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Finite type cluster algebras and Demazure crystals

Yuki Kanakubo
(Sophia university, Graduate school of Science and Technology)

Notation

Let $G = \mathrm{SL}_{r+1}(\mathbb{C})$ be a classical algebraic group of type A_r over $\mathbb{C}$ and $I = \{1, 2, \cdots, r\}$ be indices set. Let B, $(B^-) \subset G$ be the set of upper (lower) triangular matrices, $H := B \cap B^-$, $N \subset B, N^- \subset B^-$ unipotent radicals, and $W := \mathrm{Norm}_G(H)/H =< \overline{s_i}H >$ the Weyl group of G with the simple reflections $s_i = \overline{s_i}H$. Here, $\overline{s_i} \in \mathrm{Norm}_G(H)$ is defined as $\overline{s_i} := \exp(-e_i)\exp(f_i)\exp(-e_i)$. For a reduced expression $u = s_{i_1} \cdots s_{i_n} \in W$, setting $\overline{u} = \overline{s_{i_1}} \cdots \overline{s_{i_n}}$, we get

$$G = \amalg_{u,v \in W} B\overline{u}B \cap B^-\overline{v}B^-.$$

We call $G^{u,v} := B\overline{u}B \cap B^-\overline{v}B^-$ *Double Bruhat cell.*

1 Introduction

Fomin and Zelevinsky have invented cluster algebra for the study of total positivity and dual semi canonical base in 2002 [4]. It is a commutative algebra generated by so-called *cluster variables.* Choosing a part of the cluster variables properly, we can combinatorially calculate other variables from them. These chosen variables are called initial cluster variables.

It is known that cluster algebra structures appear in many algebras relevant to simple algebraic groups, which include $\mathbb{C}[G^{u,v}]$, $\mathbb{C}[N]$ and Grothendieck rings of certain category of representations of quantum affine algebras. In this way, cluster algebras are closely related to algebraic groups or its Lie algebras [6, 7].

Cluster algebras which have only finite many cluster variables are called *finite type.* In [5], cluster algebras of finite type are studied thoroughly, and they are classified by the set of Cartan matrices up to coefficients. For a fixed Cartan matrix, all the cluster variables are parametrized by the set of "almost positive roots", which is, a union of all positive roots and negative simple roots corresponding to the Cartan matrix. Thus, we define the type of such cluster algebra to be the type of the corresponding Cartan matrix. Let $c \in W$ be a Coxeter element whose length $l(c)$ satisfies $l(c^2) = 2l(c) = 2\mathrm{rank}(G)$. It is known that one can realize a cluster algebra of finite type on the coordinate ring $\mathbb{C}[G^{e,c^2}]$, whose type coincides with the Cartan-Killing type of G [1].

In [8, 9], we showed that certain cluster variables of $\mathbb{C}[G^{u,e}]$ $(u \in W)$ are realized as a sum of monomials in Demazure crystals in the case G is type A, B, C or D. Then we treated only a part of the cluster variables. In this article, we consider the case $G = \mathrm{SL}_{r+1}(\mathbb{C})$ $(r \geq 3)$ and describe all the cluster variables in $\mathbb{C}[G^{e,c^2}]$ using direct sum of certain monomial realizations of Demazure crystals and gave a new parametrization of these cluster variables different from those of [5]. We see that each initial cluster variable in $\mathbb{C}[G^{e,c^2}]$ is described as a sum of monomials in the Demazure crystal $B(\Lambda_k)_{w_k}$ with some $k \in \{1, 2, \cdots, r\}$ and $w_k \in W$. And other variables are described as sums of monomials in Demazure crystals in the forms $B(\sum_{s=a}^{b} \Lambda_s)_w \oplus \bigoplus_{t=1}^{p} B(\lambda_t)_{w_t}$ with some $w, w_t \in W$, $p, a, b \in \mathbb{Z}_{>0}$ and $\lambda_t \in \sum_{s=a}^{b} \Lambda_s - \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$. By this, we see that a natural correspondence $-\alpha_k \mapsto B(\Lambda_k)_{w_k}$, $\sum_{s=a}^{b} \alpha_s \to B(\sum_{s=a}^{b} \Lambda_s)_w \oplus \bigoplus_{t=1}^{p} B(\lambda_t)_{w_t}$

gives a parametrization of the cluster variables in $\mathbb{C}[G^{e,c^2}]$ by the set of almost positive roots.

This is joint work with T.Nakashima (Sophia university).

2 Quantum groups and crystal base

First, let us recall representation theory of quantum groups and monomial realization of crystal base. We set $\mathfrak{g} := \mathrm{Lie}(G) = \mathfrak{sl}_{r+1}(\mathbb{C})$, and let $P^\vee := \oplus_{i\in I}\mathbb{Z}h_i$ be a dual weight lattice, $(a_{i,j})_{i,j\in I}$ Cartan matrix of $\mathfrak{g}$. Let $\{\Lambda_i\}_{i\in I}$ be the set of fundamental weights, $P = \bigoplus_{i=1}^r \mathbb{Z}\Lambda_i$ and $P^+ = \bigoplus_{i=1}^r \mathbb{Z}_{\geq 0}\Lambda_i$ be the weight lattice and positive weight lattice respectively.

2.1 Quantum group and its representation

We suppose that $q \in \mathbb{C}_{\neq 0}$ is not root of unity.

Definition 2.1. A quantum group $U_q(\mathfrak{g})$ is $\mathbb{C}$-algebra generated by $e_i,\ f_i,\ q^h$ $(i \in I,\ h \in P^\vee)$ with the following defining relations:

- $q^0 = 1,\ q^h q^{h'} = q^{h+h'}\ (h, h' \in P^\vee)$,
- $q^h e_j q^{-h} = q^{\alpha_j(h)} e_j$,
- $q^h f_j q^{-h} = q^{-\alpha_j(h)} f_j$,
- $e_i f_j - f_j e_i = \delta_{i,j}\frac{q^{h_i}-q^{-h_i}}{q-q^{-1}}$,
- $\sum_{k=0}^{1-a_{i,j}} \begin{bmatrix} 1-a_{i,j} \\ k \end{bmatrix}_q (-1)^k e_i^{1-a_{i,j}-k} e_j e_i^k = 0\ (i \neq j)$,
- $\sum_{k=0}^{1-a_{i,j}} \begin{bmatrix} 1-a_{i,j} \\ k \end{bmatrix}_q (-1)^k f_i^{1-a_{i,j}-k} f_j f_i^k = 0\ (i \neq j)$.

