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No Suslin trees but a non-special Aronszajn tree exists by a side condition method

(compact version)

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Abtract

Let us fix a weakly Suslin tree T^* that is upward-absolutely Aronszajn. Let us assume $2^{\omega} = \omega_1$ and $2^{\omega_1} = \omega_2$. We construct an Aspero-Mota type iterated forcing $\langle P_\alpha \mid \alpha < \omega_2\rangle$ and take the direct limit $P^*_{\omega_2}$ of the P_αs. In the generic extensions $V^{P^*_{\omega_2}}$, we have (1) $2^{\omega} = \omega_2$, (2) every Aronszajn tree gets an uncountable antichain and so no Suslin trees exist, while (3) T^* remains weakly Suslin and Aronszajn. In particular, T^* has no specializing maps. The idea of a weakly Suslin tree that is upward-absolutely Aronszajn belongs to a work of S. Shelah. Combinatorics with Aronszajn trees, say, via $R_{1,\aleph_1}$ is due to T. Yorioka. An iterated forcing method that uses symmetric systems and markers is due to Aspero-Mota. It appears that the construction in this paper is sensitive to the length ω_2.

Introduction

Definition. Let T^* be an ω_1-tree a la Kunen. Let $\theta \geq \omega_2$ be a regular cardinal. We say T^* is *weakly Suslin* witnessed at θ, if

$$\{N \in [H_\theta]^{\omega} \mid \forall x \in T^*_{N\cap\omega_1}\ x \text{ pushdown } N\}$$

is stationary in $[H_\theta]^{\omega}$. Here x pushdown N abbreviates that for any $A \in N$, if $x \in A$, then there exists $y <_{T^*} x$ such that $y \in A$. We say T^* is *weakly Suslin*, if there exists a witness θ for T^*.

Proposition. (1) If T^* is weakly Suslin witnessed at θ, then for all regular cardinals $\lambda \geq \theta$,

$$\{N \in [H_\lambda]^{\omega} \mid \forall x \in T^*_{N\cap\omega_1}\ x \text{ pushdown } N\}$$

are stationary in $[H_\lambda]^{\omega}$.

(2) If T^* is a Suslin tree, then T^* is weakly Suslin witnessed at $\theta = \omega_2$ (with even a club) and (not yet upward-absolutely) Aronszajn.

(3) If T^* is weakly Suslin and Aronszajn, then T^* is an Aronszajn tree with no specializing maps f. Namely, $f : T^* \longrightarrow \omega$ such that whenever $x <_{T^*} y$, then $f(x) \neq f(y)$.

Lemma. (S. Shelah) Let T^* be a Suslin tree. Then there exists a proper poset P consisting of finite conditions such that $|P| = \omega_1$ and that P forces $\dot{f}$ and $\dot{h}$ such that

- $\dot{f} : \dot{C} \longrightarrow \omega_1$ such that the domin $\dot{C}$ is a club in ω_1 and for all $i, j \in \dot{C}$, if $i < j$, then $i \leq \dot{f}(i) < j$,
- $\dot{h} : T^*\lceil\text{range}(\dot{f}) \longrightarrow \omega$ such that if $x <_{T^*} y$, then $\dot{h}(x) \neq \dot{h}(y)$.

Then, in the generic extension, it holds that GCH, if we start with GCH, and that T^* remains weakly Suslin witnessed at ω_2 and upward-absolutely Aronszajn.

This sets our ground model V to start with T^*. We force ω_2-times with an Aspero-Mota type iteration over V. We iteratively add uncountable antichains to all relevant Aronzajn trees, while preserving T^* to be weakly Suslin witnessed at ω_2 and Aronszajn. In particular, we have a consistency of no Suslin trees exist yet a non-special Aronszajn tree exists with $2^{\omega} = \omega_2$, a large continuum. However, we see no generalizations of this construction to longer iterations, say, ω_3.

Question. Is it possible to form a longer Aspero-Mota type iterated forcing to get a larger continuum with the current combinatorial context ?

The finite symmetric systems P_{FAM}

We use symmetric systems of Aspero-Mota.

Definition. ($2^{\omega_1} = \omega_2$) Let $\Phi : \omega_2 \longrightarrow H_{\omega_2}$ such that for each $x \in H_{\omega_2}$,

$$\{i < \omega_2 \mid \Phi(i) = x\} \nearrow \omega_2.$$

We form a relational structure (i.e. a first-order structure with no functions)

$$(H_{\omega_2}, \in, \Phi).$$

Here, $\in$ denotes the binary relation $\in$ on the universe H_{ω_2}. We treat Φ as a single-valued partial binary relation, namely

$$(H_{\omega_2}, \in, \Phi) \models \text{“}\forall \alpha : \omega_2 \,\exists !\, y \text{ s.t. } \alpha \,\Phi\, y\text{”}.$$

Proposition. Let $X = \big(X, \in \cap (X \times X), \Phi \cap (X \times X)\big)$ be a countable elementary substructure of $(H_{\omega_2}, \in, \Phi)$. Then $X = \{\Phi(i) \mid i \in X \cap \omega_2\}$, that is denoted by $\Phi[X \cap \omega_2]$. Hence

$$X = \Phi[X \cap \omega_2].$$

In particular, if $0 \neq \alpha \in X \cap \omega_2$, there exists $\beta \in X \cap \omega_2$ such that $\beta < \omega_2$ is the least with $\Phi(\beta) : \omega_1 \longrightarrow \alpha$ onto.

Let

$$\mathcal{M}^* = \{(X, \in \cap (X \times X), \Phi \cap (X \times X)) \mid (1)\, X \in [H_{\omega_2}]^{\omega},\ (2)\, (X, \in \cap (X \times X), \Phi \cap (X \times X) \prec (H_{\omega_2}, \in, \Phi)\}$$

Let $\big(X, \in \cap (X \times X), \Phi \cap (X \times X)\big) \in \mathcal{M}^*$. Since X is closed under the function Φ, we have

$$\in \cap (X \times X) = \{(x, y) \mid x \in y, x \in X, y \in X\} = \in \cap X,$$

$$\Phi\lceil X = \{(i, \Phi(i)) \mid i \in X\} = \Phi \cap (X \times X) = \Phi \cap X.$$

Hence

$$\big(X, \in \cap (X \times X), \Phi \cap (X \times X)\big) = (X, \in \cap (X \times X), \Phi\lceil X) = (X, \in \cap X, \Phi \cap X).$$

We just write $(X, \in, \Phi)$, (X, Φ), or even X for $\big(X, \in \cap (X \times X), \Phi \cap (X \times X)\big) \in \mathcal{M}^*$.

