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Remarks on the Caffarelli-Kohn-Nirenberg inequalities of the logarithmic type

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1 Introduction

Let Ω be a domain in $\mathbb{R}^n$ with $n \geq 3$ and assume $0 \in \Omega$. The classical Hardy inequality states that the inequality

$$\left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{|f|^2}{|x|^2} dx \leq \int_\Omega |\nabla f|^2 dx \tag{1.1}$$

holds for all $f \in H_0^1(\Omega)$, where the constant $\left(\frac{n-2}{2}\right)^2$ is best-possible. It is also well-known that the inequality (1.1) admits no nontrivial extremizers, and this fact implies a possibility for (1.1) to be improved by adding some remainder terms. In fact, the authors in [9] proved that the following improved Hardy inequality

$$\left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{|f|^2}{|x|^2} dx + \Lambda \int_\Omega |f|^2 dx \leq \int_\Omega |\nabla f|^2 dx \tag{1.2}$$

holds for all $f \in H_0^1(\Omega)$ provided that Ω is bounded, where the constant Λ in (1.2) is given by $\Lambda = \Lambda(n, \Omega) = z_0^2 \omega_n^{\frac{2}{n}} |\Omega|^{-\frac{2}{n}}$, and ω_n and $|\Omega|$ denote the Lebesgue measures of the unit ball and Ω on $\mathbb{R}^n$, respectively, and the absolute constant z_0 denotes the first zero of the Bessel function $J_0(z)$. The constant Λ is optimal if Ω is a ball, but still the inequality (1.2) admits no nontrivial extremizers. More generally, the authors in [9] obtained the inequality

$$\left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{|f|^2}{|x|^2} dx + \tilde{\Lambda} \left(\int_\Omega |f|^p dx\right)^{\frac{2}{p}} \leq \int_\Omega |\nabla f|^2 dx$$

for $f \in H_0^1(\Omega)$, where $1 < p < \frac{2n}{n-2}$ and $\tilde{\Lambda}$ is a positive constant independent of u. Similar improvements have been done for the Hardy inequality not only in the L^2-setting but in L^p-setting with some remainder terms, see for instance [3, 7, 8, 21, 35].

Hardy type inequalities are known as useful mathematical tools in various fields such as real analysis, functional analysis, probability and partial differential equations. In fact, Hardy type inequalities and their improvements are applied in many contexts. For instance, Hardy type inequalities were utilized in investigating the stability of solutions of semi-linear elliptic and parabolic equations in [9, 11]. As for the existence and asymptotic behavior of solutions of the heat equation involving singular potentials, see [10, 35]. Among others we refer to [1, 5, 16, 18, 27, 31] for the concrete applications of Hardy type inequalities. We also refer to [13, 33] for a comprehensive understanding of Hardy type inequalities.

Based on the historical remarks on the Hardy type inequalities, our purpose in this paper is to establish the classical Hardy inequalities in the frame work of equalities which immediately imply the Hardy inequalities by dropping the remainder terms. At the same time, those equalities characterize the form of the vanishing remainder terms. Our method on the basis of equalities presumably provides a simple and direct understanding of the Hardy type inequalities as well as the nonexistence of nontrivial extremizers.

In what follows, we always assume $\Omega = \mathbb{R}^n$ and the standard $L^2(\mathbb{R}^n)$ norm is denoted by $\|\cdot\|_2$. Then the Hardy type inequalities in L^2-setting that we discuss in this paper are the following:

$$\left\|\frac{f}{|x|}\right\|_2 \leq \frac{2}{n-2}\left\|\frac{x}{|x|}\cdot\nabla f\right\|_2, \qquad n \geq 3, \tag{1.3}$$

$$\sup_{R>0}\left\|\frac{f-f_R}{|x|^{\frac{n}{2}}\log\frac{R}{|x|}}\right\|_2 \leq 2\left\|\frac{1}{|x|^{\frac{n}{2}-1}}\frac{x}{|x|}\cdot\nabla f\right\|_2, \qquad n \geq 2, \tag{1.4}$$

$$\int_0^\infty x^{-p-1}\left|\int_0^x f(y)dy\right|^2 dx \leq \left(\frac{2}{p}\right)^2\int_0^\infty x^{-p+1}|f(x)|^2 dx, \tag{1.5}$$