Here, $[n]_q := \frac{q^n - q^{-n}}{q-q^{-1}}$, $[n]_q! := [n]_q[n-1]_q \cdots [1]_q$ and $\begin{bmatrix} m \\ n \end{bmatrix}_q := \frac{[m]_q!}{[n]_q![m-n]_q!}$.

There are several basic facts about representations of $U_q(\mathfrak{g})$:

- A finite dimensional irreducible representation V of $U_q(\mathfrak{g})$ have highest weight vector v_λ $(\lambda \in P^+)$, that is, $V = U_q(\mathfrak{g})v_\lambda$, $q^h v_\lambda = q^{\lambda(h)} v_\lambda (h \in P^\vee)$ and $e_i v_\lambda = 0$ $(i \in I)$.
- For the finite dimensional irreducible representation $V(\lambda)$ with highest weight λ, it is decomposed to weight spaces:

$$V(\lambda) = \oplus_{\mu\in P,\ \mu\leq\lambda} V(\lambda)_\mu, \quad V(\lambda)_\mu := \{v \in V(\lambda) | q^{h_i} v = q^{\mu(h_i)} v\}.$$

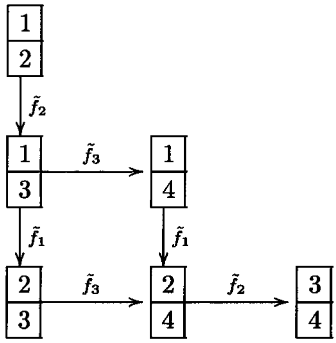
- $V(\lambda)$ has the *crystal base* $B(\lambda)$ [11].

2.2 Crystal bases

- Remark that the crystal base $B(\lambda)$ is **not** a base of $V(\lambda)$.
- $B(\lambda)$ is a set which includes the highest weight vector $\overline{v_\lambda}$.
- There exist *Kashiwara operators* $\tilde{e}_i$, $\tilde{f}_i : B(\lambda) \to B(\lambda) \cup \{0\}$. Each element in $B(\lambda)$ has *weight*, that is, there exists a function $\mathrm{wt} : B(\lambda) \to P$.

The crystal base $B(\lambda)$ is described as Young tableaux, Laurent monomials and so on. Using these descriptions, we can calculate dim $V(\lambda)_\mu$, and roughly reveal a structure of representations combinatorially.

Example 2.2. Let us consider the case $G = \mathrm{SL}_4(\mathbb{C})$. The crystal base $B(\Lambda_2)$ of the representation $V(\Lambda_2)$ is described by Young tableau as follows:



The above figure implies that the representation $V(\Lambda_2)$ has dimension 6 and it is decomposed to 1 dimensional weight spaces as follows:

$$V(\Lambda_2) = V(\Lambda_2)_{\Lambda_2} \oplus V(\Lambda_2)_{\Lambda_3-\Lambda_2+\Lambda_1} \oplus V(\Lambda_2)_{\Lambda_3-\Lambda_1} \oplus V(\Lambda_2)_{-\Lambda_3+\Lambda_1} \oplus V(\Lambda_2)_{-\Lambda_3+\Lambda_2-\Lambda_1} \oplus V(\Lambda_2)_{-\Lambda_2},$$

where we used $\mathrm{wt}\left(\begin{array}{|c|}\hline i \\ \hline j \\ \hline\end{array}\right) = \Lambda_i - \Lambda_{i-1} + \Lambda_j - \Lambda_{j-1}$.

Now, we define the *Demazure crystal* $B(\lambda)_w$ for $w \in W$.

Definition 2.3. Let u_λ be the highest weight vector of $B(\lambda)$. For the identity element e of W, we set $B(\lambda)_e := \{u_\lambda\}$. For $w \in W$, if $s_i w < w$,

$$B(\lambda)_w := \{\tilde{f}_i^k b \mid k \geq 0,\ b \in B(\lambda)_{s_i w},\ \tilde{e}_i b = 0\} \setminus \{0\}.$$

Theorem 2.4. [12] For $w \in W$, let $w = s_{i_1} \cdots s_{i_n}$ be an arbitrary reduced expression. Let u_λ be the highest weight vector of $B(\lambda)$. Then

$$B(\lambda)_w = \{\tilde{f}_{i_1}^{a(1)} \cdots \tilde{f}_{i_n}^{a(n)} u_\lambda | a(1), \cdots, a(n) \in \mathbb{Z}_{\geq 0}\} \setminus \{0\}.$$

2.3 Monomial realization of crystal base

In the previous subsection, we have seen the Young tableau description. Next, let us recall the monomial realization of crystal base [10, 13].