We later expand the relational structure $(H_{\omega_2}, \in, \Phi)$ only by unary relations $\mathcal{P}$, $\mathcal{M}$, and so forth forming

$$(H_{\omega_2}, \in, \Phi, \mathcal{P}, \mathcal{M}, \cdots).$$

Let $\big(X, \in \cap (X \times X), \Phi \cap (X \times X), \mathcal{P} \cap X, \mathcal{M} \cap X, \cdots\big)$ be an elementary substructure of the expanded structure $(H_{\omega_2}, \in, \Phi, \mathcal{P}, \mathcal{M}, \cdots)$. Then the shortened structure $\big(X, \in \cap (X \times X), \Phi \cap (X \times X)\big)$ is in $\mathcal{M}^*$. The converse may not hold.

Proposition. Let $(X_1, \in, \Phi, \mathcal{P}, \mathcal{M}, \cdots)$ and $(X_2, \in, \Phi, \mathcal{P}, \mathcal{M}, \cdots)$ be two elemetary substructures of a relational structure $(H_{\omega_2}, \in, \Phi, \mathcal{P}, \mathcal{M}, \cdots)$. Let ϕ be an isomorphism from $(X_1, \in, \Phi, \mathcal{P}, \mathcal{M}, \cdots)$ to $(X_2, \in, \Phi, \mathcal{P}, \mathcal{M}, \cdots)$. Then $\phi = \phi_{X_1 X_2}$, where $\phi_{X_1 X_2}$ denotes the unique isomorphism from $(X_1, \in)$ to $(X_2, \in)$.

□

There is no guarantee that $(X_1, \in, \Phi, \mathcal{P}, \mathcal{M}, \cdots)$ and $(X_2, \in, \Phi, \mathcal{P}, \mathcal{M}, \cdots)$ are isomorphic, even if $(X_1, \in, \Phi)$ and $(X_2, \in, \Phi)$ are isomorphic. Hence, we must employ abbreviations and suppressions to denote substructures with caution .

Definition. Let $X, Y \in \mathcal{M}^*$. We say X and Y enjoy *a finite alternation* (at the level of ω_2), if the following holds.

$(\mathrm{fa})_{\omega_2}$ For any $\xi \in S^2_0$, if $\xi = \bigcup(X \cap \xi)$ and $\xi = \bigcup(Y \cap \xi)$, then $\xi = \bigcup(X \cap Y \cap \xi)$.

Notation. Let $X, Y \in \mathcal{M}^*$. We write $X =_{\omega_1} Y$, if $X \cap \omega_1 = Y \cap \omega_1$. Similarly, $X <_{\omega_1} Y$, if $X \cap \omega_1 < Y \cap \omega_1$. Also, $X \leq_{\omega_1} Y$, if $X \cap \omega_1 \leq Y \cap \omega_1$.

Proposition. Let $X, Y \in \mathcal{M}^*$.

(1) If $\eta \in X \cap Y \cap \omega_2$, then $X \cap (\eta + 1) = Y \cap (\eta + 1)$.

(2) Let X and Y enjoy a finite alternation and $X =_{\omega_1} Y$. Let $\xi \in S^2_0$ such that $\xi = \bigcup(X \cap \xi)$ and $\xi = \bigcup(Y \cap \xi)$. Then

$$X \cap \xi = Y \cap \xi.$$

We consider finite symmetric systems of Aspero-Mota that enjoy finite alternations.

Definition. Let $\mathcal{N} \in P_{FAM}$, if

(1) $\mathcal{N}$ is a finite subset of $\mathcal{M}^*$.

(2) If $X, Y \in \mathcal{N}$ with $X =_{\omega_1} Y$, then there exists an isomorphism

$$\varphi_{XY} : (X, \in, \Phi) \longrightarrow (Y, \in, \Phi)$$

that is the identity on the intersection $X \cap Y$ and $\phi[\mathcal{N} \cap X] = \mathcal{N} \cap Y$.

(3) If $X, Y \in \mathcal{N}$ with $X <_{\omega_1} Y$, then there exists $Z \in \mathcal{N}$ such that $X \in Z =_{\omega_1} Y$.

(4) If $X, Y \in \mathcal{N}$ with $X =_{\omega_1} Y$, then X and Y enjoy a finite alternation at the level of ω_2.

Lemma. Let $\mathcal{N} \in P_{FAM}$ and let $X \in \mathcal{N}$. Let $\alpha \in X$ with $\mathrm{cf}(\alpha) \geq \omega_1$. Then there exists $\rho \in X \cap \alpha$ such that for any $Y \in \mathcal{N}$ with $Y <_{\omega_1} X$, it holds that $X \cap Y \cap \alpha \subset \rho$.

The above does not need $(\mathrm{fa})_{\omega_2}$.

Lemma. Let $Y_1, Y_2 \in \mathcal{M}^*$ such that Y_1 and Y_2 are isomorphic, the isomorphism $\phi = \phi_{Y_1 Y_2} : Y_1 \longrightarrow Y_2$ is the identity on the intersection $Y_1 \cap Y_2$, and that Y_1 and Y_2 enjoy a finite alternation. Let $\mathcal{N} \in P_{FAM}$ with $\mathcal{N} \in Y_1$. Then $\mathcal{N} \cup \phi_{Y_1 Y_2}(\mathcal{N}) \in P_{FAM}$.

Expanding relational structures and isomorphisms

We expand the relational structure $(H_{\omega_2}, \in, \Phi)$ by adding a finitely many sequences $\langle P^1_i \mid i < \alpha \rangle, \cdots$, say, $\langle P^{23}_i \mid i < \alpha \rangle$ of a common length α. Typically, P^1_i are forcing posets such that $P^1_i \subset H_{\omega_2}$ and that (CH) has the ω_2-cc. Typically, P^2_i are some forcing relations with resect to P^1_i or sets of countable elementary substructures Z of $(H_{\omega_2}, \in, \Phi, \cdots)$. These sequences are made explicit later. We present things with a single sequence for the sake of shortness.

Notation. Let $\langle P_i \mid i < \alpha \rangle$ be a sequence of non-empty subsets of H_{ω_2} with $\alpha < \omega_2$. We are primarily interested in the initial segments $\langle P_i \mid i \leq \xi \rangle$ with $\xi < \alpha$. We first code the P_is as a single subset of H_{ω_2} by a standard method. Let

$$\mathcal{P} = \mathcal{P}_{<\alpha} = \langle \langle P_i \mid i < \alpha \rangle \rangle = \{(i, x) \mid i < \alpha, x \in P_i\}.$$

We next form an associated relational structure $(H_{\omega_2}, \in, \Phi, \mathcal{P})$ that expands $(H_{\omega_2}, \in, \Phi)$ with the unary relation $\mathcal{P}$.

