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Global and local structures of oscillatory bifurcation diagrams

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1 Introduction

We first consider the following example of nonlinear eigenvalue problems

$$-u''(t) = \lambda\left(u(t) + g(u(t))\right), \quad t \in I =: (-1, 1), \tag{1.1}$$

$$u(t) > 0, \quad t \in I, \tag{1.2}$$

$$u(-1) = u(1) = 0. \tag{1.3}$$

Here, $g(u) := \frac{1}{2}\sin^k u(t)$, $k \geq 1$ is a given integer, and $\lambda > 0$ is a bifurcation parameter. We know from [15] that the solution set of (1.1)–(1.3) consists of the set

$$Q := \{(\lambda(k, \alpha), u_\alpha) |\ \text{sol. of (1.1)–(1.3) with } \|u_\alpha\|_\infty = \alpha\} \subset \mathbb{R}_+ \times C^2(\bar{I}).$$

Indeed, in this case, for any given $\alpha > 0$, there exists a unique solution pair (λ, u_α) of (1.1)–(1.3) with $\alpha = \|u_\alpha\|_\infty$ and λ is parameterized by α. So we write as $\lambda = \lambda(k, \alpha)$. If we consider the asymptotic behavior of $\lambda(k, \alpha)$ as $\alpha \to \infty$, then it seems clear that

$$\lambda(k, \alpha) \to \frac{\pi^2}{4}. \tag{1.4}$$

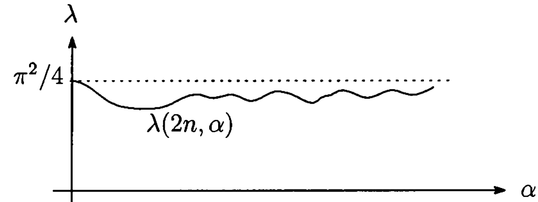

Fig.1 ([19]) **bifurcation curve for** $\lambda(2n, \alpha)$

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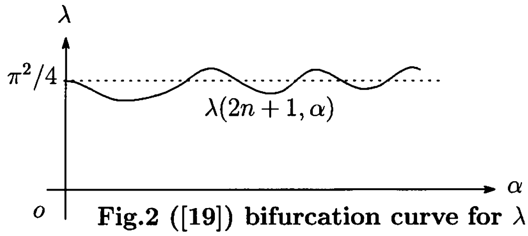

Fig.2 ([19]) bifurcation curve for $\lambda(2n+1,\alpha)$

So it is natural to expect that the rate of convergence of $\lambda(2n,\alpha)$ to $\pi^2/4$ as $\alpha \to \infty$ is the same as that of $\lambda(2n+1,\alpha)$. However, we will find that the following formula holds.

$$|\lambda(2n_1+1,\alpha)-\pi^2/4| \ll |\lambda(2n_2,\alpha)-\pi^2/4| \to 0, \tag{1.5}$$

where $n_1 \geq 1$ and $n_2 \geq 1$ are arbitrary given integers. To show (1.5), we calculate the asymptotic behavior of $\lambda(k,\alpha)$ precisely.

Theorem 1.1 ([19]). (i) *Let $k = 2n$ $(n \geq 1)$. Then as $\alpha \to \infty$*

$$\begin{aligned}\lambda(2n,\alpha) &= \frac{\pi^2}{4} - \frac{\pi}{2^{2n+1}\alpha}\binom{2n}{n} - \frac{\pi^{3/2}}{2^{2n+1}\alpha^{3/2}}\sum_{r=0}^{n-1}(-1)^{n-r}\binom{2n}{r} \\ &\quad \times \frac{1}{\sqrt{n-r}}\sin\left((2n-2r)\alpha+\frac{\pi}{4}\right)+O(\alpha^{-2}).\end{aligned} \tag{1.6}$$

(ii) *Let $k = 2n+1$ $(n \geq 0)$. Then as $\alpha \to \infty$*

$$\begin{aligned}\lambda(2n+1,\alpha) &= \frac{\pi^2}{4} - \frac{\pi^{3/2}}{2^{2n+1}\alpha^{3/2}}\sum_{r=0}^{n}(-1)^{n+r}\binom{2n+1}{r} \\ &\quad \times \sqrt{\frac{1}{2(2n-2r+1)}}\sin\left((2n-2r+1)\alpha-\frac{1}{4}\pi\right) \\ &\quad + O(\alpha^{-2}).\end{aligned} \tag{1.7}$$

We consider why this kind of difference between (1.6) and (1.7) occurs in the next section.