$$\int_0^\infty x^{p-1}\left|\int_x^\infty f(y)dy\right|^2 dx \leq \left(\frac{2}{p}\right)^2\int_0^\infty x^{p+1}|f(x)|^2 dx, \tag{1.6}$$

where $f_R(x) = f\left(R\frac{x}{|x|}\right)$ and $p > 0$. The inequalities (1.3), (1.5), and (1.6) are standard (see [19] for instance), while (1.4) is rather new (see [28, 30]). In addition, as we noticed in [30], the logarithmic Hardy inequality (1.4) has a scaling property.

We state our main theorems. We denote by ∂_r the radial derivative defined by $\partial_r = \frac{x}{|x|}\cdot\nabla = \sum_{j=1}^n \frac{x_j}{|x|}\partial_j$. The space $D^{1,2}(\mathbb{R}^n)$ denotes the completion of $C_0^\infty(\mathbb{R}^n)$ under the Dirichlet norm $\|\nabla\cdot\|_2$. Also the notation S^{n-1} denotes the unit sphere in $\mathbb{R}^n$ endowed with the Lebesgue measure σ.

2 Hardy type inequalities in the framework of equalities

In this section, we shall prove the Hardy type inequalities in the framework of equalities. Our first result states as follows:

Theorem 2.1. *Let $n \geq 3$. Then the equalities*

$$\left(\frac{n-2}{2}\right)^2\left\|\frac{f}{|x|}\right\|_2^2 = \|\partial_r f\|_2^2 - \left\|\partial_r f + \frac{n-2}{2|x|}f\right\|_2^2 \tag{2.1}$$

$$= \|\partial_r f\|_2^2 - \left\||x|^{-\frac{n-2}{2}}\partial_r(|x|^{\frac{n-2}{2}}f)\right\|_2^2 \tag{2.2}$$

hold for all $f \in D^{1,2}(\mathbb{R}^n)$. Moreover, the second term in the right hand side of (2.1) *or* (2.2) *vanishes if and only if f takes the form*

$$f(x) = |x|^{-\frac{n-2}{2}}\varphi\left(\frac{x}{|x|}\right) \tag{2.3}$$

for some function $\varphi : S^{n-1} \to \mathbb{C}$, which makes the left hand side of (2.1) *infinite unless $\int_{S^{n-1}}|\varphi(\omega)|^2 d\sigma(\omega) = 0$:*

$$\frac{|f|^2}{|x|^2} = \frac{\left|\varphi\left(\frac{x}{|x|}\right)\right|^2}{|x|^n} \notin L^1(\mathbb{R}^n). \tag{2.4}$$

We remark that as in (2.4), functions of the form (2.3) imply the nonexistence of nontrivial extremizers for (1.3). The corresponding integral diverges at both origin and infinity. A similar result to Theorem 2.1 can be found in [6, 15]. However, the essential ideas for the proofs are different. Indeed, the proof in [6] is done by direct calculations with respect to the quotient with the optimizer of a Hardy type inequality. On the other hand, we shall prove Theorem 2.1 by applying an orthogonality argument in general Hilbert space settings. More precisely, an equality

$$\left(\frac{n-2}{2}\right)^2 \left\| \frac{f}{|x|} \right\|_2^2 = \|\nabla f\|_2^2 - \left\| \nabla f + \frac{n-2}{2} \frac{x}{|x|^2} f \right\|_2^2 \tag{2.5}$$

has been observed in [6, 15]. We should remark that (2.1) and (2.5) are the same for radially symmetric functions and are not the same for nonradial functions. In fact, the Dirichlet integral is decomposed into radial and spherical components as

$$\|\nabla f\|_2^2 = \|\partial_r f\|_2^2 + \sum_{j=1}^{n} \left\| \left(\partial_j - \frac{x_j}{|x|} \partial_r \right) f \right\|_2^2 .$$

Next, we state the logarithmic Hardy type equalities in the critical weighted Sobolev spaces.