Let $X \in \mathcal{M}^*$. We consider elementary substructures $(X, \in, \Phi, \mathcal{P})$ of the expanded structure $(H_{\omega_2}, \in, \Phi, \mathcal{P})$, where we mean

$$(X, \in, \Phi, \mathcal{P}) = (X, \in \cap X, \Phi \cap X, \mathcal{P} \cap X).$$

Let $\xi < \alpha$ and write $\mathcal{P}\lceil\xi$, $\mathcal{P}_{<\xi}$, and $\mathcal{P}_{\leq\xi}$ meaning

$$\mathcal{P}\lceil\xi = \mathcal{P}_{<\xi} = \langle\langle P_i \mid i < \xi\rangle\rangle = \{(i,x) \mid i < \xi, x \in P_i\}.$$

$$\mathcal{P}_{\leq\xi} = \mathcal{P}_{<\xi+1} = \mathcal{P}\lceil(\xi+1) = \langle\langle P_i \mid i < \xi+1\rangle\rangle = \{(i,x) \mid i \leq \xi, x \in P_i\}.$$

We are interested in situations when

$$(X, \in, \Phi, \mathcal{P}_{\leq\xi})$$

is an elementary substructure of

$$(H_{\omega_2}, \in, \Phi, \mathcal{P}_{\leq\xi}).$$

Note that $(H_{\omega_2}, \in, \Phi, \mathcal{P}_{\leq\xi})$ is interpretable in $(H_{\omega_2}, \in, \Phi, \mathcal{P})$. Similarly for $(H_{\omega_2}, \in, \Phi, \mathcal{P}_{<\xi})$.

Proposition. (1) Let $\varphi(v_1, \cdots, v_n)$ be a formula appropriate for $(H_{\omega_2}, \in, \Phi, \mathcal{P}_{\leq\xi})$. Then there is a corresponding formula $\varphi^*(v, v_1, \cdots, v_n)$ such that for all $x_1, \cdots, x_n \in H_{\omega_2}$, we have

$$(H_{\omega_2}, \in, \Phi, \mathcal{P}_{\leq\xi}) \models \text{“}\varphi(x_1, \cdots, x_n)\text{”}$$

iff

$$(H_{\omega_2}, \in, \Phi, \mathcal{P}) \models \text{“}\varphi^*(\xi, x_1, \cdots, x_n)\text{”}.$$

(2) Let $\varphi(v_1, \cdots, v_n)$ be a formula appropriate for $(H_{\omega_2}, \in, \Phi, P_\xi)$. Then there is a corresponding formula $\varphi^{**}(v, v_1, \cdots, v_n)$ such that for all $x_1, \cdots, x_n \in H_{\omega_2}$, we have

$$(H_{\omega_2}, \in, \Phi, P_\xi) \models \text{“}\varphi(x_1, \cdots, x_n)\text{”}$$

iff

$$(H_{\omega_2}, \in, \Phi, \mathcal{P}_{\leq\xi}) \models \text{“}\varphi^{**}(\xi, x_1, \cdots, x_n)\text{”}.$$

Proposition. Let $X, X_1, X_2 \in \mathcal{M}^*$. Let $\xi < \alpha$ and $\xi_1 < \xi_2 < \alpha$.

(1) If $(X, \in, \Phi, \mathcal{P}_{\leq\xi})$ is an elementary substructure of $(H_{\omega_2}, \in, \Phi, \mathcal{P}_{\leq\xi})$, then $\xi \in X$.

(2) If $(X, \in, \Phi, \mathcal{P}_{\leq\xi_2})$ is an elementary substructure of $(H_{\omega_2}, \in, \Phi, \mathcal{P}_{\leq\xi_2})$ and $\xi_1 \in X$, then $(X, \in, \Phi, \mathcal{P}_{\leq\xi_1})$ is an elementary substructure of $(H_{\omega_2}, \in, \Phi, \mathcal{P}_{\leq\xi_1})$.

(3) If $(X_1, \in, \Phi, \mathcal{P})$ and $(X_2, \in, \Phi, \mathcal{P})$ are isomorphic elementary substructures of $(H_{\omega_2}, \in, \Phi, \mathcal{P})$ and $\xi \in X_1 \cap X_2$, then $(X_1, \in, \Phi, \mathcal{P}_{\leq\xi})$ and $(X_2, \in, \Phi, \mathcal{P}_{\leq\xi})$ are isomorphic elementary substructures of $(H_{\omega_2}, \in, \Phi, \mathcal{P}_{\leq\xi})$.

Lemma. (Induced structure) Let $X_1, X_2, Y \in \mathcal{M}^*$. Let $\xi \in X_1 \cap X_2 \cap \alpha$. Let $(X_1, \in, \Phi, \mathcal{P}_{\leq\xi})$, $(X_2, \in, \Phi, \mathcal{P}_{\leq\xi})$, and $(Y, \in, \Phi, \mathcal{P}_{\leq\xi})$ be all elementary substructures of $(H_{\omega_2}, \in, \Phi, \mathcal{P}_{\leq\xi})$. Let

$$\phi : (X_1, \in, \Phi, \mathcal{P}_{\leq\xi}) \longrightarrow (X_2, \in, \Phi, \mathcal{P}_{\leq\xi})$$

be the isomorphism with $\phi(\xi) = \xi$ and $Y \in X_1$. Then

(1) $(Y, \in, \Phi, \mathcal{P}_{\leq\xi}) \in X_1$.

(2) $\phi\big((Y, \in, \Phi, \mathcal{P}_{\leq\xi})\big) = (\phi(Y), \in, \Phi, \mathcal{P}_{\leq\xi})$.

(3) $\phi(Y) \in \mathcal{M}^*$.

(4) The induced copy $(\phi(Y), \in, \Phi, \mathcal{P}_{\leq\xi})$ forms an elementary substructure of $(H_{\omega_2}, \in, \Phi, \mathcal{P}_{\leq\xi})$.

(5) $(Y, \in, \Phi, \mathcal{P}_{\leq \xi})$ and $(\phi(Y), \in, \Phi, \mathcal{P}_{\leq \xi})$ are isomorphic by the restriction of ϕ.

The Basic Poset P_{BASE}

Notation. We say S is a *relation* from A to B, if $S \subseteq A \times B$. We write aSb to mean $(a,b) \in S$. Let x be *any*, we denote $S(x)$ to mean the range $\{b \mid xSb\}$. Hence if $x \notin A$, then $S(x) = \emptyset$. Let C be a subset of B, we denote aSC to mean that $C \subseteq S(a)$.

Definition. Let $p = (\mathcal{N}^p, S^p, A^p) = (\mathcal{N}, S, A) \in P_{BASE}$, if

(1) $\mathcal{N}^p \in P_{FAM}$.

(2) S^P is a relation from $\mathcal{N}^p$ to ω_2 such that for all $Y \in \mathcal{N}^p$, $S^p(Y) \subseteq Y \cap \omega_2$.

(3) A^P is a *finite* relation from ω_2 to ω_1.