2 General results

The purpose in this section is to show the reason why such kind of difference between (1.6) and (1.7) occurs. We consider (1.1)–(1.3) and we assume that $g(u)$ satisfies the following conditions.

(A.1) $g(u) \in C^1(\mathbb{R})$ and $u + g(u) > 0$ for $u > 0$.

(A.2) $g(u + 2\pi) = g(u)$ for $u \in \mathbb{R}$.

Then we know from [15] that there exists a unique solution pair (λ, u_α) of (1.1)–(1.3) with $\alpha = \|u_\alpha\|_\infty$ for any given $\alpha > 0$ under the condition (A.1). Besides, λ is parameterized by α as $\lambda(\alpha)$. Moreover, $\lambda(\alpha)$ is a continuous function of $\alpha > 0$. Then it is convenient for us to write $\lambda = \lambda(g, \alpha)$, since λ also depends on g. We note that the Fourier series of g converges uniformly to g. under the conditions (A.1) and (A.2).

Now, we introduce the notion of (OP).

(OP) *$\lambda(g, \alpha) \to \pi^2/4$ as $\alpha \to \infty$, and it intersects the line $\lambda = \pi^2/4$ infinitely many times for $\alpha \gg 1$.*

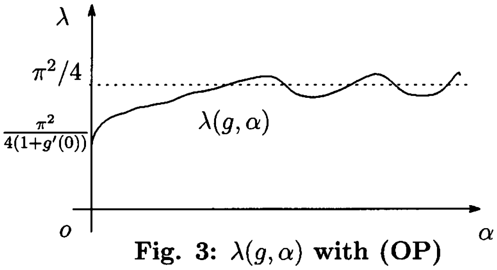

Fig. 3: $\lambda(g, \alpha)$ with (OP)

Since $g(u)$ is bounded in $\mathbb{R}$ by (A.2), it is clear that $\lambda(g, \alpha) \to \pi^2/4$ as $\alpha \to \infty$. Therefore, the essential point is to find the condition whether $\lambda(g, \alpha)$ intersects the line $\lambda = \pi^2/4$ infinitely many times for $\alpha \gg 1$. By Theorem 1.1, if $g(u) = \frac{1}{2}\sin^{2n+1}(u)$, then (OP) holds. On the other hand, if $g(u) = \frac{1}{2}\sin^{2n}(u)$, then (OP) does not hold. The purpose here is to find a simple condition, from which we understand whether $\lambda(g, \alpha)$ satisfies (OP) or not immediately.

Now we state our main results.

Theorem 2.1 ([20]). *Assume that $g(u)$ satisfies (A.1)–(A.2). Then as $\alpha \to \infty$,*

$$\lambda(g, \alpha) = \frac{\pi^2}{4} - \frac{\pi a_0}{2\alpha} - \frac{1}{\alpha}\sqrt{\frac{\pi}{2\alpha}} \sum_{n=1}^{\infty} \frac{c_n}{n^{3/2}} + O(\alpha^{-2}), \tag{2.1}$$

where

$$a_0 := \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta) d\theta, \tag{2.2}$$

$$c_n := \int_{-\pi}^{\pi} g'(\theta) \cos\left(n(\theta - \alpha) + \frac{3}{4}\pi\right) d\theta, \quad (n \in \mathbb{N}). \tag{2.3}$$

As a corollary of Theorem 2.1, we get an interesting result for the asymptotic behavior of $\lambda(g, \alpha)$.

Corollary 2.2 ([20]). *Assume that $g(u)$ satisfies (A.1)-(A.2). If $a_0 \neq 0$, then $\lambda(g,\alpha)$ does not satisfy (OP).*

By Corollary 2.2, we understand immediately the reason that, in the case of $g(u) = \frac{1}{2}\sin^{2n+1}(u)$, (OP) holds, and in the case of $g(u) = \frac{1}{2}\sin^{2n}(u)$, (OP) does not hold.

