iterated Pieri rule is the case of $p = h = 0$. When $k = p = 0$, the $((0,0),h,l)$-**Pieri rule** describes how the tensor product

$$\left(\bigotimes_{s=1}^{h} \rho_n^{(\alpha_s)}\right) \otimes \left(\bigotimes_{j=1}^{l} \rho_n^{(\beta_j)^*}\right)$$

decomposes. The algebra $\mathcal{P}(\mathrm{M}_{n,h+l})^{U_n}$ is a $((0,0),h,l)$-**Pieri algebra**.

Since

$$\mathcal{P}(\mathrm{M}_{n,h+l})^{U_n} \cong \sum_{\lambda} \left(\rho_n^{\lambda}\right)^{U_n} \otimes \mathcal{L}_{h,l}^{\lambda},$$

the ψ_n^{λ} eigenspace of A_n in $\mathcal{P}(\mathrm{M}_{n,h+l})^{U_n}$ is the realization of $\mathcal{L}_{h,l}^{\lambda}$. So we can figure out the structure of the lowest weight module by studying $((0,0),h,l)$-Pieri algebra.

6.2 Iterated Pieri Algebras for O_n and Sp_{2n}

There are analogues of the Pieri rule for $\mathrm{O}_n = \mathrm{O}_n(\mathbb{C})$ and $\mathrm{Sp}_{2n} = \mathrm{Sp}_{2n}(\mathbb{C})$. In [13], the authors construct iterated Pieri algebras for O_n and Sp_{2n}. They also determine the structure of these algebras under a stable range condition. We plan to remove the restriction based on the result of [20].

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