Theorem 2.2. *Let $n \geq 2$. Then the equalities*

$$\frac{1}{4} \left\| \frac{f - f_R}{|x|^{\frac{n}{2}} \log \frac{R}{|x|}} \right\|_2^2 = \left\| \frac{1}{|x|^{\frac{n}{2}-1}} \partial_r f \right\|_2^2 - \left\| \frac{1}{|x|^{\frac{n}{2}-1}} \left(\partial_r f + \frac{f - f_R}{2|x| \log \frac{R}{|x|}} \right) \right\|_2^2 \tag{2.6}$$

$$= \left\| \frac{1}{|x|^{\frac{n}{2}-1}} \partial_r f \right\|_2^2 - \left\| \frac{|\log \frac{R}{|x|}|^{\frac{1}{2}}}{|x|^{\frac{n}{2}-1}} \partial_r \left(\frac{f - f_R}{|\log \frac{R}{|x|}|^{\frac{1}{2}}} \right) \right\|_2^2 \tag{2.7}$$

hold for all $R > 0$ and all $f \in L^1_{loc}(\mathbb{R}^n)$ with $\frac{1}{|x|^{\frac{n}{2}-1}} \nabla f \in L^2(\mathbb{R}^n)$, where f_R is defined by $f_R(x) = f\left(R \frac{x}{|x|}\right)$. Moreover, the second term in the right hand side of (2.6) or (2.7) vanishes if and only if $f - f_R$ takes the form

$$f(x) - f_R(x) = \left| \log \frac{R}{|x|} \right|^{\frac{1}{2}} \varphi\left(\frac{x}{|x|} \right) \tag{2.8}$$

for some function $\varphi : S^{n-1} \to \mathbb{C}$, which makes the left hand side of (2.6) infinite unless $\int_{S^{n-1}} |\varphi(\omega)|^2 d\sigma(\omega) = 0$:

$$\frac{|f - f_R|^2}{|x|^n \left| \log \frac{R}{|x|} \right|^2} = \frac{\left| \varphi\left(\frac{x}{|x|} \right) \right|^2}{|x|^n \left| \log \frac{R}{|x|} \right|} \notin L^1(\mathbb{R}^n). \tag{2.9}$$

As in (2.9), functions of the form (2.8) imply the nonexistence of nontrivial extremizers for (1.4). The corresponding integral diverges at both origin and infinity and, in addition, on the sphere of radius $R > 0$.

3 Caffarelli-Kohn-Nirenberg type inequalities in the framework of equalities

In this section, we shall prove the Caffarelli-Kohn-Nirenberg type inequalities in the framework of equalities. We first recall the Caffarelli-Kohn-Nirenberg inequality: Let $n \in \mathbb{N}$, $1 \leq p < \infty$, $r > 0$, $\frac{1}{r} + \frac{\sigma}{n} > 0$. Then

$$\| |x|^{\sigma} u \|_r \leq C \| |x|^{\alpha} |\nabla u| \|_p$$

holds for all $u \in C_0^\infty(\mathbb{R}^n)$ if and only if $0 \le \alpha - \sigma \le 1$ and $\frac{1}{r} + \frac{\sigma}{n} = \frac{1}{p} + \frac{\alpha-1}{n}$. Especially, taking $p = r$, we obtain the following: Let $n \in \mathbb{N}$, $1 \le p < \infty$, $\frac{1}{p} + \frac{\alpha-1}{n} > 0$. Then there holds

$$\||x|^{\alpha-1}u\|_p \le C\||x|^\alpha|\nabla u|\|_p \tag{3.1}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$. We already proved the equality version of (3.1) with $\alpha = 0$ and $p = 2$ as follows:

Let $n \ge 3$. Then there holds

$$\left(\frac{n-2}{2}\right)^2 \left\||x|^{-1}u\right\|_2^2 = \|\partial_r u\|_2^2 - \left\|\partial_r u + \frac{n-2}{2|x|}u\right\|_2^2 \tag{3.2}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$, where $\partial_r u(x) := \frac{x}{|x|} \cdot \nabla u(x)$ for $x \in \mathbb{R}^n \setminus \{0\}$. Firstly, we extend the result of (3.2) for general α, which corresponds to the equality version of (3.1) with $p = 2$ as stated in Theorem 3.1 (i). Furthermore, we shall establish the equality version of the Caffarelli-Kohn-Nirenberg type inequality of the logarithmic form by applying Theorem 3.1 (i) as stated in Theorem 3.1 (ii).

