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BALANCED ZERO-SUM SEQUENCES AND MINIMAL INTRINSIC EXTENSIONS

神戸大学大学院システム情報学研究科　桔梗宏孝 (HIROTAKA KIKYO)
GRADUATE SCHOOL OF SYSTEM INFORMATICS, KOBE UNIVERSITY

ABSTRACT. Let a and b be positive real numbers. Assume that a/b is a rational number. Let v be a finite zero-sum sequence with entries from $\{a,-b\}$. We say that v has the positively balanced prefix property if $0<\sum u<a+b$ for any proper prefix u of v. We say that v is balanced if $|\sum u|<a+b$ for any consecutive subsequence u of v. There exists uniquely a sequence s having the positively balanced prefix property. Any rotations of s^k are balanced. Conversely, any balanced sequence is a rotation of s^k.

1. INTRODUCTION

The paper is related to Hrushovski's *ab initio* amalgamation construction. Let δ be a function on the class of finite graphs defined by $\delta(G)=|G|-\alpha e(G)$ where G is a finite graph, $|G|$ the number of the vertices of G, $e(G)$ the number of the edges of G, and α a real number with $0<\alpha<1$. We fix a language with one binary relation. We consider any graph as a structure of this language where the set of vertices is the universe and the set of edges is the interpretation of the binary relation.

Let A be a substructure of a graph B. We write $A\leq B$ if whenever $A\subseteq X\subseteq B$ then $\delta(A)\leq\delta(X)$. We write $A<B$ if whenever $A\subsetneq X\subseteq B$ then $\delta(A)<\delta(X)$.

We say that B is a zero-extension of A if $A\leq B$ and $\delta(A)=\delta(B)$. We say that B is a minimal zero-extension of A if B is a proper zero-extension of A and is minimal with this property. In this case, $A\subsetneq X\subsetneq B$ implies $\delta(A)<\delta(X)$.

We say that B is an *minimal intrinsic extension* of A if $A<X$ for any proper substructure X of B with $A<X$, but $A\not<B$. A minimal zero-extension of A is a minimal intrinsic extension of A.

In a paper [7], we defined twigs and wreaths. A twig has a path and leaves where each leaf is adjacent to a vertex in the path. A wreath has a cycle and leaves where each leaf is adjacent to a vertex in the cycle. Each twig or wreath is a minimal zero-extension of the set of their leaves.

In [7], we showed existence of a zero-sum sequence with the positively balanced prefix property in order to construct twigs and wreaths. Twigs must be associated to a sequence having the positively balanced prefix property, but wreaths can be associated to any balanced sequences. In this paper, we show that balanced sequences come from the sequence with the positively balanced prefix property. With this result, we can see that any wreaths for α with a cycle of the same size are isomorphic.

Suppose $(K,<)$ is an free amalgamation class with $\emptyset \in K$. Then any twigs belong to K and wreaths with a sufficiently large cycle belong to K.

2. BALANCED ZERO-SUM SEQUENCES

First half of this section appears in [7]. We state and prove some properties of finite zero-sum sequences. We define what we mean by a finite sequence first.

Definition 2.1. Let $\mathbb{Z}$ be the set of integers, and n a positive integer. $[n]$ denotes the set $\{i \in \mathbb{Z} \mid 0 \leq i < n\}$. Let Y be a set. A *Y-sequence of length n* is a map from $[n]$ to Y. If s is a Y-sequence of length m and t a Y-sequence of length n then a *concatenation* of s and t is a Y-sequence u of length $m+n$ such that $u(i) = s(i)$ for $0 \leq i < m$ and $u(m+j) = t(j)$ for $0 \leq j < n$. st denotes the concatenation of s and t. s^n with a positive integer n is defined by induction as follows: $s^1 = s$ and $s^n = s^{n-1}s$ for $n > 1$.

Definition 2.2. Let $\mathbb{R}$ be the set of real numbers. and s a $\mathbb{R}$-sequence of length l. $\sum s$ is the value $\sum_{i=0}^{l-1} s(i)$. If $s = uv$ then vu is called a *rotation* of s.

If $s = uvw$, u is called a *prefix* of s, w a *suffix* of s and v a *consecutive subsequence* of s. A *rotational consecutive subsequence* of s is a consecutive subsequence of some rotation of s. If u is a rotational consecutive subsequence of s then uv is a rotation of s for some sequence v. v is also a rotational consecutive subsequence of s. v is called a *rotational complement* of u. u is a rotational complement of v.

$\langle y \rangle$ is a sequence s of length 1 such that $s(0) = y$.

Suppose that s has an entry other than x. If $s^2 = u\langle y \rangle \langle x \rangle^k \langle z \rangle v$ with $x \neq y, z$ then k is called a *rotational run length* of x in s.

Let c be a real number. $c \cdot s$ is a sequence obtained by multiplying c to each entry of s.