First, for a sequence $(i_1, i_2, \cdots, i_r)$ such that $\{i_1, i_2, \cdots, i_r\} = \{1, 2, \cdots, r\}$, let $p = (p_{j,i})_{j,i \in I,\, j \neq i}$ be integers such that

$$p_{i_a, i_b} = \begin{cases} 1 & \text{if} \quad a > b, \\ 0 & \text{if} \quad a < b. \end{cases}$$

Second, for doubly-indexed variables $\{Y_{s,i} \mid i \in I,\, s \in \mathbb{Z}\}$,

$$\mathcal{Y} := \left\{ Y = \prod_{s \in \mathbb{Z},\, i \in I} Y_{s,i}^{\zeta_{s,i}} \,\middle|\, \zeta_{s,i} \in \mathbb{Z}, \quad \text{only finitely many } \zeta_{s,i} \neq 0 \right\}.$$

For $Y = \prod_{s \in \mathbb{Z},\, i \in I} Y_{s,i}^{\zeta_{s,i}} \in \mathcal{Y}$, $\quad \mathrm{wt}(Y) := \sum_{i,s} \zeta_{s,i} \Lambda_i,$

$$\varphi_i(Y) := \max \left\{ \sum_{k \leq s} \zeta_{k,i} \mid s \in \mathbb{Z} \right\}, \quad \varepsilon_i(Y) := \varphi_i(Y) - \mathrm{wt}(Y)(h_i).$$

Setting

$$A_{s,i} := Y_{s,i} Y_{s+1,i} \prod_{j \neq i} Y_{s+p_{j,i},j}^{a_{j,i}}.$$

Define the Kashiwara operators as

$$\tilde{f}_i Y = \begin{cases} A_{n_{f_i},i}^{-1} Y & \text{if} \quad \varphi_i(Y) > 0, \\ 0 & \text{if} \quad \varphi_i(Y) = 0, \end{cases} \qquad \tilde{e}_i Y = \begin{cases} A_{n_{e_i},i} Y & \text{if} \quad \varepsilon_i(Y) > 0, \\ 0 & \text{if} \quad \varepsilon_i(Y) = 0, \end{cases}$$

where

$$n_{f_i} := \min \left\{ n \,\middle|\, \varphi_i(Y) = \sum_{k \leq n} \zeta_{k,i} \right\}, \quad n_{e_i} := \max \left\{ n \,\middle|\, \varphi_i(Y) = \sum_{k \leq n} \zeta_{k,i} \right\}.$$

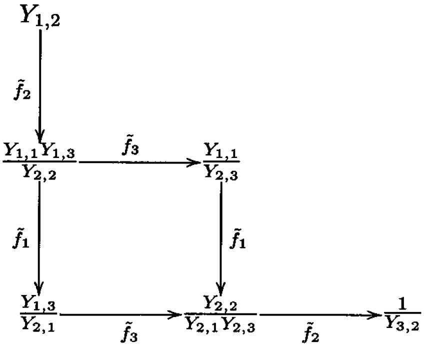
Theorem 2.5. [10, 13]

(i) For the set $p = (p_{j,i})$ as above, $(\mathcal{Y}, \mathrm{wt}, \varphi_i, \varepsilon_i, \tilde{f}_i, \tilde{e}_i)_{i \in I}$ is a crystal.

(ii) If a monomial $Y \in \mathcal{Y}$ satisfies $\varepsilon_i(Y) = 0$ for all $i \in I$, then the connected component containing Y is isomorphic to $B(\mathrm{wt}(Y))$.

A monomial realization of crystal base is determined by a monomial Y satisfying $\varepsilon_i(Y) = 0$. Note that if Y has no negative power, then $\varepsilon_i(Y) = 0$ $(i \in I)$.

Example 2.6. Let us consider the case $G = \mathrm{SL}_4(\mathbb{C})$. A monomial realization of crystal base $B(\Lambda_2)$ is described as follows.



We can verify $\mathrm{wt}(Y_{1,2}) = \Lambda_2$ and $\varepsilon_i(Y_{1,2}) = 0$ $(i = 1, 2, 3)$.

3 Cluster algebra structures of coordinate rings

In this section, we shall recall the definition of cluster algebra. We will refer to a relation between certain cluster variables on Double Bruhat cells and crystal bases in the next section. First, let us see an example. For $l \in \mathbb{Z}_{>0}$, we set $[1, l] := \{1, 2, \cdots, l\}$ and $[-1, -l] := \{-1, -2, \cdots, -l\}$.

3.1 Example

Example 3.1. Let us consider the case $G = \mathrm{SL}_4(\mathbb{C})$, $v := s_2s_1s_3s_2s_1s_3 \in W$ and the double Bruhat cell $G^{e,v} = B \cap B^- vB^- \subset B$. For $x = (x_{i,j})$, let $D_{12,24}(x)$ denote the minor $\det\begin{pmatrix} x_{1,2} & x_{1,4} \\ x_{2,2} & x_{2,4} \end{pmatrix} \in \mathbb{C}[G^{e,v}]$ and so on. We can obtain generators of $\mathbb{C}[G^{e,v}]$ over $\mathbb{C}[D^{\pm 1}_{123,234}, D^{\pm 1}_{12,34}, D^{\pm 1}_{1,4}, D^{\pm 1}_{123,123}, D^{\pm 1}_{12,12}, D^{\pm 1}_{1,1}]$ from 3-tuple $(D_{12,24}, D_{1,2}, D_{123,124})$. First,

$$(D_{12,12}D_{13,34}) = \frac{D_{12,12}D_{1,4}D_{123,234} + D_{1,2}D_{123,124}}{D_{12,24}},$$

$$D_{12,14} = \frac{D_{12,24}D_{1,1} + D_{1,4}D_{12,12}}{D_{1,2}}, \quad D_{12,23} = \frac{D_{12,24}D_{123,123} + D_{123,234}D_{12,12}}{D_{123,124}}.$$

Note that the denominator of $(D_{12,12}D_{13,34})$ is $D_{12,24}$, the numerator is binomial of $D_{1,2}$, $D_{123,124}$ and the coefficients. The same is true of $D_{12,14}$ and $D_{12,23}$. Replacing $D_{12,24}$ with $(D_{12,12}D_{13,34})$, we get new tuple $((D_{12,12}D_{13,34}), D_{1,2}, D_{123,124})$.

We can also obtain $D_{12,12}D_{3,4}$ and $D_{1,1}D_{23,34}$ from$((D_{12,12}D_{13,34}), D_{1,2}, D_{123,124})$:

$$D_{123,134} = \frac{(D_{12,12}D_{13,34}) + D_{123,234}D_{1,1}}{D_{1,2}},$$

$$D_{1,3} = \frac{(D_{12,12}D_{13,34}) + D_{1,4}D_{123,123}}{D_{123,124}}.$$

The denominator of $D_{123,134}$ is $D_{1,2}$, the numerator is binomial of $(D_{12,12}D_{13,34})$, $D_{123,124}$ and coefficients. The denominator of $D_{1,3}$ is $D_{123,124}$, the numerator is binomial of $(D_{12,12}D_{13,34})$, $D_{1,2}$ and coefficients.