Theorem 3.1. *(i) Let $n \in \mathbb{N}$ and $\alpha > \frac{2-n}{2}$. Then there holds*

$$\left(\frac{n-2+2\alpha}{2}\right)^2 \left\||x|^{\alpha-1}u\right\|_2^2 = \||x|^\alpha \partial_r u\|_2^2 - \left\||x|^\alpha \partial_r u + \frac{n-2+2\alpha}{2}|x|^{\alpha-1}u\right\|_2^2 \tag{3.3}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$.

(ii) Let $n \in \mathbb{N}$, $1 < \gamma < 3$ and $R > 0$. Then there holds

$$\left(\frac{\gamma-1}{2}\right)^2 \left\|\left|\log\left(\frac{R}{|x|}\right)\right|^{-\frac{\gamma}{2}} |x|^{-\frac{n}{2}}(u - u_R)\right\|_2^2 = \left\|\left|\log\left(\frac{R}{|x|}\right)\right|^{\frac{2-\gamma}{2}} |x|^{\frac{2-n}{2}} \partial_r u\right\|_2^2$$
$$- \left\|\left|\log\left(\frac{R}{|x|}\right)\right|^{\frac{2-\gamma}{2}} |x|^{\frac{2-n}{2}} \left(\partial_r u + \frac{\gamma-1}{2}|x|^{-1}\left|\log\left(\frac{R}{|x|}\right)\right|^{-1}(u - u_R)\right)\right\|_2^2 \tag{3.4}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$, where $u_R(x) := u(R\frac{x}{|x|})$ for $x \in \mathbb{R}^n \setminus \{0\}$.

Proof. (i) It is enough to show

$$\left\||x|^{\alpha-1}u\right\|_2^2 = -\frac{2}{n-2+2\alpha} \operatorname{Re} \int_{\mathbb{R}^n} |x|^{2\alpha-1} u\, \overline{\partial_r u}\, dx.$$

Indeed, by integration by parts, we see

$$\begin{aligned}
\left\||x|^{\alpha-1}u\right\|_2^2 &= \int_{\mathbb{R}^n} |x|^{2\alpha-2}|u|^2 dx = \int_0^\infty r^{2\alpha+n-3} \int_{S^{n-1}} |u(r\omega)|^2 d\omega dr \\
&= -\frac{2}{2\alpha+n-2} \operatorname{Re} \int_0^\infty r^{2\alpha+n-2} \int_{S^{n-1}} u(r\omega)\overline{\omega \cdot \nabla u(r\omega)} d\omega dr \\
&= -\frac{2}{2\alpha+n-2} \operatorname{Re} \int_{\mathbb{R}^n} |x|^{2\alpha-1} u\, \overline{\partial_r u}\, dx,
\end{aligned}$$

where we used $2\alpha + n - 2 > 0$.

(ii) First, we establish

$$\left(\frac{\gamma-1}{2}\right)^2 \int_{B_R(0)} |u - u_R|^2 \left[\log\left(\frac{R}{|x|}\right)\right]^{-\gamma} \frac{dx}{|x|^n} = \int_{B_R(0)} |\partial_r u|^2 \left[\log\left(\frac{R}{|x|}\right)\right]^{2-\gamma} |x|^{2-n} dx$$

$$- \int_{B_R(0)} \left| \partial_r u + \frac{\gamma-1}{2} |x|^{-1} \left[\log \left(\frac{R}{|x|} \right) \right]^{-1} (u - u_R) \right|^2 \left[\log \left(\frac{R}{|x|} \right) \right]^{2-\gamma} \frac{dx}{|x|^{n-2}} \tag{3.5}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$.