Definition 2.3. Let s be a finite $\mathbb{R}$-sequence. s is a *zero-sum* sequence if $\sum s = 0$.

Let $c > 0$ be a real number. s is *c-balanced* if whenever u is a consecutive subsequence of s then $|\sum u| < c$.

s has the *positively c-balanced prefix property* if whenever u is a nonempty prefix of s with $u \neq s$ then $0 < \sum u < c$.

We state some easy facts first.

Lemma 2.4. *Let s be a zero-sum $\mathbb{R}$-sequence of length l, c and c' positive real numbers, and n a positive integer.*

(1) *If s is c-balanced and $s = uwv$ then $|\sum u + \sum v| < c$.*
(2) *s^n is a periodic sequence with period l. It is a zero-sum sequence.*
(3) *Any consecutive subsequence of s^n of length l is a zero-sum sequence.*
(4) *If s is c-balanced then s^n is also c-balanced.*
(5) *If s is c-balanced, then any rotation of s is c-balanced.*
(6) *If s has the positively c-balanced prefix property then s is c-balanced.*
(7) *If s is c-balanced and c' is a non-zero real number then $c' \cdot s$ is $|cc'|$-balanced.*
(8) *Suppose $c' > 0$. s has the positively c-balanced prefix property if and only if $c' \cdot s$ has the positively cc'-balanced prefix property.*

Proof. (2), (7), and (8) are clear.

(1) Suppose s is c-balanced and $s = uwv$. We have $|\sum w| < c$ because s is c-balanced. Since s is a zero-sum sequence, we have $\sum u + \sum w + \sum v = 0$. Hence, $\sum u + \sum v = -\sum w$. Therefore, $|\sum u + \sum v| = |-\sum w| = |\sum w| < c$.

(5) follows from (1).

(3) Let s' be a consecutive subsequence of s^n of length l. Since the length of s' is equal to the length of s, s' is a consecutive subsequence of s^2. Hence $ss = us'v$ for some sequences u, v. Since the length of s' is l, the length of uv is also l. Because u is a prefix of s, v is a suffix of s, we have $uv = s$. So, we have $\sum u + \sum v = \sum s = 0$. Hence, $0 = \sum s + \sum s = \sum ss = \sum us'v = \sum u + \sum s' + \sum v = \sum s'$.

(4) Let s' be a consecutive subsequence of s^n. Since any subsequence of s' of length l has zero-sum by (2), we can assume that the length of s' is less than l. Hence, s' is a subsequence of s^2, and thus we can write $s' = vu$ where v is a suffix of s and u a prefix of s. Since the length of s' is less than l, we can write $s = uwv$. By (1), we have $|\sum s'| = |\sum u + \sum v| < c$.

(6) Let v be a consecutive subsequence of s. Then uv is a prefix of s for some prefix u of s. Since s has the positively c-balanced prefix property, $0 < \sum u < c$ and $0 < \sum uv < c$. We have $\sum v = \sum uv - \sum u$. Hence, $|\sum v| < c$. □ □

Proposition 2.5. (1) *Let a and b be positive real numbers such that a/b is a rational number. Let p, q be relatively prime positive integers such that $a/b = p/q$. Then there exists uniquely a zero-sum $\{a, -b\}$-sequence which has the positively $(a+b)$-balanced prefix property. The length of such a sequence is $p+q$.*

(2) *Let b be a non-zero real number. Then $\langle 0 \rangle$ is the unique zero-sum $\{0,b\}$-sequence which has the positively $|b|$-balanced prefix property.*

Proof. (1) By Lemma 2.4 (7), it is enough to show the statement in the case that $a = p$ and $b = q$. We first show that if a $\{p,-q\}$-sequence s has the positively $(p+q)$-balanced prefix property, then s is uniquely defined.

Since $s(0)$ must be positive, we have $s(0) = p$.

Suppose $s(i)$ is defined for $i < n$.

If $\sum_{i=0}^{n-1} s(i) \geq q$ then $s(n)$ cannot be p because $\sum_{i=0}^{n} s(i)$ will be $p+q$ or more. Therefore, $s(n)$ must be $-q$.

If $\sum_{i=0}^{n-1} s(i) < q$, then $s(n)$ cannot be $-q$ because $\sum_{i=0}^{n} s(i)$ will be negative. Therefore, $s(n)$ must be p.

Let s be a sequence defined as follows:

(i) $s(0) = p$.

(ii) If $\sum_{i=0}^{n-1} s(i) \geq q$ then $s(n) = -q$. Otherwise, $s(n) = p$.