The method to study the local behavior of $\lambda(g, \alpha)$ has been already obtained in [17, 18], because the time-map method and Taylor expansion work very well to study the local structure of $\lambda(g, \alpha)$.

Theorem 2.3 ([20]). *Assume (A.1)-(A.2). Furthermore, assume that $g \in C^2$ near $u = 0$.*
(i) Assume that $g(0) \neq 0$. Then as $\alpha \to 0$,

$$\lambda(g,\alpha) = \frac{2\alpha}{g(0)}\left\{1 + A_1\alpha + A_2\alpha^2 + o(\alpha^2)\right\}, \tag{2.4}$$

where

$$A_1 = -\frac{5}{6g(0)}(1 + g'(0)), \quad A_2 = \frac{32}{45}\frac{(1+g'(0))^2}{g(0)^2} - \frac{11}{30}\frac{g''(0)}{g(0)}. \tag{2.5}$$

(ii) Assume that $g(0) = 0$ and $g'(0) > -1$. Then as $\alpha \to 0$,

$$\lambda(g,\alpha) = \frac{1}{1+g'(0)}\left(\frac{\pi^2}{4} - \frac{\pi g''(0)}{3(1+g'(0))}\alpha + o(\alpha)\right). \tag{2.6}$$

3 Global behavior of $\lambda(g, \alpha)$

The proof of Theorem 2.1 is given by the combination of time-map method, Fourier expansion and the asymptotic formulas for some special functions. The proof is given by several steps. In this section, let $\alpha \gg 1$. For simplicity, we write $\lambda = \lambda(g, \alpha)$. Moreover, we denote by C the various positive constants independent of α. Let

$$G(u) := \int_0^u g(s)ds. \tag{3.1}$$

We know that if $(u_\alpha, \lambda) \in C^2(\bar{I}) \times \mathbb{R}_+$ satisfies (1.1)–(1.3), then the following properties hold.

$$u_\alpha(t) = u_\alpha(-t), \quad 0 \le t \le 1, \tag{3.2}$$

$$u_\alpha(0) = \max_{-1\le t\le 1} u_\alpha(t) = \alpha, \tag{3.3}$$

$$u'_\alpha(t) > 0, \quad -1 < t < 0. \tag{3.4}$$

Step 1. The well known time-map (3.7) below is constructed as follows. By (1.1),

$$\{u_\alpha''(t) + \lambda\left(u_\alpha(t) + g(u_\alpha(t))\right)\} u_\alpha'(t) = 0.$$

By this and putting $t = 0$, we have

$$\frac{1}{2}u_\alpha'(t)^2 + \lambda\left(\frac{1}{2}u_\alpha(t)^2 + G(u_\alpha(t))\right) = \text{constant} = \lambda\left(\frac{1}{2}\alpha^2 + G(\alpha)\right).$$

By this and (3.4), for $-1 \le t \le 0$, we have

$$u_\alpha'(t) = \sqrt{\lambda}\sqrt{\alpha^2 - u_\alpha(t)^2 + 2(G(\alpha) - G(u_\alpha(t)))}. \tag{3.5}$$

It is clear from (A.2) that $|g(u)| \le C$ for $u \in \mathbb{R}$. Then for $0 \le s \le 1$,

$$\left|\frac{G(\alpha) - G(\alpha s)}{\alpha^2(1-s^2)}\right| = \left|\frac{\int_{\alpha s}^{\alpha} g(t)dt}{\alpha^2(1-s^2)}\right| \le \frac{C\alpha(1-s)}{\alpha^2(1-s^2)} \le C\alpha^{-1}. \tag{3.6}$$