In this way, from $(D_{12,24},\ D_{1,2},\ D_{123,124})$, we can constitute elements of $\mathbb{C}[G^{e,v}]$ one after another. We call these elements *cluster variables.* All cluster variables included in $\mathbb{C}[G^{e,v}]$ are

$$D_{1,2},\quad D_{12,24},\quad D_{123,124},\quad D_{123,134},\quad (D_{12,12}D_{13,34}),$$

$$D_{1,3},\quad D_{12,14},\quad D_{12,23}\quad D_{12,13}.$$

In the following subsections, we shall give details of it.

3.2 Cluster algebras of geometric type

Let $\tilde{B}=(b_{ij})_{1\leq i\leq n+m,\ 1\leq j\leq n}$ be an $(n+m)\times n$ integer matrix $(n,m\in\mathbb{Z}_{>0})$. The *principal part* B of $\tilde{B}$ is obtained from $\tilde{B}$ by deleting the last m rows. For $\tilde{B}$ and $k\in[1,n]$, the new $(n+m)\times n$ integer matrix $\mu_k(\tilde{B})=(b'_{ij})$ is defined by

$$b'_{ij}:=\begin{cases}-b_{ij} & \text{if } i=k \text{ or } j=k,\\ b_{ij}+\frac{|b_{ik}|b_{kj}+b_{ik}|b_{kj}|}{2} & \text{otherwise.}\end{cases}$$

One calls $\mu_k(\tilde{B})$ the *matrix mutation* in direction k of $\tilde{B}$. If there exists a positive integer diagonal matrix D such that DB is skew symmetric, we say B is *skew symmetrizable.* It is easily verified that if $\tilde{B}$ has a skew symmetrizable principal part then $\mu_k(\tilde{B})$ also has a skew symmetrizable principal part. We can also verify that $\mu_k\mu_k(\tilde{B})=\tilde{B}$. Define $\mathbf{x}:=(x_1,\cdots,x_{n+m})$ and we call the pair $(\mathbf{x},\tilde{B})$ *initial seed.* Let $\mathcal{F}:=\mathbb{C}(x_1,\cdots,x_n,x_{n+1},\cdots,x_{n+m})$. For $1\leq k\leq n$, a new cluster variable x'_k is defined by

$$x_kx'_k=\prod_{1\leq i\leq n+m,\ b_{i,k}>0}x_i^{b_{i,k}}+\prod_{1\leq i\leq n+m,\ b_{i,k}<0}x_i^{-b_{i,k}}.$$

Let $\mu_k(\mathbf{x})$ be the set of variables obtained from $\mathbf{x}$ by replacing x_k by x'_k. Ones call the pair $(\mu_k(\mathbf{x}),\mu_k(\tilde{B}))$ the *mutation* in direction k of the seed $(\mathbf{x},\tilde{B})$.

Now, we can repeat this process of mutation and obtain a set of seeds inductively. Hence, each seed consists of an r-tuple of variables and a matrix. Ones call this r-tuple and matrix *cluster* and *exchange matrix* respectively. Variables in cluster are called *cluster variables,* and $x_{n+1},\ x_{n+2},\cdots,x_{n+m}$ are called *frozen variables.*

Definition 3.2. [3] Let $\tilde{B}$ be a integer matrix whose principal part is skew symmetrizable and $\Sigma=(\mathbf{x},\tilde{B})$ a seed. We set $\mathbb{A}:=\mathbb{Z}[x_{n+1}^{\pm1},\cdots,x_{n+m}^{\pm1}]$. The cluster algebra (of geometric type) $\mathcal{A}=\mathcal{A}(\Sigma)$ over $\mathbb{A}$ associated with seed Σ is defined as the $\mathbb{A}$-subalgebra of $\mathcal{F}$ generated by all cluster variables in all seeds which can be obtained from Σ by sequences of mutations.

3.3 Cluster algebra $\mathcal{A}(\mathbf{i})$ and generalized minors.

Let $A=(a_{i,j})$ be the Cartan matrix of G. For a reduced word $\mathbf{i}=(i_1,\cdots,i_{l(v)})$ of v, we define the cluster algebra $\mathcal{A}(\mathbf{i})$, which obtained from $\mathbf{i}$. It satisfies that

$\mathcal{A}(\mathbf{i}) \otimes \mathbb{C}$ is isomorphic to the coordinate ring $\mathbb{C}[G^{e,v}]$ of the double Bruhat cell [1]. Let i_k ($k \in [1, l(v)]$) be the k-th index of $\mathbf{i}$ from the left. For $t \in [-1, -r]$, we set $i_t := t$.

For $k \in [-1, -r] \cup [1, l(v)]$, we denote by k^+ the smallest index l such that $k < l$ and $|i_l| = |i_k|$. For example, if $\mathbf{i} = (1, 2, 3, 1, 2)$ then, $1^+ = 4$, $2^+ = 5$ and 3^+ is not defined. We define a set e($\mathbf{i}$) as

$$e(\mathbf{i}) := \{k \in [1, l(v)] | k^+ \text{ is well} - \text{defined}\}.$$

Following [1], we define a quiver $\Gamma_{\mathbf{i}}$ as follows. The vertices of $\Gamma_{\mathbf{i}}$ are the numbers $[-1, -r] \cup [1, l(v)]$. For two vertices $k \in [-1, -r] \cup [1, l(v)]$ and $l \in [1, l(v)]$ with $k < l$, there exists an arrow $k \to l$ (resp. $l \to k$) if and only if $l = k^+$ (resp. $l < k^+ < l^+$ and $a_{i_k, i_l} < 0$). Next, let us define a matrix $\tilde{B} = \tilde{B}(\mathbf{i})$.

