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On the domain of a Schrödinger operator with complex potential – Old and New –

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The aim of this talk is to review and compare the spectral properties of (the closed extension of) $-\Delta + U$ ($U \geq 0$) and $-\Delta + iV$ in $L^2(\mathbb{R}^d)$ for C^∞ potentials U or V with polynomial behavior.
The case with magnetic field is also considered. More precisely, we would like to compare the criteria for:

- essential selfadjointness (**esa**) or maximal accretivity (**maxacc**)
- Compactness of the resolvent.
- Maximal inequalities,

for these operators.
By L^2-maximal inequalities, we mean the existence of $C > 0$ s. t.

$$||u||^2_{H^2} + ||Uu||^2_{L^2} \leq C \left(||(-\Delta + U)u||^2_{L^2} + ||u||^2_{L^2} \right) \,, \forall u \in C_0^\infty(\mathbb{R}^d)\,, \tag{0.1}$$

or

$$||u||^2_{H^2} + ||Vu||^2 \leq C \left(||(-\Delta + iV)u||^2 + ||u||^2 \right) \,, \forall u \in C_0^\infty(\mathbb{R}^d)\,. \tag{0.2}$$

We will also discuss the magnetic case:

$$P_{\mathbf{A},V} = -\Delta_A + W := \sum_{j=1}^{d} (D_{x_j} - A_j(x))^2 + W(x)\,,$$

(with $W = U + iV$) and the notion of maximal regularity is expressed in terms of the magnetic Sobolev spaces:

$$\begin{aligned} &||(D - \mathbf{A})u||^2_{L^2(\mathbb{R}^d,\mathbb{C}^d)} \\ &\quad + \textstyle\sum_{j,\ell} ||(D_j - A_j)(D_\ell - A_\ell)u||^2_{L^2(\mathbb{R}^d)} \\ &\qquad + ||\,|W|u||^2_{L^2(\mathbb{R}^d)} \\ &\qquad\quad \leq C \left(||P_{\mathbf{A},W}u||^2_{L^2(\mathbb{R}^d)} + ||u||^2_{L^2(\mathbb{R}^d)} \right), \end{aligned} \tag{0.3}$$

The question of analyzing $-\Delta + iV$ or more generally $P_{\mathbf{A},iV} := -\Delta_A + iV$ appears in many situations:

- Time dependent Ginzburg-Landau theory leads for example to the spectral analysis of
$$D_x^2 + (D_y - \frac{x^2}{2})^2 + iy$$
Here curl $\mathbf{A} = x$ vanishes along a line.

- Control theory

- Bloch-Torrey (complex Airy) equation
$$-\Delta + ix$$

- Spectral analysis of the complex harmonic oscillator.

Moreover, in some of the applications, V does not satisfy necessarily a sign condition $V \leq 0$ as for dissipative systems.
After reviewing all the main results devoted to this question in the selfadjoint case, we will show that similar results can be proved in the case of a complex potential. These recent results have been obtained in collaboration with Y. Almog and J. Nourrigat.
Below, we give a selected non exhaustive bibliography.

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