By (3.5), (3.6), putting $s := u_\alpha(t)/\alpha$ and Taylor expansion, we have

$$\begin{aligned} \sqrt{\lambda} &= \int_{-1}^{0} \frac{u_\alpha'(t)}{\sqrt{\alpha^2 - u_\alpha(t)^2 + 2(G(\alpha) - G(u_\alpha(t)))}}dt \qquad (3.7)\\ &= \int_0^1 \frac{1}{\sqrt{1 - s^2 + 2(G(\alpha) - G(\alpha s))/\alpha^2}}ds \\ &= \int_0^1 \frac{1}{\sqrt{1-s^2}}\frac{1}{\sqrt{1 + 2(G(\alpha) - G(\alpha s))/(\alpha^2(1-s^2))}}ds \\ &= \int_0^1 \frac{1}{\sqrt{1-s^2}}\left\{1 - \frac{G(\alpha) - G(\alpha s)}{\alpha^2(1-s^2)} + O(\alpha^{-2})\right\}ds \\ &:= \frac{\pi}{2} - \frac{1}{\alpha^2}K(\alpha) + O(\alpha^{-2}). \end{aligned}$$

Here,

$$K(\alpha) := \int_0^1 \frac{G(\alpha) - G(\alpha s)}{(1-s^2)^{3/2}}ds. \tag{3.8}$$

Step 2. We calculate $K(\alpha)$ by the asymptotic formulas for several special functions as $\alpha \to \infty$. We know that under the conditions (A.1)–(A.2),

$$g(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \tag{3.9}$$

holds for $x \in \mathbb{R}$ and the right hand side of (3.9) converges to $g(x)$ uniformly on $\mathbb{R}$. Here,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\cos n\theta d\theta = -\frac{1}{n\pi}\int_{-\pi}^{\pi} g'(\theta)\sin n\theta d\theta \qquad (3.10)\\ &:= -\frac{1}{n}\tilde{a}_n \quad (n \in \mathbb{N}_0), \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\sin n\theta d\theta = \frac{1}{n\pi}\int_{-\pi}^{\pi} g'(\theta)\cos n\theta d\theta \qquad (3.11)\\ &:= \frac{1}{n}\tilde{b}_n \quad (n \in \mathbb{N}). \end{aligned}$$

Step 3.

Lemma 3.1 ([20]). *As $\alpha \to \infty$,*

$$K(\alpha) = \frac{1}{2}a_0\alpha + \frac{1}{\pi}\sqrt{\frac{\pi\alpha}{2}}\sum_{n=1}^{\infty}\frac{c_n}{n^{3/2}} + O(\alpha^{-1/2}). \qquad (3.12)$$

Proof. We put $s = \sin\theta$ in (3.8). Then we obtain

$$K(\alpha) = \alpha\int_0^{\pi/2} g(\alpha\sin\theta)\sin\theta d\theta. \qquad (3.13)$$

We use here the integration by parts and l'Hôpital's rule. For $n = \mathbb{N}$, let

$$U_n := \int_0^{\pi/2}\cos(n\alpha\sin\theta)\sin\theta d\theta, \qquad (3.14)$$

$$V_n := \int_0^{\pi/2}\sin(n\alpha\sin\theta)\sin\theta d\theta. \qquad (3.15)$$

By (3.13)–(3.15),

$$\begin{aligned} K(\alpha) &= \alpha\int_0^{\pi/2} g(\alpha\sin\theta)\sin\theta d\theta \qquad (3.16)\\ &= \alpha\int_0^{\pi/2}\left\{\frac{1}{2}a_0 + \sum_{n=1}^{\infty}a_n\cos(n\alpha\sin\theta)\right.\\ &\qquad\left. + \sum_{n=1}^{\infty}b_n\sin(n\alpha\sin\theta)\right\}\sin\theta d\theta\\ &= \alpha\left\{\frac{1}{2}a_0 + \sum_{n=1}^{\infty}a_n\int_0^{\pi/2}\cos(n\alpha\sin\theta)\sin\theta d\theta\right.\\ &\qquad\left. + \sum_{n=1}^{\infty}b_n\int_0^{\pi/2}\sin(n\alpha\sin\theta)\sin\theta d\theta\right\}\\ &= \alpha\left\{\frac{1}{2}a_0 - \sum_{n=1}^{\infty}\frac{1}{n}\tilde{a}_n U_n + \sum_{n=1}^{\infty}\frac{1}{n}\tilde{b}_n V_n\right\}. \end{aligned}$$