By using polar coordinates, we have

$$\int_{B_R(0)} |u - u_R|^2 \left[\log \left(\frac{R}{|x|} \right) \right]^{-\gamma} \frac{dx}{|x|^n} = \int_{S^{n-1}} \int_0^R |u(t\omega) - u(R\omega)|^2 \left[\log \left(\frac{R}{t} \right) \right]^{-\gamma} \frac{dt}{t} d\omega.$$

Changing variables $t = t(r) := R \exp(-r^{-\frac{1}{\gamma}})$ for $r \in (0, \infty)$, we see

$$\int_{S^{n-1}} \int_0^R |u(t\omega) - u(R\omega)|^2 \left[\log \left(\frac{R}{t} \right) \right]^{-\gamma} \frac{dt}{t} d\omega = \frac{1}{\gamma} \int_{S^{n-1}} \int_0^\infty |u(t(r)\omega) - u(R\omega)|^2 r^{-\frac{1}{\gamma}} dr d\omega$$

$$= \frac{1}{2\pi\gamma} \int_{S^{n-1}} \int_{\mathbb{R}^2} |u(t(|x|_2)\omega) - u(R\omega)|^2 |x|_2^{-\frac{1}{\gamma}-1} dx d\omega = \frac{1}{2\pi\gamma} \int_{S^{n-1}} \left\| |x|_2^{\frac{\gamma-1}{2\gamma}-1} f_\omega \right\|_{L^2(\mathbb{R}^2)}^2 d\omega,$$

where $|x|_2$ denotes the two-dimensional Euclidean norm and $f_\omega(x) := u(t(|x|_2)\omega) - u(R\omega)$ for $\omega \in S^{n-1}$ and $x \in \mathbb{R}^2$. Then applying (3.3) for the function f_ω with the dimension 2 and $\alpha = \frac{\gamma-1}{2\gamma} > 0$, we obtain

$$\frac{\pi(\gamma-1)^2}{2\gamma} \int_{B_R(0)} |u - u_R|^2 \left[\log \left(\frac{R}{|x|} \right) \right]^{-\gamma} \frac{dx}{|x|^n}$$

$$= \int_{S^{n-1}} \left\| |x|_2^{\frac{\gamma-1}{2\gamma}} \partial_{2,r} f_\omega \right\|_{L^2(\mathbb{R}^2)}^2 d\omega - \int_{S^{n-1}} \left\| |x|_2^{\frac{\gamma-1}{2\gamma}} \partial_{2,r} f_\omega + \frac{\gamma-1}{2\gamma} |x|_2^{\frac{\gamma-1}{2\gamma}-1} f_\omega \right\|_{L^2(\mathbb{R}^2)}^2 d\omega, \tag{3.6}$$

where $\partial_{2,r} f_\omega$ denotes the two-dimensional radial derivative, i.e., $\partial_{2,r} f_\omega(x) := \frac{x}{|x|_2} \cdot \nabla_2 f_\omega(x)$ for $x \in \mathbb{R}^2 \setminus \{0\}$ and $\nabla_2 := (\partial_1, \partial_2)$. By a direct computation, we see for $x \in \mathbb{R}^2 \setminus \{0\}$,

$$\partial_{2,r} f_\omega(x) = \frac{R}{\gamma} \partial_r u(t(|x|_2)\omega) \exp(-|x|_2^{-\frac{1}{\gamma}}) |x|_2^{-\frac{1}{\gamma}-1},$$

and then by changing variables $r = r(t) := \left[\log(\frac{R}{t}) \right]^{-\gamma}$ for $t \in (0, R)$,

$$\int_{S^{n-1}} \left\| |x|_2^{\frac{\gamma-1}{2\gamma}} \partial_{2,r} f_\omega \right\|_{L^2(\mathbb{R}^2)}^2 d\omega = \frac{R^2}{\gamma^2} \int_{S^{n-1}} \int_{\mathbb{R}^2} |\partial_r u(t(|x|_2)\omega)|^2 \exp(-2|x|_2^{-\frac{1}{\gamma}}) |x|_2^{-\frac{3}{\gamma}-1} dx d\omega$$

$$= \frac{2\pi R^2}{\gamma^2} \int_{S^{n-1}} \int_0^\infty |\partial_r u(t(r)\omega)|^2 \exp(-2r^{-\frac{1}{\gamma}}) r^{-\frac{3}{\gamma}} dr d\omega$$

$$= \frac{2\pi}{\gamma} \int_{S^{n-1}} \int_0^R |\partial_r u(t\omega)|^2 \left[\log \left(\frac{R}{t} \right) \right]^{2-\gamma} t \, dt d\omega = \frac{2\pi}{\gamma} \int_{B_R(0)} |\partial_r u|^2 \left[\log \left(\frac{R}{|x|} \right) \right]^{2-\gamma} |x|^{2-n} dx. \tag{3.7}$$