According to our notational convention, we write $\xi A^p t$ for $(\xi, t) \in A^p$. We also write $A^p(\xi)$ to mean $\{t < \omega_1 \mid \xi A^p t\}$. Hence

$$A^p = \bigcup \{\{\xi\} \times A^p(\xi) \mid \xi \in \mathrm{dom}(A^p)\}.$$

It is clear that

$$P_{BASE} \subset H_{\omega_2}.$$

Let $p = (\mathcal{N}, S, A) \in P_{BASE}$ and $\alpha < \omega_2$. We define the usual restriction $p\lceil\alpha = (\mathcal{N}, S, A)\lceil\alpha = (\mathcal{N}, S\lceil\alpha, A\lceil\alpha)$, where

$$S\lceil\alpha = S \cap (\mathcal{N} \times \alpha),$$

$$A\lceil\alpha = A \cap (\alpha \times \omega_1).$$

Hence $S\lceil\alpha$ is a relation from $\mathcal{N}$ to α and $A\lceil\alpha$ is a finite relation from α to ω_1 such that

- For any Y, we have $S^{p\lceil\alpha}(Y) = S^p(Y) \cap \alpha$.
- For any $\xi < \alpha$, we have $A^{p\lceil\alpha}(\xi) = A^p(\xi)$.

If $\alpha_1 \leq \alpha_2 < \omega_2$, then

$$((\mathcal{N}, S, A)\lceil\alpha_2)\lceil\alpha_1 = (\mathcal{N}, S, A)\lceil\alpha_1.$$

For $p, q \in P_{BASE}$, let $q \leq p$, if $\mathcal{N}^q \supseteq \mathcal{N}^p$, $S^q \supseteq S^p$, and $A^q \supseteq A^p$.

We construct a $\subset_{\mathrm{reg}}$-increasing sequence $\langle P_\alpha \mid \alpha < \omega_2 \rangle$ of subposets of P_{BASE}.

Remark. If you are sort of familiar with the markers of Aspero-Mota, what we roughly intend is the following.

- If $YS^p\eta$, then Y is well-closed with respect to P_η and $p\lceil\eta$ is (P_η, Y)-g.
- If $YS^p\eta$ and Y is well-closed with respect to $P_{\eta+1}$, then $p\lceil(\eta+1)$ is $(P_{\eta+1}, Y)$-g.
- $S^p(Y)$ is an initial segment of $Y \cap \omega_2$.
- $Y\Delta^p\beta$ iff $YS^p(Y \cap \beta)$, though we do not introduce the *finite* relation Δ^p of Aspero-Mota.
- If $YS^p(Y \cap \eta)$ and Y is well-closed with respect to P_η, then $p\lceil\eta$ is (P_η, Y)-g.
- S^p is usually an infinite relation. But $\{(Y, S^p(Y)) \mid Y \in \mathcal{N}^p\}$ is a finite set that discerns S^p.

Hence we prepared the predicate S^P to argue things point-wise, namely, we worry $YS^p\eta$ or not.

$$\begin{array}{ccccccc} YS^p(\eta+1) & \Longrightarrow & YS^p\eta + Y \prec P_{\eta+1} & & \Longrightarrow & & (P_{\eta+1}, Y) - \mathrm{g} \\ & & \Downarrow & & & & \Downarrow \\ Y\Delta^p(\eta+1) + \eta \in Y & \Longleftrightarrow & YS^p\eta & \Longrightarrow & YS^p(Y \cap \eta) + Y \prec P_\eta & \Longrightarrow & (P_\eta, Y) - \mathrm{g} \end{array}$$

We prepare a sort of second-order treatment of forcing posets that has a right chain condition. In the following, we may think of $\kappa = \omega_2$. We stick to the only universe H_κ and prepare a variety of unary predicates on it, resulting a variety of clubs in $[H_\kappa]^\omega$.

Definition. Let κ be a regular uncountable cardinal. Let $(P, \leq_P)$ be a poset such that $P \subset H_\kappa$ and P has the κ-cc. We consider a relational structure

$$(H_\kappa, \in, P, \leq_P, R^P_=, R^P_\in, H^P_\kappa, \cdots),$$

where

- $R^P_= = \{(p, \tau, \pi) \in (P \times V^P \times V^P) \cap H_\kappa \mid p \Vdash_P \text{“}\tau = \pi\text{”}\}$,
- $R^P_\in = \{(p, \tau, \pi) \in (P \times V^P \times V^P) \cap H_\kappa \mid p \Vdash_P \text{“}\tau \in \pi\text{”}\}$,
- $H^P_\kappa = V^P \cap H_\kappa$.

We are interested in countable elementary substructures

$$(X, \in \cap X^2, P \cap X, \leq_P \cap X^2, R^P_= \cap X^3, R^P_\in \cap X^3, H^P_\kappa \cap X, \cdots)$$

of

$$(H_\kappa, \in, P, \leq_P, R^P_=, R^P_\in, H^P_\kappa, \cdots).$$

Lemma. Let

$$(X, \in \cap X^2, P \cap X, \leq_P \cap X^2, R^P_= \cap X^3, R^P_\in \cap X^3, H^P_\kappa \cap X, \cdots)$$

be a countable elementary substucture of

$$(H_\kappa, \in, P, \leq_P, R^P_=, R^P_\in, H^P_\kappa, \cdots).$$

Let G be P-generic over the ground model V. Then in $V[G]$, we have

$$(X[G], \in \cap X[G]^2, H^V_\kappa \cap X[G], G \cap X[G], P \cap X[G], \leq_P \cap X[G]^2, R^P_= \cap X[G]^3, R^P_\in \cap X[G]^3, H^P_\kappa \cap X[G], \cdots)$$

is a countable elementary substructure of

$$(H^{V[G]}_\kappa, \in, H^V_\kappa, G, P, \leq_P, R^P_=, R^P_\in, H^P_\kappa, \cdots).$$

Definition. Let P be a poset such that $P \subset H_\kappa$ and P has the κ-cc. Let

$$X \prec (H_\kappa, \in, P, \leq_P, R^p_=, R^p_\in, H^p_\kappa).$$

We define that $p \in P$ is *(P, X)-generic*, if for each predense subset $D \in X$ of P, $D \cap X$ is predense below p in P. Hence, we use maximal antichains rather than dense subsets that would be too large to belong to H_κ.

Lemma. Let P be a poset such that $P \subset H_\kappa$ and P has the κ-cc. Let

$$X \prec (H_\kappa, \in, P, \leq_P, R^p_=, R^p_\in, H^p_\kappa)$$

and $p \in P$. The following are equivalent

(1) p is (P, X)-generic.

(2) For any maximal antichain $A \in X$ of P, $A \cap X$ is predense below p in P.

(3) $p \Vdash_P \text{“}X[\dot{G}] \cap H^V_\kappa = X\text{”}$.

(4) $p \Vdash_P \text{“}X[\dot{G}] \cap \kappa = X \cap \kappa\text{”}$.

Definition. Let T^* be weakly Suslin witnessed at κ. Let P be a poset such that $P \subset H_\kappa$ and P has the κ-cc. We say P is *T^*-preserving,* if there exists a club many X such that if $X \in \mathcal{M}^*$ and $p \in P \cap X$, then there exists $q \leq p$ in P such that

- q is (P, X)-generic,
- For any $x \in T^*_{X \cap \omega_1}$, if x pushdown X, then $q \Vdash_P$"x pushdown $X[\dot{G}]$".