Definition 3.3. Let $\tilde{B}(\mathbf{i})$ be an integer matrix with rows labelled by all the indices in $[-1, -r] \cup [1, l(v)]$ and columns labelled by all the indices in $e(\mathbf{i})$. For $k \in [-1, -r] \cup [1, l(v)]$ and $l \in e(\mathbf{i})$, an entry $b_{k,l}$ of $\tilde{B}(\mathbf{i})$ is determined as follows: If there exists an arrow $k \to l$ (resp. $l \to k$) in $\Gamma_{\mathbf{i}}$, then

$$b_{k,l} := \begin{cases} 1 \text{ (resp. } -1) & \text{if } |i_k| = |i_l|, \\ -a_{|i_k||i_l|} \text{ (resp. } a_{|i_k||i_l|}) & \text{if } |i_k| \neq |i_l|. \end{cases}$$

If there exist no arrows between k and l, we set $b_{k,l} = 0$. The principal part $B(\mathbf{i})$ of $\tilde{B}(\mathbf{i})$ is submatrix $(b_{i,j})_{i,j \in e(\mathbf{i})}$.

Proposition 3.4. [1, *Proposition* 2.6] The principal part of $\tilde{B}(\mathbf{i})$ is skew symmetrizable.

By Definition 3.2 and Proposition 3.4, we can construct the cluster algebra:

Definition 3.5. We denote this cluster algebra by $\mathcal{A}(\mathbf{i})$.

Set $\mathcal{A}(\mathbf{i})_{\mathbb{C}} := \mathcal{A}(\mathbf{i}) \otimes \mathbb{C}$. It is known that the coordinate ring $\mathbb{C}[G^{e,v}]$ of the double Bruhat cell is isomorphic to $\mathcal{A}(\mathbf{i})_{\mathbb{C}}$ (Theorem 3.7). To describe this isomorphism explicitly, we need generalized minors.

We set $G_0 := N_- H N$, and let $x = [x]_-[x]_0[x]_+$ with $[x]_- \in N_-$, $[x]_0 \in H$, $[x]_+ \in N$ be the corresponding decomposition.

Definition 3.6. For $i \in [1, r]$ and $w \in W$, the generalized minor $\Delta_{\Lambda_i, w\Lambda_i}$ is a regular function on G whose restriction to the open set $G_0 w^{-1}$ is given by $\Delta_{\Lambda_i, w\Lambda_i}(x) = ([xw]_0)^{\Lambda_i}$.

In the case $G = \mathrm{SL}_{r+1}(\mathbb{C})$, it is coincide with an ordinary minor. In fact, $\Delta_{\Lambda_i, w\Lambda_i}(x) = D_{\{1,\cdots,i\}, w\{1,\cdots,i\}}$. For example, if $G = \mathrm{SL}_4(\mathbb{C})$, $v = s_2 s_1 s_3 s_2 s_1 s_3$, since $W \cong \mathfrak{S}_3$,

$$D_{\{1,2\}, v_{>2}\{1,2\}} = D_{\{1,2\}, s_3 s_1 s_2 s_3 \{1,2\}} = D_{\{1,2\},\{2,4\}},$$

where the notation $v_{>2}$ will be define in the next subsection (1).

3.4 Cluster algebras on Double Bruhat cells

For $v = s_{i_1} s_{i_2} \cdots s_{i_n}$ $(n := l(v))$ and $k \in [1, n]$, we set

$$v_{>k} = v_{>k}(\mathbf{i}) := s_{i_n} s_{i_{n-1}} \cdots s_{i_{k+1}}. \tag{1}$$

For $k \in [-1, -r]$, we set $v_{>k} := v^{-1}$ and $i_k := k$. For $k \in [-1, -r] \cup [1, n]$, we define

$$\Delta(k; \mathbf{i})(x) := \Delta_{\Lambda_{|i_k|}, v_{>k}\Lambda_{|i_k|}}(x).$$

Finally, we set

$$F(\mathbf{i}) := \{\Delta(k; \mathbf{i})(x) | k \in [-1, -r] \cup [1, n]\}.$$

It is known that the set $F(\mathbf{i})$ is an algebraically independent generating set for the field of rational functions $\mathbb{C}(G^{e,v})$ [3, Theorem 1.12]. Then, we have the following theorem.

Theorem 3.7. [1, 6, 7] The isomorphism of fields $\varphi : \mathcal{F} \to \mathbb{C}(G^{e,v})$ defined by $\varphi(x_k) = \Delta(k; \mathbf{i})$ $(k \in [-1, -r] \cup [1, n])$ restricts to an isomorphism of algebras $\mathcal{A}(\mathbf{i})_{\mathbb{C}} \to \mathbb{C}[G^{e,v}]$.

In Example 3.1, $\mathbf{i} = (2, 1, 3, 2, 1, 3)$ and

- $(\tilde{B}(\mathbf{i}), (D_{12,24}, D_{1,2}, D_{123,124}))$ is an initial seed, where

$$\tilde{B}(\mathbf{i}) = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

  Here, the row is labeled by 1,2,3,4,5,6,-1,-2,-3 from top to bottom, the column is labeled by 1,2,3 from left to right. The isomorphism φ is given by $x_1 \mapsto D_{12,24}$, $x_2 \mapsto D_{1,2}$, $x_3 \mapsto D_{123,124}$, $x_4 \mapsto D_{1,1}$, $x_5 \mapsto D_{123,123}$, $x_6 \mapsto D_{12,12}$, $x_7 \mapsto D_{1,4}$, $x_8 \mapsto D_{12,34}$ and $x_9 \mapsto D_{123,234}$.

- $D_{12,24}$, $D_{1,2}$, $D_{123,124}$ are initial cluster variables.

- $D_{12,24}$, $D_{1,2}$, $D_{123,124}$, $D_{123,134}$, $(D_{12,12}D_{13,34})$, $D_{1,3}$, $D_{12,14}$, $D_{12,23}$, $D_{12,13}$ are cluster variables.