Similarly, we see

$$\int_{S^{n-1}} \left\| |x|_2^{\frac{\gamma-1}{2\gamma}} \partial_{2,r} f_\omega + \frac{\gamma-1}{2\gamma} |x|_2^{\frac{\gamma-1}{2\gamma}-1} f_\omega \right\|_{L^2(\mathbb{R}^2)}^2 d\omega$$

$$= \int_{S^{n-1}} \int_{\mathbb{R}^2} \left| \frac{R}{\gamma} \partial_r u(t(|x|_2)\omega) \exp(-|x|_2^{-\frac{1}{\gamma}}) |x|_2^{-\frac{3}{2\gamma}-\frac{1}{2}} + \frac{\gamma-1}{2\gamma} |x|_2^{\frac{\gamma-1}{2\gamma}-1} \left(u(t(|x|_2)\omega) - u(R\omega) \right) \right|^2 dx d\omega$$

$$= 2\pi \int_{S^{n-1}} \int_0^\infty \left| \frac{R}{\gamma} \partial_r u(t(r)\omega) \exp(-r^{-\frac{1}{\gamma}}) r^{-\frac{3}{2\gamma}-\frac{1}{2}} + \frac{\gamma-1}{2\gamma} r^{\frac{\gamma-1}{2\gamma}-1} \left(u(t(r)\omega) - u(R\omega) \right) \right|^2 r dr d\omega$$

$$= \frac{2\pi}{\gamma} \int_{S^{n-1}} \int_0^R \left| \partial_r u(t\omega) + \frac{\gamma-1}{2} t^{-1} \left[\log\left(\frac{R}{t}\right) \right]^{-1} (u(t\omega) - u(R\omega)) \right|^2 \left[\log\left(\frac{R}{t}\right) \right]^{2-\gamma} t\, dt d\omega$$

$$= \frac{2\pi}{\gamma} \int_{B_R(0)} \left| \partial_r u + \frac{\gamma-1}{2} |x|^{-1} \left[\log\left(\frac{R}{|x|}\right) \right]^{-1} (u - u_R) \right|^2 \left[\log\left(\frac{R}{|x|}\right) \right]^{2-\gamma} \frac{dx}{|x|^{n-2}}. \qquad (3.8)$$

Plugging (3.7) and (3.8) into (3.6), we obtain (3.5).

In the same way as above by replacing the change of variables by $r = r(t) := \left[\log\left(\frac{t}{R}\right)\right]^{-\gamma}$ for $t \in (R, \infty)$ and $t = t(r) := R\exp(r^{-\frac{1}{\gamma}})$ for $r \in (0, \infty)$, we obtain

$$\left(\frac{\gamma-1}{2}\right)^2 \int_{B_R(0)^c} |u - u_R|^2 \left[\log\left(\frac{|x|}{R}\right) \right]^{-\gamma} \frac{dx}{|x|^n} = \int_{B_R(0)^c} |\partial_r u|^2 \left[\log\left(\frac{|x|}{R}\right) \right]^{2-\gamma} |x|^{2-n} dx$$

$$- \int_{B_R(0)^c} \left| \partial_r u + \frac{\gamma-1}{2} |x|^{-1} \left[\log\left(\frac{|x|}{R}\right) \right]^{-1} (u - u_R) \right|^2 \left[\log\left(\frac{|x|}{R}\right) \right]^{2-\gamma} \frac{dx}{|x|^{n-2}} \qquad (3.9)$$

for all $u \in C_0^\infty(\mathbb{R}^n)$. Finally, adding the both sides of the equalities (3.5) and (3.9) yields (3.4). □

