Calculation of invariant rings and their divisor class groups by cutting semi-invariants

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Abstract

Let G be an affine connected algebraic group acting regularly on an affine Krull scheme $X = \mathrm{Spec}(R)$ over an algebraically closed field K of any characteristic. We study on the minimal calculation of the ring R^G of invariants of G in R and their class groups by cutting prime semi-invariants which form free modules over R^G.

MSC: primary 13A50, 14R20, 20G05; secondary 14L30, 14M25
Keywords: algebraic torus; pseudo-reflection; Krull domain; semi-invariant; divisor class group

1 Introduction

Let G be an affine algebraic group over an algebraically closed field K of arbitrary characteristic p. Let R be an integral domain containing K as a subfield. We say that (R, G) a K-regular action of G on R, if G acts on R as a rational G-module over K which induces the homomorphism $G \to \mathrm{Aut}_{K-algebra}(R)$ (e.g., [12]). Let $\mathrm{U}(R)$ denote the group of all units in R and $\mathrm{U}_K(R)$ the quotient group of $\mathrm{U}(R)$ by the multiplicative group $\mathrm{U}(K) = K^\times$ of K. In general $\mathrm{U}_K(R)$ is torsion-free, as K is algebraically closed. We say that a non-zero element f of R is said to be a non-zero semi-invariant of R relative to χ, if the map

$$\chi : G \ni \sigma \mapsto \frac{\sigma(f)}{f} \in \mathrm{U}(K)$$

is a rational character of G. In order to calculate rings of invariants and their class groups, we can cut some prime semi-invariants and explain this viewpoint in the following example:

*Partially supported by Grant No. 26400019: the Japan Society for the Promotion of Sciences.

Example 1.1 *Let* $\boldsymbol{C}[X_1, X_2, X_3]$ *be the* 3*-dimensional polynomial ring over the complex number field* $\boldsymbol{C}$*. Let* $\boldsymbol{G}_m$ *be the multiplicative group* $\boldsymbol{C}^\times$ *whose action on this algebra in such a way that* $\boldsymbol{G}_m \ni t$ *acts on* $\{X_1, X_2, X_3\}$ *by*

$$\mathrm{diag}[t^2, t^{-1}, t^{-1}].$$

Then we have
(1) $\boldsymbol{C}[X_1, X_2, X_3]^{\boldsymbol{G}_m} = \boldsymbol{C}[X_1X_2^2, X_1X_2X_3, X_1X_3^2]$.
(2) The stabilizer $(\boldsymbol{G}_m)_{X_1} = \langle \mathrm{diag}[1, -1, -1] \rangle$ *of* $\boldsymbol{G}_m$ *at* X_1 *on* $\{X_1, X_2, X_3\}$.
(3) $\boldsymbol{C}[X_1, X_2, X_3]^{(\boldsymbol{G}_m)_{X_1}} = \boldsymbol{C}[X_1, X_2^2, X_2X_3, X_3^2]$.
(4) The divisor class group $\mathrm{Cl}(\boldsymbol{C}[X_1, X_2, X_3]^{\boldsymbol{G}_m}) \cong \boldsymbol{Z}/2\boldsymbol{Z}$ *which is isomorphic to*

$$\mathrm{Hom}((\boldsymbol{G}_m)_{X_1}, \boldsymbol{C}^*) \cong \mathrm{Cl}(\boldsymbol{C}[X_1, X_2, X_3]^{(\boldsymbol{G}_m)_{X_1}}).$$

(5) There is the isomorphism

$$\boldsymbol{C}[X_1, X_2, X_3]^{(\boldsymbol{G}_m)_{X_1}}/(X_1 - 1) \cong \boldsymbol{C}[X_1, X_2, X_3]^{\boldsymbol{G}_m}$$

induced by

$$\psi : \boldsymbol{C}[X_1, X_2, X_3] \to \boldsymbol{C}[X_1, X_2, X_3]$$

$(\psi(X_1) = 1, \psi(X_2) = X_2, \psi(X_3) = X_3)$.

The purpose of this paper is to generalize the assertion of this example to in the case of factorial (or Krull) domains with affine algebraic group actions in characteristic-free.