- $D_{123,234}$, $D_{12,34}$, $D_{1,4}$, $D_{123,123}$, $D_{12,12}$, $D_{1,1}$ are frozen variables.

- The coordinate ring $\mathbb{C}[G^{e,v}]$ is generated by all the cluster variables and frozen variables.

3.5 Cluster algebra of finite type

A cluster algebra is said to be *finite type* if it has only finite many cluster variables. For an $n \times n$ matrix $B = (b_{i,j})$, *Cartan counter part* $A(B) = (a_{i,j})$ of B is defined as

$$a_{i,j} := \begin{cases} 2 & \text{if } i = j, \\ -|b_{i,j}| & \text{if } i \neq j. \end{cases}$$

Theorem 3.8. [5]

1. A cluster algebra $\mathcal{A}$ is finite type if and only if there exists a seed $\Sigma = (\mathbf{y}, \tilde{B})$ such that $A(B)$ is a Cartan matrix C and $\mathcal{A} \cong \mathcal{A}(\Sigma)$ as $\mathbb{C}-algebra$, where B is the principal part of $\tilde{B}$. This Cartan matrix is uniquely determined.

2. Let $\mathcal{A}(\Sigma)$ be a cluster algebra of finite type, and C be a corresponding Cartan matrix. Let $\Phi_{\geq -1} := \Phi_{>0} \cup \{-\alpha_i | i \in I\}$ be the set of *almost positive roots* of C, which is a union of the set of positive roots $\Phi_{>0}$ and negative simple roots $\{-\alpha_i | i \in I\}$ of C. Then the number of cluster variables included in $\mathcal{A}(\Sigma)$ is equal to $\#\Phi_{\geq -1}$.

3. Let c be a Coxeter element such that $l(c^2) = 2l(c)$. Then there exists a seed $\Sigma = (\mathbf{y}, \tilde{B})$ such that $A(B)$ coincides with the Cartan matrix of G and
$$\mathbb{C}[G^{e,c^2}] \cong \mathcal{A}(\Sigma).$$

3.6 A coordinate transformation of cluster variables

For $a = \mathrm{diag}(a_1, a_2, \cdots, a_{r+1}) \in H$, $v = s_{i_1} \cdots s_{i_n} \in W$ and its reduced expression $\mathbf{i} = (i_1, \cdots, i_n)$, we define a map $X_{\mathbf{i}}$ as

$$X_{\mathbf{i}} : H \times (\mathbb{C}_{\neq 0})^n \longrightarrow G \tag{2}$$

$$(a; t_1, \cdots, t_n) \mapsto a(\exp(t_1 e_{i_1}) t_1^{h_{i_1}}) \cdots (\exp(t_n e_{i_n}) t_n^{h_{i_n}}).$$

Theorem 3.9. [2, 3] The map $X_{\mathbf{i}}$ is a biregularly isomorphism from $H \times (\mathbb{C}_{\neq 0})^n$ to a Zariski open subset of $G^{e,v}$.

Example 3.10. In the case $G = \mathrm{SL}_4(\mathbb{C})$, $v = s_2 s_1 s_3 s_2 s_1 s_3$ and $\mathbf{i} = (2,1,3,2,1,3)$, let us calculate $X_{\mathbf{i}}$. In this case, we have

$$\begin{aligned} \mathrm{Lie}(G) &= < \quad e_i \quad , \quad f_i \quad , \quad h_i > \\ &= < E_{i,i+1},\ E_{i+1,i},\ E_{i,i} - E_{i+1,i+1} >, \end{aligned}$$

where $\{E_{i,j}\}_{1 \leq i,j \leq 4}$ are the matrix units. Then

$$\exp(te_1)\ t^{h_1} = \begin{pmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t & 0 & 0 & 0 \\ 0 & t^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} t & 1 & 0 & 0 \\ 0 & t^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus, $X_{\mathbf{i}} : H \times (\mathbb{C}_{\neq 0})^6 \xrightarrow{\sim} G^{e,s_2s_1s_3s_2s_1s_3}$ is given by

$$X_{\mathbf{i}}(a; Y_{1,2}, Y_{1,1}, Y_{1,3}, Y_{2,2}, Y_{2,1}, Y_{2,3}) =$$
$$a \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & Y_{1,2} & 1 & 0 \\ 0 & 0 & Y_{1,2}^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} Y_{1,1} & 1 & 0 & 0 \\ 0 & Y_{1,1}^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & Y_{2,3} & 1 \\ 0 & 0 & 0 & Y_{2,3}^{-1} \end{pmatrix}$$
$$= a \cdot \begin{pmatrix} Y_{1,1}Y_{2,1} & Y_{1,1} + \frac{Y_{2,2}}{Y_{2,1}} & Y_{2,3} & 1 \\ 0 & \frac{Y_{1,2}Y_{2,2}}{Y_{1,1}Y_{2,1}} & \frac{Y_{1,2}Y_{2,3}}{Y_{1,1}} + \frac{Y_{1,3}Y_{2,3}}{Y_{2,2}} & \frac{Y_{1,2}}{Y_{1,1}} + \frac{Y_{1,3}}{Y_{2,2}} + \frac{1}{Y_{2,3}} \\ 0 & 0 & \frac{Y_{1,3}Y_{2,3}}{Y_{1,2}Y_{2,2}} & \frac{Y_{1,3}}{Y_{1,2}Y_{2,2}} + \frac{1}{Y_{1,2}Y_{2,3}} \\ 0 & 0 & 0 & \frac{1}{Y_{1,3}Y_{2,3}} \end{pmatrix}.$$

Recall that in Example 3.1, the minor $D_{12,24}$ is one of the initial cluster variables in $\mathbb{C}[G^{e,v}]$. Now, using above $X_{\mathbf{i}}$, we can regard $D_{12,24}$ as a function on $H \times (\mathbb{C}_{\neq 0})^6$.