By induction, we can see that $0 \leq \sum_{i=0}^{k} s(i) < p+q$ for any k. Also, we can see that p appears q times in s eventually. Let j be the index such that $s(j)$ is the q'th p in s. If $k < j$, then $\sum_{i=0}^{k} s(i) = lp - l'q$ with $l < q$. Since p and q are relatively prime, $lp - l'q$ cannot be zero. Hence, $\sum_{i=0}^{k} s(i) > 0$ for $k < j$. We also have $\sum_{i=0}^{j} s(i) > 0$ because $s(j) = p > 0$. $0 < \sum_{i=0}^{j} s(i) = qp - l''q = (p - l'')q$ for some integer l''. $s|[p+q]$ is a zero-sum $\{p,-q\}$-sequence with the positively $(p+q)$-balanced prefix property.

(2) $\langle 0 \rangle$ is a zero-sum $\{0,b\}$-sequence which has the positively $|b|$-balanced prefix property by definition.

Let s be a zero-sum $\{0,b\}$-sequence which has the positively $|b|$-balanced prefix property. Let k be the length of s. Since s is $|b|$-balanced by Lemma 2.4 (4), $s(i) \neq b$ for $i < k$. Hence, $s(i) = 0$ for any $i < k$. If $k > 1$ then the prefix of s of length 1 violates the definition of the positively $|b|$-balanced prefix property. □ □

Proposition 2.6. *Let a and b be positive real numbers such that a/b is a rational number. Let p, q be relatively prime positive integers such that $a/b = p/q$.*

Let s be a $(a+b)$-balanced zero-sum $\{a,-b\}$-sequence. Then s is balanced: Suppose $a \geq b$ and let k be an integer with $a - kb \geq 0$ and $a - (k+1)b < 0$. Then any run length of a in s is 1 and any rotational run length of $-b$ in s is k or $k+1$. If $a = kb$, then any rotational run length of $-b$ in s is k.

If $a < b$ then similar statement holds by considering $-1 \cdot s$.

Proof. Suppose $\langle a \rangle \langle -b \rangle^n \langle a \rangle$ is a subsequence in a rotation of s. Then $-a - b < -nb$ since s is $(a+b)$-balanced. Hence, $(n-1)b < a$. Thus, $n - 1 \leq k$.

So, $n \leq k+1$. Also, $2a - nb < a+b$. Hence, $a - (n+1)b < 0$. Thus, $k+1 \leq n+1$. So, $k \leq n$.

If $a = kb$ then n cannot be $k+1$, because $\sum\langle -b\rangle^n = -nb = -(k+1)b = -(a+b)$ violates the definition of an $(a+b)$-balanced sequence. □

Definition 2.7. Let s be an $\{a, -b\}$-sequence. Assume that any entry a is always followed by $-b$, or any entry $-b$ is always preceded by a in s. Replace every consecutive subsequence of s of the form $\langle a, -b\rangle$ by $\langle a-b\rangle$. Let s' be the resulting sequence. We call s' a *reduct* of s.

Proposition 2.8. *Let s be an $(a+b)$-balanced zero-sum $\{a, -b\}$-sequence with $a \neq b$.*

(1) *Assume that $a > b$ and any entry a is always followed by $-b$ in s. Then a reduct of s is an a-balanced zero-sum $\{a-b, -b\}$-sequence.*
(2) *Assume that $a < b$ and any entry $-b$ is always preceded by a in s. Then a reduct of s is an b-balanced zero-sum $\{a, a-b\}$-sequence.*

Proof. Let s' be the reduct of s.

(1) Let u be a rotational consecutive subsequence of s'.

Case $\sum u \geq 0$. In this case, u must have $a-b$ as an entry. So, we can write $u = u'\langle a-b\rangle\langle -b\rangle^k$. We have $\sum u \leq \sum u' + a - b$. Since s is $(a+b)$-balanced, we have $\sum u' + a < a+b$. Hence, $\sum u' + a - b < a$. Therefore, $\sum u < a$.

Case $\sum u < 0$. Let v be the rotational complement of u in s'. Then $0 < |\sum u| = -\sum u = \sum v$ since s' is a zero-sum sequence. We have $\sum v < a$ by the case above.

(2) Let u be a rotational consecutive subsequence of s'. Case $\sum u \leq 0$. In this case, u must have $a-b$ as an entry. So, we can write $u = \langle a\rangle^k\langle a-b\rangle u'$. We have $a - b + \sum u' \leq \sum u$. Since s is $(a+b)$-balanced, we have $-a-b < -b + \sum u'$. Hence, $-b < a - b + \sum u'$. Therefore, $-b < \sum u$.

Case $\sum u > 0$. Let v be the rotational complement of u in s'. Then $0 > -\sum u = \sum v$ since s' is a zero-sum sequence. We have $\sum v > -b$ by the case above. Therefore, $\sum u < b$. □

Definition 2.9. Let s be an $\{a, -b\}$-sequence. Replace every subsequence of s of the form $\langle a\rangle$ by $\langle a+b, -b\rangle$. Let s' be the resulting sequence. We call s' a *positive expansion* of s.