2 Preliminaries

Let $\mathcal{Q}(A)$ denote the total quotient ring of a ring A and

$$\mathrm{Ht}_1(A) := \{\mathfrak{P} \in \mathrm{Spec}(A) \mid \mathrm{ht}(\mathfrak{P}) = 1\}.$$

For an integral domain A and a subring B of A such that $B = \mathcal{Q}(B) \cap A$ and $\mathcal{Q}(B) \subseteq \mathcal{Q}(A)$, we denote by

$$\mathrm{Ht}_1(A, B) := \{\mathfrak{P} \in \mathrm{Ht}_1(A) \mid \mathfrak{P} \cap B \in \mathrm{Ht}_1(B)\},$$

$$\mathrm{Ht}_1^{(2)}(A, B) := \{\mathfrak{P} \in \mathrm{Ht}_1(A) \mid \mathrm{ht}(\mathfrak{P} \cap B) \geq 2)\}$$

and, for $\mathfrak{p} \in \mathrm{Ht}_1(B)$, by

$$\mathrm{Over}_{\mathfrak{p}}(A) := \{\mathfrak{P} \in \mathrm{Ht}_1(A) \mid \mathfrak{P} \cap B = \mathfrak{p}\}.$$

Especially suppose that A is a Krull domain (e.g., [1]). Let $\mathrm{v}_{A,\mathfrak{P}}$ be the discrete valuation defined by $\mathfrak{P} \in \mathrm{Ht}_1(A)$ of A. Denote by $\mathrm{Div}(A)$ (resp. $\mathrm{PDiv}(A)$, $\mathrm{Cl}(A)$) the divisor group (resp. the group of principal divisors, the divisor class group) of A. For a subring B of A such that $B = \mathcal{Q}(B) \cap A$, B is a Krull domain (e.g., [1, 3]) and every $\mathrm{Over}_{\mathfrak{p}}(A)$ is non-empty and finite. Let $\mathrm{e}(\mathfrak{P}, \mathfrak{p}) = \mathrm{v}_{A,\mathfrak{P}}(\mathfrak{p}A)$ be the ramification index of $\mathfrak{P} \in \mathrm{Over}_{\mathfrak{p}}(A)$ for a prime ideal $\mathfrak{p} \in \mathrm{Ht}_1(B)$. If all ramification indices of minimal prime ideals are equal to 1, the extension $B \to A$ is said to be divisorially unramified (cf. [7]).

Consider an action of a group G on a ring R as automorphisms. For a prime ideal $\mathfrak{P}$ of R, let

$$\mathcal{I}_G(\mathfrak{P}) = \{\sigma \in G \mid \sigma(x) - x \in \mathfrak{P} \ (x \in R)\}$$

which is referred to as the *inertia group* of $\mathfrak{P}$ under this action (for the classical case, see [5]). Let $Z^1(G, \mathrm{U}(R))$ be the group of 1cocycles of G on the unit group $\mathrm{U}(R)$ of R. For a 1-cocycle χ,

$$R_\chi := \{x \in R \mid \sigma(x) = \chi(\sigma)x \ (\sigma \in G)\},$$

which is a module over the invariant subring R^G.

The next theorem is a generalization of [11] and is fundamental in this paper:

Theorem 2.1 (cf. [7]) *Let R be a Krull domain acted by a group G as automorphisms. For a cocycle $\chi \in Z^1(G, \mathrm{U}(R))$, R_χ is a free R^G-module if and only if the following conditions are satisfied:*

(i) $\dim \mathcal{Q}(R^G) \otimes_{R^G} R_\chi = 1$

(ii) There is a nonzero element $f \in R_\chi$ satisfying

$$\forall \mathfrak{p} \in \mathrm{Ht}_1(R^G) \Rightarrow \exists \mathfrak{P} \in \mathrm{Over}_{\mathfrak{p}}(R) \textit{ such that } \mathrm{v}_{R,\mathfrak{P}}(f) < \mathrm{v}_{R,\mathfrak{P}}(\mathfrak{p}R)).$$

Here the condition *(i)* holds, if $R_\chi \cdot R_{-\chi} \neq \{0\}$.

Algebraic groups are affine and defined over a fixed algebraically closed field K of an arbitrary characteristic p. Let $\mathfrak{X}(G)$ be the group of rational characters of an algebraic group G expressed as an additive group with zero. The K-algebras R are not necessarily finite generated as algebras over K.

A subset N of a set M with an action of G is said to be G-invariant, if N is invariant under the action of G on M. In this case $G|_N$ denote the group consisting of the restriction $\sigma|_N$ of all $\sigma \in G$ to N, which is called *the group G on N*.

Pseudo-reflections on finite-dimensional vector spaces are defined in [2] and should be generalized as follows:

Definition 2.2 (Pseudo-reflections of actions) *Suppose that R is a Krull K-domain with (R, G) a regular action of an algebraic group G. Define the subgroup*

$$\mathfrak{R}(R, G) := \left\langle \bigcup_{\mathfrak{P} \in \mathrm{Ht}_1(R, R^G)} \mathcal{I}_G(\mathfrak{P}) \right\rangle$$

of G which is called the pseudo-reflection group of the action (R, G).

Finiteness of pseudo-reflections of regular actions characterize reductivity of algebraic groups. We have

Theorem 2.3 (cf. [8]) *Let G^0 be the identity component of an algebraic group G. Then the following conditions are equivalent:*

(i) G^0 is reductive.