Remark 3.2. *(i) We easily see that the integrals in* (3.5) *diverge for some function in* $C_0^\infty(\mathbb{R}^n)$ *when* $\gamma \le 1$ *or* $\gamma \ge 3$. *On the other hand, the integrals converge for any function in* $C_0^\infty(\mathbb{R}^n)$ *when* $1 < \gamma < 3$. *Indeed, by using the estimates* $\log(\frac{R}{t}) \ge \frac{R-t}{R}$ *for* $0 < t < R$, *and* $|u(R\omega) - u(t\omega)| \le \|\nabla u\|_\infty (R-t)$ *for* $\omega \in S^{n-1}$ *and* $0 < t < R$, *we see*

$$\int_{B_R(0)} |u - u_R|^2 \left[\log\left(\frac{R}{|x|}\right) \right]^{-\gamma} \frac{dx}{|x|^n} = \int_{S^{n-1}} \int_0^R |u(t\omega) - u(R\omega)|^2 \left[\log\left(\frac{R}{t}\right) \right]^{-\gamma} \frac{dt}{t} d\omega$$

$$= \int_{S^{n-1}} \int_0^{\frac{R}{2}} |u(t\omega) - u(R\omega)|^2 \left[\log\left(\frac{R}{t}\right) \right]^{-\gamma} \frac{dt}{t} d\omega + \int_{S^{n-1}} \int_{\frac{R}{2}}^R |u(t\omega) - u(R\omega)|^2 \left[\log\left(\frac{R}{t}\right) \right]^{-\gamma} \frac{dt}{t} d\omega$$

$$\le 4\,\omega_{n-1} \|u\|_\infty^2 \int_0^{\frac{R}{2}} \left[\log\left(\frac{R}{t}\right) \right]^{-\gamma} \frac{dt}{t} + 2R^{\gamma-1} \omega_{n-1} \|\nabla u\|_\infty^2 \int_{\frac{R}{2}}^R (R-t)^{2-\gamma} dt < +\infty$$

since $1 < \gamma < 3$. *Also we have*

$$\int_{B_R(0)} |\partial_r u|^2 \left[\log\left(\frac{R}{|x|}\right) \right]^{2-\gamma} |x|^{2-n} dx \le \omega_{n-1} \|\nabla u\|_\infty^2 \int_0^R \left[\log\left(\frac{R}{t}\right) \right]^{2-\gamma} t\, dt < +\infty$$

since $\gamma < 3$.

(ii) By restricting the functions into $C_0^\infty(B_R(0))$, *we can remove the condition* $\gamma < 3$. *Precisely, the integrals in* (3.5) *converge for any function in* $C_0^\infty(B_R(0))$ *and the inequality* (3.5) *holds in* $C_0^\infty(B_R(0))$ *for all* $\gamma > 1$.

References

[1] A. Ancona, On strong barriers and an inequality of Hardy for domains in $\mathbb{R}^n$, J. London Math. Soc., **34** (1986), 274–290.

[2] A. Balinsky, W. D. Evans, D. Hundertmark and R. T. Lewis, On inequalities of Hardy-Sobolev type, Banach J. Math. Anal., **2** (2008), 94–106.

[3] G. Barbatis, S.Filippas and A. Tertikas, A unified approach to improved L^p Hardy inequalities with best constants, Trans. Am. Math. Soc., **356** (2004), 2169–2196.

[4] W. Beckner, Pitt's inequality and the fractional Laplacian: sharp error estimates, Forum Math., **24** (2012), 177–209.

[5] K. Bogdan and B. Dyda, The best constant in a fractional Hardy inequality, Math. Nachr., **284** (2011), 629–638.

[6] K. Bogdan, B. Dyda and P. Kim, Hardy inequalities and non-explosion results for semigroups, Potential Anal., **44** (2016), 229–247.

[7] H. Brezis and M. Marcus, Hardy's inequality revisited, Ann. Scuola. Norm. Sup. Pisa, **25** (1997), 217–237.

[8] H. Brezis, M. Marcus and I. Shafrir, Extremal functions for Hardy's inequality with weight, J. Funct. Anal., **171** (2000), 177–191.

[9] H. Brezis and J. L. Vázquez, Blow-up solutions of some nonlinear elliptic problems, Rev. Mat. Univ. Complut. Madrid., **10** (1997), 443–469.

[10] X. Cabré and Y. Martel, Existence versus explosion instantanée pour des équations de la chaleur linéaires avec potential singulier, C. R. Acad. Sci. Paris Sér. I, **329** (1999), 973–978.