Lemma. Let T^* be weakly Suslin witnessed at κ. Let P be T^*-preserving. Then $1 \Vdash_P$"T^* is weakly Suslin witnessed at κ".

Iteration

Definition. Let $\Phi : \omega_2 \longrightarrow H_{\omega_2}$ be an onto map such that for each $a \in H_{\omega_2}$, $\{\xi < \omega_2 \mid \Phi(\xi) = a\}$ is cofinal in ω_2. We may assume that

$$T^* = \Phi(0).$$

We construct a sequence of posets $\langle P_\alpha \mid \alpha < \omega_2 \rangle$ by recursion on $\alpha < \omega_2$. Let us assume that $\alpha < \omega_2$ and we have constructed $\langle P_\xi \mid \xi < \alpha \rangle$ together with $\langle (\dot{K}_0^\xi, \dot{K}_1^\xi) \mid \xi < \alpha \rangle$ and $\langle M_\xi \mid \xi < \alpha \rangle$. Let us identify the finitely many sequences of subsets of H_{ω_2}.

$$\mathcal{P} = \langle \langle P_i \mid i < \alpha \rangle \rangle,$$

$$\mathcal{K}_0 = \langle \langle \dot{K}_0^i \mid i < \alpha \rangle \rangle,$$
$$\mathcal{K}_1 = \langle \langle \dot{K}_1^i \mid i < \alpha \rangle \rangle,$$
$$\mathcal{R}_=^P = \langle \langle R_=^{P_i} \mid i < \alpha \rangle \rangle,$$
$$\mathcal{R}_\in^P = \langle \langle R_\in^{P_i} \mid i < \alpha \rangle \rangle,$$
$$\mathcal{H}^P = \langle \langle H_{\omega_2}^{P_i} \mid i < \alpha \rangle \rangle,$$
$$\mathcal{M} = \langle \langle M_i \mid i < \alpha \rangle \rangle.$$

We begin to state induction hypothesis, where we suppress mentioning the sequences except $\mathcal{P}$ and $\mathcal{M}$, as it is tidy. We write $X \prec P_\xi$ for

$$X \prec (H_{\omega_2}, \in, \Phi, P_\xi, R_=^{P_\xi}, R_\in^{P_\xi}, H_{\omega_2}^{P_\xi}).$$

We also write $X \prec (\mathcal{P}_{\leq \xi}, \mathcal{M}_{\leq \xi})$ for

$$X \prec (H_{\omega_2}, \in, \Phi, \mathcal{P}_{\leq \xi}, \mathcal{M}_{\leq \xi}).$$

We assume recursively that for each $\xi < \alpha$

- $P_\xi \subset \{p \in P_{BASE} \mid p\lceil\xi = p\} \subset H_{\omega_2}$ and (CH) P_ξ has the ω_2-cc,
- $\Vdash_{P_\xi}$"$\dot{K}_0^\xi \dot{\cup} \dot{K}_1^\xi$ is a partition of $[\omega_1]^2$ that is $R_{1,\aleph_1}$".
- If $\Phi(\xi)$ is a P_ξ-name, then $\Vdash_{P_\xi}$"if $\Phi(\xi)$ is an Aronszajn tree, then $\dot{K}_0^\xi \dot{\cup} \dot{K}_1^\xi$ is the induced partition".
- $M_\xi = \{X \in \mathcal{M}^* \mid (1)\ X \prec P_\xi;\ (2)$ For all $\eta \in X \cap \xi$, $X \prec (\mathcal{P}_{\leq \eta}, \mathcal{M}_{\leq \eta})\}$

  Let $p = (\mathcal{N}^p, S^p, A^p) = (\mathcal{N}, S, A) \in P_\alpha$, if

  (ob) • $\mathcal{N} \in P_{FAM}$.
  - S is a relation from $\mathcal{N}$ to α such that for all $Y \in \mathcal{N}$, $S(Y)$ are initial segments of $Y \cap \alpha$.
  - A is a finite relation from α to ω_1.

  (el) If $Y S \eta$, then $(Y)_{\leq \eta} \downarrow$, this abbreviates

$$Y \prec (\mathcal{P}_{\leq\eta}, \mathcal{M}_{\leq\eta}).$$

(ho) If $Y_1 S\eta =_{\omega_1} Y_2 S\eta$, then $(Y_1)_{\leq\eta} \sim (Y_2)_{\leq\eta}$, this abbreviates

$$(Y_1, \in, \Phi, \mathcal{P}_{\leq\eta}, \mathcal{M}_{\leq\eta}) \sim (Y_2, \in, \Phi, \mathcal{P}_{\leq\eta}, \mathcal{M}_{\leq\eta}).$$

(up) If $Y_2 S\eta$, $Y_3 S\eta$, $Y_3 <_{\omega_1} Y_2$, then there is $Y_1 \in \mathcal{N}$ such that $Y_3 \in Y_1 =_{\omega_1} Y_2$ and $Y_1 S\eta$.

(down) If $Y_1 S\eta =_{\omega_1} Y_2 S\eta$, $Y_3 S\eta$, and $Y_3 \in Y_1$, then $\phi_{Y_1 Y_2}(Y_3) S\eta$.

($\underline{*}$) If $p\lceil\xi \in P_\xi$, then $p\lceil\xi \Vdash_{P_\xi}$ "$A(\xi)$ is $\dot{K}_0^\xi$-homo".

(g) If $\xi A t$ and $Y S\xi$, then either

- $t <_{\omega_1} Y$, or
- There exists Z such that $S(Z) \supseteq Z \cap \xi$, $Z \prec (\mathcal{P}_{\leq\xi}, \mathcal{M}_{\leq\xi})$, and $Y \in Z \leq_{\omega_1} t$.

For $p, q \in P_\alpha$, set $q \leq p$ in P_α, if $\mathcal{N}^q \supseteq \mathcal{N}^p$, $S^q \supseteq S^p$, and $A^q \supseteq A^p$.

Hence P_α is a suborder of P_{BASE}.

Lemma. (The restrictions) Let $p \in P_\alpha$ and $\rho < \alpha$. Then $p\lceil\rho \in P_\rho$.

Lemma. (The projection) Let $\rho < \alpha$. The map $P_\alpha \longrightarrow P_\rho$ defined by $p \mapsto p\lceil\rho$ is a projection in the following sense.

(1) If $p, q \in P_\alpha$ with $q \leq p$ in P_α, then $q\lceil\rho \leq p\lceil\rho$ in P_ρ.
(2) If $h \in P_\rho$ with $h \leq p\lceil\rho$, then $h^+ = (\mathcal{N}^h, S^h \cup S^p, A^h \cup A^p) \in P_\alpha$ such that $h^+\lceil\rho = h$ and $h^+ \leq p$ in P_α.