(ii) $\mathfrak{R}(R, G)$ is finite on R for any Krull K-domain R with a regular action of G.

3 The abstract descent of class groups

In this section, suppose that A is Krull. For a subset Γ of $\mathcal{Q}(A)$ satisfying $\gamma \cdot \Gamma \subset A$ for some $\gamma \in A$, let $\mathrm{div}_A(\Gamma)$ be the divisor of ΓA on A. On the other hand, let $\mathrm{I}_A(D)$ be the divisorial fractional ideal of A defined by the divisor D on A. Consider a K-subalgebra B of A satisfying $\mathcal{Q}(B) \cap A = B$. For each $\mathfrak{p} \in \mathrm{Ht}_1(B)$, set

$$d_{\mathfrak{p}} = \sum_{\mathfrak{P} \in \mathrm{Over}_{\mathfrak{p}}(A)} \mathrm{v}_{A,\mathfrak{P}}(\mathfrak{p}A)\mathrm{div}_A(\mathfrak{P}) \in \mathrm{Div}(A).$$

Define the subgroup

$$E^*(A, B) := \Big(\bigoplus_{\mathfrak{p} \in \mathrm{Ht}_1(B)} \boldsymbol{Z} d_{\mathfrak{p}} \Big) \oplus \mathrm{Bup}(A, B)$$

of $\mathrm{Div}(A)$ where $\mathrm{Bup}(A, B) = \bigoplus_{\mathfrak{P} \in \mathrm{Ht}_1(A), \mathrm{ht}(\mathfrak{P} \cap B) \geqq 2} \boldsymbol{Z} \mathrm{div}_A(\mathfrak{P})$. Let

$$\Phi^*_{A,B} : E^*(A, B) \to \mathrm{Div}(B)$$

be the homomorphism defined by the composite of the projection

$$E^*(A, B) \to \bigoplus_{\mathfrak{p} \in \mathrm{Ht}_1(B)} \boldsymbol{Z} d_{\mathfrak{p}}$$

and the isomorphism

$$\bigoplus_{\mathfrak{p} \in \mathrm{Ht}_1(B)} \boldsymbol{Z} d_{\mathfrak{p}} \ni \sum_{\mathfrak{p}} a_{\mathfrak{p}} d_{\mathfrak{p}} \mapsto \sum_{\mathfrak{p}} a_{\mathfrak{p}} \mathrm{div}_S(\mathfrak{p}) \in \mathrm{Div}(B)$$

Set $\mathrm{Div}_A(\mathrm{U}(\mathcal{Q}(B))) := \{\mathrm{div}_A(\gamma) \mid \gamma \in \mathrm{U}(\mathcal{Q}(B))\} \subset \mathrm{PDiv}(A)$. Then

$$\mathrm{PDiv}(B) \ni D \mapsto \mathrm{div}_A(\mathrm{I}_B(D)) \in \mathrm{Div}_A(\mathrm{U}(\mathcal{Q}(S)))$$

is an isomorphism whose inverse is the restriction $\Phi^*_{R,S}|_{\mathrm{Div}_A(\mathrm{U}(\mathcal{Q}(B)))}$. Since

$$\mathrm{Div}_A(\mathrm{U}(\mathcal{Q}(B))) \subset E^*(A, B),$$

we define $E(A, B) := E^*(A, B)/\mathrm{Div}_A(\mathrm{U}(\mathcal{Q}(B)))$. Moreover define the subgoup

$$L(A, B) := \{f \in \mathrm{U}(\mathcal{Q}(A)) \mid \mathrm{div}_A(f) \in E^*(A, B)\}$$

of $\mathrm{U}(\mathcal{Q}(A))$. Then:

Theorem 3.1 *Under the circumstances as above, we obtain the sequences*

$$0 \to (\mathrm{Div}_A(\mathrm{U}(\mathcal{Q}(B))) + \mathrm{Bup}(A, B))/\mathrm{Div}_A(\mathrm{U}(\mathcal{Q}(B)) \to E(A, B) \to \mathrm{Cl}(B) \to 0$$

$$0 \to \frac{L(A, B)/\mathrm{U}(\mathcal{Q}(B))}{\mathrm{U}(A)/\mathrm{U}(B)} \to E(A, B) \to \mathrm{Cl}(A)$$

which are exact.

We introduce the concept of redundant prime elements which partially generate the subring C of A over B as follows:

Definition 3.2 (Paralleled linear hulls) *Consider an intermediate subring C of A such that $C = \mathcal{Q}(C) \cap A$ and $B \subseteq C$. The pair $(C, \{f_1, \dots, f_m\})$ is defined to be a paralleled linear hull of B with respect to f_i $(1 \le i \le m)$, if the composite of the inclusion and the canonical epimorphism*

$$\begin{array}{ccc} B & \xrightarrow{\ \subseteq\ } & C \\ & \underset{\cong}{\searrow} \quad \circlearrowleft & \downarrow \text{can.} \\ & & C/(\sum_{i=1}^{m} C(f_i - 1)) \end{array}$$

induces an isomorphism, f_i $(1 \le i \le m)$ are algebraically independent over $\mathcal{Q}(B)$ and

$$\mathrm{Cl}(B) \cong \mathrm{Cl}(C).$$

Note in general $C \neq B[f_1, \dots, f_m]$.