Put $\theta = \pi/2 - \phi$ in (3.22). Then by (3.9)–(3.12), (3.14), (3.15) and [9, p.425], we obtain

$$\begin{aligned} U_n &= \int_0^{\pi/2}\cos(n\alpha\cos\phi)\cos\phi\, d\phi \qquad (3.17)\\ &= \frac{\pi}{4}(\mathbf{E}_1(n\alpha) - \mathbf{E}_{-1}(n\alpha))\\ &= \frac{\pi}{4}(-Y_1(n\alpha) + Y_{-1}(n\alpha) + O((n\alpha)^{-2})\\ &= \frac{\pi}{4}\left(-\sqrt{\frac{2}{n\pi\alpha}}\sin\left(n\alpha - \frac{3}{4}\pi\right) + \sqrt{\frac{2}{n\pi\alpha}}\sin\left(n\alpha + \frac{1}{4}\pi\right)\right) \end{aligned}$$

$$
\begin{aligned}
& + O((n\alpha)^{-3/2}) \\
= & -\sqrt{\frac{\pi}{2n\alpha}} \sin\left(n\alpha - \frac{3}{4}\pi\right) + O((n\alpha)^{-3/2}).
\end{aligned}
$$

Here, $\mathbf{E}_\nu(z)$ are Weber functions and $Y_\nu(z)$ are Neumann functions. Moreover,

$$
\begin{aligned}
V_n &= \int_0^{\pi/2} \sin(n\alpha\cos\phi)\cos\phi\, d\phi \qquad (3.18) \\
&= \frac{\pi}{4}\{\mathbf{J}_1(n\alpha) - \mathbf{J}_{-1}(n\alpha)\} \\
&= \frac{\pi}{4}\{J_1(n\alpha) - J_{-1}(n\alpha)\} \\
&= \frac{\pi}{4}\left\{\sqrt{\frac{2}{n\pi\alpha}}\cos\left(n\alpha - \frac{3}{4}\pi\right) - \sqrt{\frac{2}{n\pi\alpha}}\cos\left(n\alpha + \frac{1}{4}\pi\right)\right\} \\
&\quad + O((n\alpha)^{-3/2}) \\
&= \sqrt{\frac{\pi}{2n\alpha}}\cos\left(n\alpha - \frac{3}{4}\pi\right) + O((n\alpha)^{-3/2}).
\end{aligned}
$$

Here, $\mathbf{J}_\nu(z)$ are Anger functions and $J_\nu(z)$ are Bessel functions) By (3.14)–(3.18), we obtain

$$
\begin{aligned}
K(\alpha) &= \alpha\left\{\frac{1}{2}a_0 + \sqrt{\frac{\pi}{2\alpha}}\sum_{n=1}^{\infty}\left(\tilde{a}_n \sin\left(n\alpha - \frac{3}{4}\pi\right)\right.\right. \\
&\qquad \left.\left. + \tilde{b}_n\cos\left(n\alpha - \frac{3}{4}\pi\right)\right)\frac{1}{n^{3/2}}\right\} \\
&\quad + O\left(\alpha^{-1/2}\sum_{n=1}^{\infty}\frac{1}{n^{5/2}}\right) \\
&= \alpha\left\{\frac{1}{2}a_0 + \frac{1}{\pi}\sqrt{\frac{\pi}{2\alpha}}\sum_{n=1}^{\infty}\frac{c_n}{n^{3/2}}\right\} + O(\alpha^{-1/2}).
\end{aligned}
$$

Thus the proof is complete. ▮

By (3.7) and Lemma 3.1, we obtain Theorem 2.1. ▮

We introduce the Special functions and their asymptotic behavior here. For $z \gg 1$, we have (cf. [9, p. 929, p. 958])

$$
\begin{aligned}
J_1(z) &= \sqrt{\frac{2}{\pi z}}\left\{[1 + R_1]\cos\left(z - \frac{3}{4}\pi\right)\right. \\
&\qquad \left. - \left[\frac{1}{2z}\frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} + R_2\right]\sin\left(z - \frac{3}{4}\pi\right)\right\}, \qquad (3.19)
\end{aligned}
$$

$$
J_{-1}(z) = \sqrt{\frac{2}{\pi z}}\left\{[1 + R_1]\cos\left(z + \frac{1}{4}\pi\right)\right.
$$