Lemma. (Complete suborders) Let $\rho < \alpha$. Then

(1) P_ρ is a suborder of P_α.
(2) For $p, q \in P_\rho$, p and q are incompatible in P_ρ iff in P_α.
(3) Any maximal antichain A in P_ρ remains in P_α.
(4) $p \leq p\lceil\rho$ in P_α.
(5) If G_α is P_α-generic over V, then

$$G_\alpha \cap P_\rho = G_\alpha\lceil\rho = \{g\lceil\rho \mid g \in G_\alpha\}$$

is P_ρ-generic over V.

Lemma. (1) $P_\alpha \subset \{p \in P_{BASE} \mid p\lceil\alpha = p\} \subset H_{\omega_2}$.

(2) (CH) P_α has the ω_2-cc.

Here is the main lemma proved by induction on $\alpha < \omega_2$.

Lemma. (MAIN) Let $p \in P_\alpha$ and X be such that

(1) $X \prec (\mathcal{P}_{\leq\alpha}, \mathcal{M}_{\leq\alpha})$,
(2) $X \cap \alpha = S^p(X)$.

Then p is (P_α, X)-gg. Namely,

(1) p is (P_α, X)-g,
(2) If $x \in T^*_{X\cap\omega_1}$ with x pushdown X, then $p \Vdash_{P_\alpha}$ "x pushdown $X[\dot{G}_\alpha]$".

Lemma. (Start) Let $\alpha < \omega_2$ and $p \in P_\alpha$. Let

(1) $p \in X \prec (\mathcal{P}_{\leq\alpha}, \mathcal{M}_{\leq\alpha})$,

Then there exists $q \in P_\alpha$ such that $q \leq p$ and that q satisfies the assumption of lemma (main).

In the forcing construction, it suffices to deal with those Aronszajn trees T such that

(1) T has a single root.

(2) Every node of T has infinitely many successors on every higher level of T.

In particular, for every finite K_0-homogeneous set (namely, antichain) A with respect to T, we have $\{t \in T \mid A \cup \{t\}$ is K_0-homogeneous$\}$ is uncountable. Details based on [K].

Lemma. (Add Domain, Add a new Element) Let $p \in P_{\alpha+1}$. Let $Z \prec (\mathcal{P}_{\leq\alpha}, \mathcal{M}_{\leq\alpha})$ such that $p \in Z$. Then there exists (h^+, t) such that

(1) $h^+ \in P_{\alpha+1}$,

(2) $h^+ \leq p$ in $P_{\alpha+1}$,

(3) $Z \in \mathcal{N}^{h^+}$,

(4) $Z <_{\omega_1} t$,

(5) $A^{h^+}(\alpha) = A^p(\alpha) \cup \{t\}$.

Lemma. ($\alpha + 1 \models$ Ext) Let $X \prec (\mathcal{P}_{\leq\alpha+1}, \mathcal{M}_{\leq\alpha+1})$, $p \in P_{\alpha+1}$, $XS^p\alpha$, and $\alpha \in \mathrm{dom}(A^p)$. Then there is $(\mathcal{Z}, S^*)$ such that

(1) $\mathcal{N}^p \cap X$, $\mathrm{rang}(A^p) \cap X <_{\omega_1} \mathcal{Z} <_{\omega_1} X$.

(2) $S^* \subseteq \mathcal{Z} \times \alpha$.

(3) $(\mathcal{N}^p \cup \mathcal{Z}, S^p \cup S^*, A^p) \in P_{\alpha+1}$.

(4) If $Y <_{\omega_1} X$ and $YS^p\alpha$, then there is (Z, X') such that

- $Z \in \mathcal{Z}$, $S^*(Z) = Z \cap \alpha$, and $Z \prec (\mathcal{P}_{\leq\alpha}, \mathcal{M}_{\leq\alpha})$.
- $X'S^p\alpha$, and $Y \in Z \in X' =_{\omega_1} X$.

Proof of main lemma out-lined. Details in use with $R_{1,\aleph_1}$ provided along [Y].

By induction on $\alpha < \omega_2$. Let $p \in P_\alpha$, $S^p(X) = X \cap \alpha$, and $X \prec (\mathcal{P}_{\leq\alpha}, \mathcal{M}_{\leq\alpha})$.

Case 0.0. $\alpha = 0$ and want p is (P_0, X)-g.

Recall $p \in P_0$ iff $p = (\mathcal{N}^p, \emptyset, \emptyset)$ and $\mathcal{N}^p \in P_{FAM}$.
We have $q \leq p$ iff $\mathcal{N}^q \supseteq \mathcal{N}^p$. Hence, P_0 and P_{FAM} are isomorphic.
Let $D \in X$ be a predense subset of P_0, $q \leq p$, $q \leq d$, and $d \in D$.
Get q' and d' such that

- $q' \in P_0 \cap X$, $q' \leq d'$ in P_0, and $d' \in D \cap X$,
- $\mathcal{N}^{q'} \supseteq \mathcal{N}^q \cap X$.

Let $h^+ \in P_0$ such that $\mathcal{N}^{h^+} \supset \mathcal{N}^q \cup \mathcal{N}^{q'}$. Then $h^+ \leq q$, $h^+ \leq d'$, and $d' \in D \cap X$.

Case 0.1. $\alpha = 0$ and want p is (P_0, X)-gg.

Let $x \in T^*_{X \cap \omega_1}$ and x pushdown X. Let $\dot{A} \in X$ be a P_0-name. Let $q \leq p$ and $q \Vdash_{P_0}$ "$x \in \dot{A}$".
Get q' and $y <_{T^*} x$ such that

- $q' \in P_0 \cap X$,
- $q' \Vdash_{P_0}$ "$y \in \dot{A}$",

- $\mathcal{N}^{q'} \supseteq \mathcal{N}^q \cap X$.

  Let $h^+ \in P_0$ such that $\mathcal{N}^{h^+} \supset \mathcal{N}^q \cup \mathcal{N}^{q'}$. Then $h^+ \leq q$ and $h^+ \Vdash_{P_0}$ "$y \in \dot{A}$".

Case 1.0. suc($\alpha = \alpha + 1$) and want p is $(P_{\alpha+1}, X)$-g.

Let $D \in X$ be a predense subset of $P_{\alpha+1}$, $q \leq p$, $q \leq d$, $d \in D$, and $\alpha \in \mathrm{dom}(A^q)$. We may assume that $A^q(\alpha) \not\subset X$ by lemma (add domain, add a new element) and that q is as in lemma (ext).