$$D_{12,24} \circ X_{\mathbf{i}}(a; Y_{1,2}, Y_{1,1}, Y_{1,3}, Y_{2,2}, Y_{2,1}, Y_{2,3}) = a_1a_2 \left(Y_{1,2} + \frac{Y_{1,1}Y_{1,3}}{Y_{2,2}} + \frac{Y_{1,3}}{Y_{2,1}} + \frac{Y_{2,2}}{Y_{2,1}Y_{2,3}} + \frac{Y_{1,1}}{Y_{2,3}} \right).$$

Comparing with the figure in Example 2.6, we can easily check that the set $\{Y_{1,2}, \frac{Y_{1,1}Y_{1,3}}{Y_{2,2}}, \frac{Y_{1,3}}{Y_{2,1}}, \frac{Y_{2,2}}{Y_{2,1}Y_{2,3}}, \frac{Y_{1,1}}{Y_{2,3}}\}$ of terms in $D_{12,24} \circ X_{\mathbf{i}}$ coincides with monomial realization of Demazure crystal $B(\Lambda_2)_{s_3s_1s_2}$. Similarly, we get

$$D_{1,2} \circ X_{\mathbf{i}}(a; \mathbf{Y}) = a_1(Y_{1,1} + \frac{Y_{2,2}}{Y_{2,1}}), \quad D_{123,124} \circ X_{\mathbf{i}}(a; \mathbf{Y}) = a_1a_2a_3(Y_{1,3} + \frac{Y_{2,2}}{Y_{2,3}}),$$

which coincide with the total sums of monomials in Demazure crystals $B(\Lambda_1)_{s_3s_1s_2}$, $B(\Lambda_3)_{s_3s_1s_2}$ respectively. All other cluster variables in $\mathbb{C}[G^{e,c^2}]$ are

$$(D_{12,12}D_{13,34}) \circ X_{\mathbf{i}} = a_1^2a_2a_3Y_{2,2}, \quad D_{12,14} \circ X_{\mathbf{i}} = a_1a_2(Y_{1,2}Y_{2,1} + \frac{Y_{1,1}Y_{1,3}Y_{2,1}}{Y_{2,2}} + \frac{Y_{1,1}Y_{2,1}}{Y_{2,3}}),$$

$$D_{12,23} \circ X_{\mathbf{i}} = a_1a_2(Y_{1,2}Y_{2,3} + \frac{Y_{1,1}Y_{1,3}Y_{2,3}}{Y_{2,2}} + \frac{Y_{1,3}Y_{2,3}}{Y_{2,1}}), \quad D_{1,3} \circ X_{\mathbf{i}} = a_1Y_{2,3},$$

$$D_{123,134} \circ X_{\mathbf{i}} = a_1a_2a_3Y_{2,1}, \quad D_{12,13} \circ X_{\mathbf{i}} = a_1a_2(Y_{1,2}Y_{2,1}Y_{2,3} + \frac{Y_{1,1}Y_{1,3}Y_{2,1}Y_{2,3}}{Y_{2,2}}),$$

which coincide with the total sums of monomials in Demazure crystals $B(\Lambda_2)_e$, $B(\Lambda_1 + \Lambda_2)_{s_3s_2}$, $B(\Lambda_2 + \Lambda_3)_{s_1s_2}$, $B(\Lambda_3)_e$, $B(\Lambda_1)_e$ and $B(\Lambda_1 + \Lambda_2 + \Lambda_3)_{s_2}$ respectively. Thus, a correspondence $-\alpha_i \mapsto B(\Lambda_i)_{s_3s_1s_2}$, $\alpha_i \mapsto B(\Lambda_i)_e$ $(i = 1, 2, 3)$, $\alpha_1 + \alpha_2 \mapsto B(\Lambda_1 + \Lambda_2)_{s_3s_2}$, $\alpha_2 + \alpha_3 \mapsto B(\Lambda_2 + \Lambda_3)_{s_1s_2}$, and $\alpha_1 + \alpha_2 + \alpha_3 \mapsto B(\Lambda_1 + \Lambda_2 + \Lambda_3)_{s_2}$ yields the alternative parametrization of all cluster variables in $\mathbb{C}[G^{e,c^2}]$ by the set of almost positive roots, which differs from the one in [5].

In the next section, we generalize this example.

4 Cluster variables in $\mathbb{C}[G^{e,c^2}]$ and crystal base

4.1 Main theorem

In this section, we shall describe all the cluster variables in $\mathbb{C}[G^{e,c^2}]$ as monomial realizations of Demazure crystals. Let $G = \mathrm{SL}_{r+1}(\mathbb{C})$ $(r \geq 3)$, and $c \in W$ be the following Coxeter element:

$$c := \begin{cases} s_2s_4\cdots s_rs_1s_3\cdots s_{r-1} & \text{if } r \text{ is even,} \\ s_2s_4\cdots s_{r-1}s_1s_3\cdots s_r & \text{if } r \text{ is odd,} \end{cases}$$

and $\mathbf{i}$ be the following reduced word of c^2:

$$\mathbf{i} := \begin{cases} (2,4,\cdots,r,1,3,\cdots,r-1,2,4,\cdots,r,1,3,\cdots,r-1) & \text{if } r \text{ is even,} \\ (2,4,\cdots,r-1,1,3,\cdots,r,2,4,\cdots,r-1,1,3,\cdots,r) & \text{if } r \text{ is odd.} \end{cases}$$

Along the above $\mathbf{i}$, we set the variables $\mathbf{Y} \in (\mathbb{C}^\times)^{2r}$ as

$$\mathbf{Y} := \begin{cases} (Y_{1,2},Y_{1,4},\cdots,Y_{1,r},Y_{1,1},Y_{1,3},\cdots,Y_{1,r-1}, \\ \qquad Y_{2,2},Y_{2,4},\cdots,Y_{2,r},Y_{2,1},Y_{2,3},\cdots,Y_{2,r-1}) & \text{if } r \text{ is even,} \\ (Y_{1,2},Y_{1,4},\cdots,Y_{1,r-1},Y_{1,1},Y_{1,3},\cdots,Y_{1,r}, \\ \qquad Y_{2,2},Y_{2,4},\cdots,Y_{2,r-1},Y_{2,1},Y_{2,3},\cdots,Y_{2,r}) & \text{if } r \text{ is odd.} \end{cases}$$