[11] X. Cabré and Y. Martel, Weak eigenfunctions for the linearization of extremal elliptic problems, J. Funct. Anal., **156** (1998), 30–56.

[12] D. G. Costa, Some new and short proofs for a class of Caffarelli-Kohn-Nirenberg type inequalities, J. Math. Anal. Appl., **337** (2008), 311–317.

[13] E. B. Davies, A review of Hardy inequalities, Oper. Theory Adv. Appl., **110** (1999), 55–67.

[14] Y. Di, L. Jiang, S. Shen and Y. Jin, A note on a class of Hardy-Rellich type inequalities, J. Inequal. Appl., (2013), 2013:84.

[15] J. Dolbeault and B. Volzone, Improved Poincaré inequalities, Nonlinear Anal., **75** (2012), 5985–6001.

[16] B. Dyda, Fractional calculus for power functions and eigenvalues of the fractional Laplacian, Fract. Calc. Appl. Anal., **15** (2012), 536–555.

[17] S. Filippas and A. Tertikas, Optimizing improved Hardy inequalities, J. Funct. Anal., **192** (2002), 186–233.

[18] P. J. Fitzsimmons, Hardy's inequality for Dirichlet forms, J. Math. Anal. Appl., **250** (2000), 548–560.

[19] G. B. Folland, "Real analysis," Second edition, Pure and Applied Mathematics (New York), John Wiley & Sons, Inc., New York, (1999).

[20] R. Frank and R. Seiringer, Non-linear ground state representations and sharp Hardy inequalities, J. Funct. Anal., **255** (2008), 3407–3430.

[21] F. Gazzola, H. C. Grunau and E. Mitidieri, Hardy inequalities with optimal constants and remainder terms, Trans. Amer. Math. Soc., **356** (2003), 2149–2168.

[22] N. Ghoussoub and A. Moradifam, Bessel pairs and optimal Hardy and Hardy-Rellich inequalities, Math. Ann., **349** (2011), 1–57.

[23] N. Ghoussoub and A. Moradifam, On the best possible remaining term in the Hardy inequality, Proc. Natl. Acad. Sci. USA, **1305** (2008), 13746–13751.

[24] G. H. Hardy, J. E. Littlewood and G. Pólya, "Inequalities," Cambridge University Press, (1952).

[25] N. Ioku and M. Ishiwata, A scale invariant form of a critical Hardy inequality, Int. Math. Res. Not. IMRN, **18** (2015), 8830–8846.

[26] N. Ioku, M. Ishiwata and T. Ozawa, Sharp remainder of a critical Hardy inequality, Arch. Math. (Basel), **106** (2016), 65–71.

[27] A. Kalamajska and K. Pietruska-Paluba, On a variant of the Gagliardo-Nirenberg inequality deduced from the Hardy inequality, Bull. Pol. Acad. Sci. Math., **59** (2011), 133–149.

[28] S. Machihara, T. Ozawa and H. Wadade, Hardy type inequalities on balls, Tohoku Math. J. (2), **65** (2013), 321–330.

[29] S. Machihara, T. Ozawa and H. Wadade, Generalizations of the logarithmic Hardy inequality in critical Sobolev-Lorentz spaces, J. Inequal. Appl., (2013), 2013:381.

[30] S. Machihara, T. Ozawa and H. Wadade, Scaling invariant Hardy inequalities of multiple logarithmic type on the whole space, J. Inequal. Appl., (2015), 2015:281.

[31] V. Maz'ya, "Sobolev spaces with applications to elliptic partial differential equations," Springer, Heidelberg, (2011).

[32] E. Mitidieri, A simple approach to Hardy inequalities, Mathematical notes, **67** (2000), 563–572.

[33] B. Opic and A. Kufner, "Hardy-type inequalities," Longman Scientific & Technical, (1990).

[34] F. Takahashi, A simple proof of Hardy's inequality in a limiting case, Arch. Math. (Basel)., **104** (2015), 77–82.

[35] J. L. Vazquez and E. Zuazua, The Hardy inequality and the asymptotic behavior of the heat equation with and inverse-square potential, J. Funct. Anal., **173** (2000), 103–153.