Get q' and d' such that

- $q' \in P_{\alpha+1} \cap X$, $q' \leq d' \in D \cap X$,
- $\alpha \in \mathrm{dom}(A^{q'})$ and $A^{q'}(\alpha) \sqsupset_{\mathrm{end}} (A^q(\alpha) \cap X)$,
- $S^{q'}(Y) \subseteq Y \cap \alpha$ or $S^{q'}(Y) = Y \cap (\alpha + 1)$, (not essential)
- If $Y \in X \cap \mathcal{N}^q$, then $S^q(Y) = S^{q'}(Y)$,
- $h \in P_\alpha$,
- $h \leq q\lceil\alpha, q'\lceil\alpha$,
- $h \Vdash_{P_\alpha}$ "$(A^q(\alpha) \setminus X) \cup (A^{q'}(\alpha) \setminus (A^q(\alpha) \cap X))$ is $\dot{K}_0^\alpha$-homo".

Let

$$h^+ = (\mathcal{N}^h, S^h \cup S^q \cup S^{q'} \cup S^+, A^h \cup A^q \cup A^{q'}).$$

Then $h^+ \in P_{\alpha+1}$ and $h^+ \leq q, q'$.

Here for $Y \in \mathcal{N}^h$ and $\eta = \alpha \in [\alpha, \alpha+1) \cap Y$, we set $YS^+\alpha$, whenever there exists (X', W) such that $X =_{\omega_1} X' \in \mathcal{N}^q$, $X'S^q\alpha$, $W \in X$, $WS^{q'}\alpha$, and $Y = \phi_{XX'}(W)$.

$$\begin{array}{ccc} XS^q\alpha & \sim & X'S^q\alpha \\ | & & | \\ WS^{q'}\alpha & \sim & YS^+\alpha \quad \geq \alpha \end{array}$$

Here we may think of that $\rho = \alpha$ and $\alpha + 1 = (\alpha+1)_X = \mathrm{ssup}(X \cap (\alpha+1))$, in view of later cases.

Case 1.1. suc($\alpha = \alpha + 1$) and want p is $(P_{\alpha+1}, X)$-gg.

Let $x \in T^*_{X\cap\omega_1}$ and x pushdown X. Let $\dot{A} \in X$ be a $P_{\alpha+1}$-name. Let $q \leq p$, $q \Vdash_{P_{\alpha+1}}$ "$x \in \dot{A}$", and $\alpha \in \mathrm{dom}(A^q)$. We may assume that $A^q(\alpha) \not\subset X$ by lemma (add domain, add a new element) and that q is as in lemma (ext).

Get q' and $y <_{T^*} x$ such that

- $q' \in P_{\alpha+1} \cap X$ and $q' \Vdash_{P_{\alpha+1}}$ "$y \in \dot{A}$"
- $\alpha \in \mathrm{dom}(A^{q'})$ and $A^{q'}(\alpha) \sqsupset_{\mathrm{end}} (A^q(\alpha) \cap X)$,
- $S^{q'}(Y) \subseteq Y \cap \alpha$ or $S^{q'}(Y) = Y \cap (\alpha + 1)$, (not essential)
- If $Y \in X \cap \mathcal{N}^q$, then $S^q(Y) = S^{q'}(Y)$,
- $h \in P_\alpha$,
- $h \leq q\lceil\alpha, q'\lceil\alpha$,
- $h \Vdash_{P_\alpha}$ "$(A^q(\alpha) \setminus X) \cup (A^{q'}(\alpha) \setminus (A^q(\alpha) \cap X))$ is $\dot{K}_0^\alpha$-homo".

Let

$$h^+ = (\mathcal{N}^h, S^h \cup S^q \cup S^{q'} \cup S^+, A^h \cup A^q \cup A^{q'}).$$

Then $h^+ \in P_{\alpha+1}$ and $h^+ \leq q, q'$.

Case 2.0. cf(α) $= \omega$ and want p is (P_α, X)-g.

Let $D \in X$ be a predense subset of P_α, $q \leq p$, $q \leq d$, and $d \in D$.
Let ρ be an ordinal such that

- $\rho \in X \cap \alpha$.
- $\mathrm{dom}(A^q) \subset \rho$.
- If $S^q(Y)$ is bounded below α, then $S^q(Y) \subset \rho$.

Get q', d', and h such that

- $q' \in P_\alpha \cap X$, $d' \in D \cap X$, and $q' \leq d'$ in P_α,
- $\mathrm{dom}(A^{q'}) \subset \rho$,
- $S^{q'}(Y) \subset Y \cap \rho$ or $S^{q'}(Y) = Y \cap \alpha$, (not essential)
- If $Y \in X \cap \mathcal{N}^q$, then $S^q(Y) = S^{q'}(Y)$,
- $h \in P_\rho$,
- $h \leq q\lceil\rho, q'\lceil\rho$.

Let

$$h^+ = (\mathcal{N}^h, S^h \cup S^q \cup S^{q'} \cup S^+, A^h).$$

Then $h^+ \in P_\alpha$ and $h^+ \leq q, q'$ in P_α.

Here for $Y \in \mathcal{N}^h$ and $\eta \in [\rho, \alpha) \cap Y$, we set $YS^+\eta$, whenever there exists (X', W) such that $X =_{\omega_1} X' \in \mathcal{N}^q$, $W \in X$, $\rho \leq \eta \in W$, $X'S^q\eta$, $WS^{q'}\eta$, and $\phi_{XX'}(W) = Y$.

$$\begin{array}{ccccc} XS^q\eta & \sim & X'S^q\eta & & \\ | & & | & & \\ WS^{q'}\eta & \sim & YS^+\eta & \geq \rho \end{array}$$

If this is the case, then we have $X \cap \alpha = X' \cap \alpha$ and even $\alpha \in X \cap X'$. This is because, $S^q(X') \cap \alpha$ is cofinal below α and so, by $(\mathrm{fa})_{\omega_2}$, $X \cap \alpha = X' \cap \alpha$. Since $\mathrm{cf}(\alpha) = \omega$, we then even have $\phi_{XX'}(\alpha) = \alpha \in X'$.

Case 2.1. $\mathrm{cf}(\alpha) = \omega$ and want p is (P_α, X)-gg.

Let $x \in T^*_{X \cap \omega_1}$ and x pushdown X. Let $\dot{A} \in X$ be a P_α-name. Let $q \leq p$ and $q \Vdash_{P_\alpha}$ "$x \in \dot{A}$".
Let ρ be an ordinal such that

- $\rho \in X \cap \alpha$.
- $\mathrm{dom}(A^q) \subset \rho$.
- If $S^q(Y)$ is bounded below α, then $S^q(Y) \subset \rho$.

Get q' and $y <_{T^*} x$ such that

- $q' \in P_\alpha \cap X$ and $q' \Vdash_{P_\alpha}$ "$y \in \dot{A}$",
- $\mathrm{dom}(A^{q'}) \subset \rho$,
- $S^{q'}(Y) \subset Y \cap \rho$ or $S^{q'}(Y) = Y \cap \alpha$, (not essential)
- If $Y \in X \cap \mathcal{N}^q$, then $S^q(Y) = S^{q'}(Y)$,
- $h \in P_\rho$,
- $h \leq q\lceil\rho, q'\lceil\rho$.