Let j_k be the k-th number of $\mathbf{i}$ from the right. For example, if r is even, then $j_1 = r-1, j_2 = r-3, \cdots, j_r = 2$. Recall that the minors $D_{\{1,2,\cdots,j_k\},s_{j_1}\cdots s_{j_k}\{1,2,\cdots,j_k\}}$ are initial cluster variables in $\mathbb{C}[G^{e,c^2}]$. Let Ξ be the set of the non-frozen cluster variables in $\mathbb{C}[G^{e,c^2}]$, and we consider the monomial realization associated with the sequence $(j_r, j_{r-1},\cdots,j_1)$ (see 2.3).

Theorem 4.1. (1) $D_{\{1,2,\cdots,j_k\},s_{j_1}\cdots s_{j_k}\{1,2,\cdots,j_k\}} \circ X_{\mathbf{i}}$ is the total sum of the monomial realization of Demazure crystal $B(\Lambda_{j_k})_{s_{j_1}\cdots s_{j_k}}$ with coefficients 1, where the map $X_{\mathbf{i}} : H \times (\mathbb{C}_{\neq 0})^{2l(c)} \longrightarrow G^{e,c^2}$ is the one in Theorem 3.9.

(2) For each non-initial cluster variable ξ in $\mathbb{C}[G^{e,c^2}]$, there uniquely exist $p \geq 0$, $w, w_i \in W$, $\lambda := \sum_{j=a}^{b} \Lambda_j$ $(1 \leq a \leq b \leq r)$ and $\lambda_i \in P^+$ such that $\lambda - \lambda_i \in \oplus_{s\in I}\mathbb{Z}_{\geq 0}\alpha_s$ and $\xi \circ X_{\mathbf{i}}$ is the total sum of monomials in Demazure crystals in the form

$$B(\lambda)_w \oplus \bigoplus_{i=1}^{p} B(\lambda_i)_{w_i}.$$

Then, let ξ_λ denote this non-initial cluster variable ξ. In particular, the map $\Phi_{\geq -1} \to \Xi$,

$$-\alpha_{j_k} \mapsto D_{\{1,2,\cdots,j_k\},s_{j_1}\cdots s_{j_k}\{1,2,\cdots,j_k\}}, \quad \sum_{j=a}^{b} \alpha_j \mapsto \xi_{\sum_{j=a}^{b}\Lambda_j}$$

is a bijection between the set $\Phi_{\geq -1}$ of almost positive roots and Ξ.

Remark 4.2. The correspondence between $\Phi_{\geq -1}$ and the set Ξ of cluster variables in this theorem is different from the one of [5].

4.2 Examples

Example 4.3. Let us consider the case $G = \mathrm{SL}_4(\mathbb{C})$ and $c = s_2 s_1 s_3 \in W$. As have seen in Example 3.10, all the cluster variables in $\mathbb{C}[G^{e,c^2}]$ are described as Demazure crystals

$$B(\Lambda_i)_{s_3 s_1 s_2}, \quad B(\Lambda_i)_e \quad (i = 1, 2, 3),$$

$$B(\Lambda_1 + \Lambda_2)_{s_3 s_2}, \ B(\Lambda_2 + \Lambda_3)_{s_1 s_2}, \ B(\Lambda_1 + \Lambda_2 + \Lambda_3)_{s_2}.$$

Example 4.4. Let us consider the case $G = \mathrm{SL}_5(\mathbb{C})$ and $c = s_2 s_4 s_1 s_3 \in W$. All the cluster variables in $\mathbb{C}[G^{e,c^2}]$ are described as Demazure crystals

$$B(\Lambda_i)_{s_3 s_1 s_4 s_2}, \quad B(\Lambda_i)_e \quad (i = 1, 2, 3, 4),$$

$$B(\Lambda_1 + \Lambda_2)_{s_3 s_2}, \ B(\Lambda_2 + \Lambda_3)_{s_1 s_2}, \ B(\Lambda_3 + \Lambda_4)_{s_4},$$

$$B(\Lambda_1 + \Lambda_2 + \Lambda_3)_{s_2}, \ B(\Lambda_2 + \Lambda_3 + \Lambda_4)_{s_1 s_4 s_2} \oplus B(\Lambda_1 + \Lambda_3)_{s_1},$$

$$B(\Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4)_{s_2 s_4} \oplus B(2\Lambda_1 + \Lambda_3)_e.$$

Thus the correspondence

$$-\alpha_i \mapsto B(\Lambda_i)_{s_3 s_1 s_4 s_2}, \quad \alpha_i \mapsto B(\Lambda_i)_e \quad (i = 1, 2, 3, 4),$$

$$\alpha_1 + \alpha_2 \mapsto B(\Lambda_1 + \Lambda_2)_{s_3 s_2}, \quad \alpha_2 + \alpha_3 \mapsto B(\Lambda_2 + \Lambda_3)_{s_1 s_2},$$

$$\alpha_3 + \alpha_4 \mapsto B(\Lambda_3 + \Lambda_4)_{s_4}, \quad \alpha_1 + \alpha_2 + \alpha_3 \mapsto B(\Lambda_1 + \Lambda_2 + \Lambda_3)_{s_2},$$

$$\alpha_2 + \alpha_3 + \alpha_4 \mapsto B(\Lambda_2 + \Lambda_3 + \Lambda_4)_{s_1 s_4 s_2} \oplus B(\Lambda_1 + \Lambda_3)_{s_1},$$

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \mapsto B(\Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4)_{s_2 s_4} \oplus B(2\Lambda_1 + \Lambda_3)_e$$

gives a parametrization of the cluster variables in $\mathbb{C}[G^{e,c^2}]$ by the set of almost positive roots.

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