Replace every subsequence of s of the form $\langle -b\rangle$ by $\langle a, -(a+b)\rangle$. Let s'' be the resulting sequence. We call s'' a *negative expansion* of s.

Proposition 2.10. *Let s be an $(a+b)$-balanced zero-sum $\{a, -b\}$-sequence s.*

(1) *An positive expansion of s is an $(a+2b)$-balanced zero-sum $\{a+b, -b\}$-sequence.*

(2) *An negative expansion of s is a $(2a+b)$-balanced zero-sum $\{a, -(a+b)\}$-sequence.*

Proof. (1) Let s' be the positive expansion of s. Let u be a rotational consecutive subsequence of s'.

Case $\sum u \geq 0$. In this case, u must have $a+b$ as an entry. So, we can write $u = u'\langle a+b\rangle\langle -b\rangle^k$. We have $\sum u \leq \sum u' + a + b$. Every entry of u' of the form $a+b$ is followed by $-b$. Hence, $\sum u' + a$ is a sum of a rotational consecutive subsequence of s. Since s is $(a+b)$-balanced, we have $\sum u' + a < a+b$. Therefore, $\sum u' + a + b < a + 2b$.

Case $\sum u < 0$. Let v be the rotational complement of u in s'. Then $0 < |\sum u| = -\sum u = \sum v$ since s' is a zero-sum sequence. We have $\sum v < a+2b$ by the case above.

(2) Let s'' be the negative expansion of s. Let u be a rotational consecutive subsequence of s''.

Case $\sum u \leq 0$. In this case, u must have $-(a+b)$ as an entry. So, we can write $u = \langle a\rangle^k\langle -(a+b)\rangle u'$. We have $-a-b+\sum u' \leq \sum u$. Every entry of u' of the form $-(a+b)$ is preceded by a. Hence, $-b+\sum u'$ is a sum of a rotational consecutive subsequence of s. Since s is $(a+b)$-balanced, we have $-a-b < -b+\sum u'$. Therefore, $-2a-b < -a-b+\sum u'$.

Case $\sum u > 0$. Let v be the rotational complement of u in s'. Then $0 > -\sum u = \sum v$ since s' is a zero-sum sequence. We have $\sum v > -2a-b$ by the case above. Therefore, $\sum u < 2a+b$. $\square$

The following three lemmas are straightforward.

Lemma 2.11. *Let s be a zero-sum $\{a, -b\}$-sequence with $a \neq b$. Suppose that a reduct s' of s exists.*

(1) *If $a > b$ then s is a positive expansion of s'.*
(2) *If $a < b$ then s is a negative expansion of s'.*

Lemma 2.12. *Let s be a zero-sum $\{a, -b\}$-sequence. Suppose s has the positively $(a+b)$-balanced prefix property.*

(1) *A positive expansion of s has the positively $(a+2b)$-balanced prefix property.*
(2) *A negative expansion of s has the positively $(2a+b)$-balanced prefix property.*

Lemma 2.13. (1) *Taking a rotation and taking a positive expansion commute.*
(2) *Taking a rotation and taking a negative expansion commute.*

Theorem 2.14. *Suppose $a/b = p/q$ where p and q are relatively prime positive integers. Let s be a zero-sum $\{a, -b\}$-sequence with the positively*

$(a+b)$-balanced prefix property. All $(a+b)$-balanced zero-sum $\{a,-b\}$-sequences are the rotations of s^k for some $k \geq 1$.

Proof. We can assume that a and b are relatively prime positive integers. Let u be an $(a+b)$-balanced $\{a,-b\}$-sequence. If $a=b$, then $a=b=1$. In this case, $u=\langle 1,-1\rangle^k$ or $\langle -1,1\rangle^k$ for some k by Proposition 2.6. u is a rotation of $\langle 1,-1\rangle^k$ anyway.

Let u_0 be u. Some rotation of u_0 has a reduct by Proposition 2.6. Let u_1 be a reduct of some rotation of u_0. u_1 is an $(a'+b')$-balanced zero-sum $\{a',-b'\}$-sequence with a', b' positive real numbers.

Iterating this process, eventually we get an $(x+1)$-balanced Y-sequence u_l with $Y=\{x,-1\}$ or $\{-x,1\}$ for some positive integer x. Then by Proposition 2.6, u_l is a rotation of $(\langle 1\rangle^x\langle -x\rangle)^k$ or $(\langle x\rangle\langle -1\rangle^x)^k$ for some k. $\langle 1\rangle^x\langle -x\rangle$ and $\langle x\rangle\langle -1\rangle^x$ have the positively $(1+x)$-balanced prefix property.

We can recover $u=u_0$ from u_l by a combination of taking rotations and taking positive or negative expansions. By lemmas above, we can see that u is a rotation of s^k. □

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