Let

$$h^+ = (\mathcal{N}^h, S^h \cup S^q \cup S^{q'} \cup S^+, A^h).$$

Then $h^+ \in P_\alpha$ and $h^+ \leq q, q'$ in P_α.

Case 3.0. $\mathrm{cf}(\alpha) \geq \omega_1$ and want p is (P_α, X)-g.

Let $D \in X$ be a predense subset of P_α, $q \leq p$, $q \leq d$, and $d \in D$.
Let ρ be an ordinal such that

- $\rho \in \alpha \cap X$,
- $\mathrm{dom}(A^q) \cap \sup(X \cap \alpha) \subset \rho$,
- If $S^q(Y) \cap \sup(X \cap \alpha)$ is bounded below $\sup(X \cap \alpha)$, then $S^q(Y) \cap \sup(X \cap \alpha) \subset \rho$,
- If $Y \in \mathcal{N}^q$ and $Y <_{\omega_1} X$, then $Y \cap X \cap \alpha \subset \rho$.

Get q' and d' such that

- $q' \in P_\alpha \cap X$, $d' \in D \cap X$, and $q' \leq d'$ in P_α,
- If $Y \in X \cap \mathcal{N}^q$, then $S^q(Y) = S^{q'}(Y)$,
- $h \in P_\rho$,

- $h \leq q\lceil\rho, q'\lceil\rho$.

  Let
$$h^+ = (\mathcal{N}^h, S^h \cup S^q \cup S^{q'} \cup S^+, A^h \cup A^q \cup A^{q'}).$$

  Then $h^+ \in P_\alpha$ and $h^+ \leq q, q'$ in P_α.

Here for $Y \in \mathcal{N}^h$ and $\eta \in [\rho, \alpha_X) \cap Y$, $\alpha_X = \sup(X \cap \alpha)$, we set $YS^+\eta$, whenever there exists (X', W) such that $X =_{\omega_1} X' \in \mathcal{N}^q$, $W \in X$, $\rho \leq \eta \in W$, $X'S^q\eta$, and $\phi_{XX'}(W) = Y$.

$$\begin{array}{ccccc} XS^q\eta & \sim & X'S^q\eta & & \\ | & & | & & \\ WS^{q'}\eta & \sim & YS^+\eta & \geq \rho & \end{array}$$

If this is the case, then we have $X \cap \alpha_X = X' \cap \alpha_X$.

Case 3.1. $\mathrm{cf}(\alpha) \geq \omega_1$ and want p is (P_α, X)-gg.

Let $x \in T^*_{X\cap\omega_1}$ and x pushdown X. Let $\dot{A} \in X$ be a P_α-name. Let $q \leq p$ and $q \Vdash_{P_\alpha}$ "$x \in \dot{A}$".
Let ρ be an ordinal such that

- $\rho \in \alpha \cap X$,
- $\mathrm{dom}(A^q) \cap \sup(X \cap \alpha) \subset \rho$,
- If $S^q(Y) \cap \sup(X \cap \alpha)$ is bounded below $\sup(X \cap \alpha)$, then $S^q(Y) \cap \sup(X \cap \alpha) \subset \rho$,
- If $Y \in \mathcal{N}^q$ and $Y <_{\omega_1} X$, then $Y \cap X \cap \alpha \subset \rho$.

Get q' and $y <_{T^*} x$ such that

- $q' \in P_\alpha \cap X$ and $q' \Vdash_{P_\alpha}$ "$y \in \dot{A}$",
- If $Y \in X \cap \mathcal{N}^q$, then $S^q(Y) = S^{q'}(Y)$,
- $h \in P_\rho$,
- $h \leq q\lceil\rho, q'\lceil\rho$.

  Let
$$h^+ = (\mathcal{N}^h, S^h \cup S^q \cup S^{q'} \cup S^+, A^h \cup A^q \cup A^{q'}).$$

Then $h^+ \in P_\alpha$ and $h^+ \leq q, q'$ in P_α.

□

The Final Stage $P^*_{\omega_2}$

We gave a uniform definition of the P_αs and we did not define P_{ω_2}. The reason was that if $\alpha < \omega_2$ and $Y \prec (H_{\omega_2}, \mathcal{P}_{\leq\alpha})$, then $\alpha \in Y$, while $\omega_2 \not< \omega_2$. We did not want to argue P_αs and P_{ω_2} in the previous sections separatedly.

Now, we form the direct limit $P^*_{\omega_2}$ of $\langle P_\alpha \mid \alpha < \omega_2\rangle$. If we had defined P_{ω_2} as in the P_αs, then $P_{\omega_2} = P^*_{\omega_2}$. Hence we pay back here by somewhat repeating relevants.

Definition. $P^*_{\omega_2} = \bigcup\{P_\alpha \mid \alpha < \omega_2\}$. For $p, q \in P^*_{\omega_2}$, let $q \leq p$ in $P^*_{\omega_2}$, if there exists $\alpha < \omega_2$ such that $p, q \in P_\alpha$ and $q \leq p$ in P_α.

The choices of α are irrelevant and $q \leq p$ in $P^*_{\omega_2}$ iff $q \leq p$ in P_{BASE} iff $\mathcal{N}^q \supseteq \mathcal{N}^p$, $S^q \supseteq S^p$, and $A^q \supseteq A^p$.

Lemma. (1) $P^*_{\omega_2} \subset P_{BASE} \subset H_{\omega_2}$.

(2) For each $\alpha < \omega_2$, P_α is a complete suborder of $P^*_{\omega_2}$.

(3) For each $\alpha < \omega_2$, the map $p \mapsto p\lceil\alpha$ from $P^*_{\omega_2}$ to P_α is a projection.

(4) Let G be a $P^*_{\omega_2}$-generic filter over V. Then $G\lceil\alpha = \{g\lceil\alpha \mid g \in G\}$ is P_α-generic filter over V and we have
$$G\lceil\alpha = G \cap P_\alpha.$$

(5) (CH) $P^*_{\omega_2}$ has the ω_2-cc.

Lemma. (1) Let $p \in P^*_{\omega_2}$ and $p \in X \prec (\mathcal{P}_{<\omega_2}, \mathcal{M}_{<\omega_2})$. Then there exists $q \in P^*_{\omega_2}$ such that $q \leq p$ in $P^*_{\omega_2}$ and $X \cap \omega_2 = S^q(X)$.

(2) Let $X \prec (\mathcal{P}_{<\omega_2}, \mathcal{M}_{<\omega_2})$. Let $q \in P^*_{\omega_2}$ such that $X \cap \omega_2 = S^q(X)$. Then q is $(P^*_{\omega_2}, X)$-gg.

Lemma. $\Vdash_{P^*_{\omega_2}}$ "T^* remains weakly Suslin and Aronszajn".

Lemma. $\Vdash_{P^*_{\omega_2}}$ "For any Aronszajn tree T, there exists an uncountable antichain $A \subset T$".

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