$$
\begin{aligned}
& - \left[\frac{1}{2z} \frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} + R_2 \right] \sin\left(z + \frac{1}{4}\pi \right) \Bigg\}, && (3.20) \\
Y_1(z) &= \sqrt{\frac{2}{\pi z}} \left\{ [1 + R_1] \sin\left(z - \frac{3}{4}\pi \right) \right. \\
& \left. + \left[\frac{1}{2z} \frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} + R_2 \right] \cos\left(z - \frac{3}{4}\pi \right) \right\}, && (3.21) \\
Y_{-1}(z) &= \sqrt{\frac{2}{\pi z}} \left\{ [1 + R_1] \sin\left(z + \frac{1}{4}\pi \right) \right. \\
& \left. + \left[\frac{1}{2z} \frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} + R_2 \right] \cos\left(z + \frac{1}{4}\pi \right) \right\}, && (3.22)
\end{aligned}
$$

where

$$
|R_1| < \left| \frac{\Gamma\left(\frac{7}{2}\right)}{8\Gamma\left(-\frac{1}{2}\right) z^2} \right|, \quad |R_2| < \left| \frac{\Gamma\left(\frac{9}{2}\right)}{48\Gamma\left(-\frac{3}{2}\right) z^3} \right|, \tag{3.23}
$$

$$
\begin{aligned}
\mathbf{J}_{\pm 1}(z) &= J_{\pm 1}(z), && (3.24) \\
\mathbf{E}_{\pm 1}(z) &= -Y_{\pm 1}(z) \mp \frac{2}{\pi z^2} + O(z^{-4}). && (3.25)
\end{aligned}
$$

4 Special case

Finally, we are interested in the case $g(u) = \sin\sqrt{u}$. In this case, the equation (1.1)–(1.3) has been proposed in Cheng [5] as a model problem which has arbitrary many solutions near $\lambda = \pi^2/4$.

Theorem 4.0.([5]) *Let $g_1(u) = \sin\sqrt{u}$ $(u \geq 0)$. Then for any integer $r \geq 1$, there is $\delta > 0$ such that if $\lambda \in (\lambda_1 - \delta, \lambda_1 + \delta)$, then (1.1)–(1.3) has at least r distinct solutions.*

Theorem 4.0 gives us the imformation about the solution set of (1.1)–(1.3), and we expect that $\lambda(g_1, \alpha)$ satisfies (OP). The purpose here is to prove the expectation above is valid.

Theorem 4.1 ([21]). *Let $g(u) = g_1(u) = \sin\sqrt{u}$. Then as $\alpha \to \infty$,*

$$
\lambda(g_1, \alpha) = \frac{\pi^2}{4} - \pi^{3/2} \alpha^{-5/4} \cos\left(\sqrt{\alpha} - \frac{3}{4}\pi \right) + o(\alpha^{-5/4}). \tag{4.1}
$$

We next give the asymptotic behavor of $\lambda(g_1, \alpha)$ as $\alpha \to 0$.

Theorem 4.2 ([21]). *Let $g(u) = g_1(u) = \sin\sqrt{u}$.*
(i) As $\alpha \to 0$, the following asymptotic formula for $\lambda(g_1, \alpha)$ holds:

$$
\lambda(g_1, \alpha) = \frac{3}{4} C_1^2 \sqrt{\alpha} + \frac{3}{2} C_1 C_2 \alpha + O(\alpha^{3/2}), \tag{4.2}
$$

where

$$C_1 := \int_0^1 \frac{1}{\sqrt{1-s^{3/2}}} ds, \quad C_2 := -\frac{3}{8}\int_0^1 \frac{1-s^2}{\sqrt{1-s^{3/2}}} ds. \tag{4.3}$$

(ii) *Let* v_0 *be a unique classical solution of the following equation*

$$-v_0''(t) = \frac{3}{4} C_1^2 \sqrt{v_0(t)}, \quad t \in I, \tag{4.4}$$

$$v_0(t) > 0, \quad t \in I, \tag{4.5}$$

$$v_0(-1) = v_0(1) = 0. \tag{4.6}$$

Furthermore, let $v_\alpha(t) := u_\alpha(t)/\alpha$. *Then* $v_\alpha \to v_0$ *in* $C^2(I)$ *as* $\alpha \to 0$.

The proofs also depend on time-map method.

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