4 Graded structures and paralleled linear hulls

Let S be an integral domain which is a $\boldsymbol{Z}^m$-graded algebra

$$S = \bigoplus_{\boldsymbol{i} \in \boldsymbol{Z}^m} S_{\boldsymbol{i}}$$

over $S_{\boldsymbol{0}}$. Then if S is Krull, so is $S_{\boldsymbol{0}}$, because $S_{\boldsymbol{0}} = \mathcal{Q}(S_{\boldsymbol{0}}) \cap S$.

Definition 4.1 (half primary $\boldsymbol{Z}^m$-freeness) *We say that S is half primary $\boldsymbol{Z}^m$-free with respect to $\{f_1, \ldots, f_m\}$, if*

$$S_{\boldsymbol{i}} = S_{(i_1, \ldots, i_m)} = S_{\boldsymbol{0}} \prod_{j=1}^{m} f_j^{i_j}$$

for any $i_j \geqq 0$ and f_j, $1 \leqq j \leqq m$, is homogeneous prime element in S of degree $(0, \ldots, 0, 1, 0, \ldots, 0)$ having 1 at the j-th part.

Theorem 4.2 *Suppose that S is a $\boldsymbol{Z}^m$-graded Krull domain. If S is half primary $\boldsymbol{Z}^m$-free with respect to $\{f_1, \ldots, f_m\}$, then $(S, \{f_1, \ldots, f_m\})$ is a paralleled linear hull of $S_{\boldsymbol{0}}$.*

Put $\boldsymbol{Z}_{\leq 0} := \{k \in \boldsymbol{Z} \mid k \leq 0\}$ and let $\boldsymbol{Z}^m_{\leq 0}$ be the direct product of k-copies of $\boldsymbol{Z}_{\leq 0}$. For a subset W of S, let W^{hom} be the set consisting homogenous elements of W in S. Let

$$U_S := \{h \in S^{\text{hom}} \mid h \neq 0, \deg(h) \in \boldsymbol{Z}^m_{\leq 0}\}.$$

For a subset Ω of Spec S, let Ω^{hom} be the set of all homogeneous prime ideals in Ω. A divisor

$$D = \sum_{\mathfrak{P} \in \text{Ht}_1(S)} a_{\mathfrak{P}} \operatorname{div}_S(\mathfrak{P})$$

of $\text{Div}(S)$ is said to be homogeneous, if all prime ideals in

$$\text{supp}_S(D) := \{\mathfrak{P} \in \text{Ht}_1(S) \mid a_{\mathfrak{P}} \neq 0\}$$

are homogeneous. For a subset of $\mathcal{D}$ of $\text{Div}(S)$, we put

$$\mathcal{D}^{\text{hom}} := \{D \in \mathcal{D} \mid D \text{ is homogeneous}\},$$

$$\text{Ht}_1(S)^{\text{hom}}_0 := \text{Ht}_1(S)^{\text{hom}} \backslash \{Sf_1, \ldots, Sf_m\}$$

and

$$\text{Div}(S)^{\text{hom}}_0 := \{D \in \text{Div}(S)^{\text{hom}} \mid \text{supp}_S(D) \cap \{Sf_1, \ldots, Sf_m\} = \emptyset\}.$$

Lemma 4.3 *Under the circumstances as above we have*

(i) $\mathrm{Cl}(U_S^{-1}S) = \{0\}$

(ii) $\mathrm{Div}(S)_0^{\mathrm{homo}} \longrightarrow \mathrm{Cl}(S)$ *is an epimorphism.*

(iii) $\mathrm{Ht}_1(S)_0^{\mathrm{hom}} \ni \mathfrak{P} \longmapsto \mathfrak{P} \cap S_{\mathbf{0}} \in \mathrm{Ht}(S_{\mathbf{0}})$ *is bijective and* $\mathrm{e}(\mathfrak{P}, \mathfrak{P} \cap S_{\mathbf{0}}) = 1$.

(iv) The composite $\mathrm{Div}(S)_0^{\mathrm{homo}} \hookrightarrow E^*(S, S_{\mathbf{0}}) \underset{\Phi^*_{S,S_0}}{\longrightarrow} \mathrm{Div}(S_{\mathbf{0}})$ *is an isomorphism and induces*

$$\mathrm{PDiv}(S) \cap \mathrm{Div}(S)_0^{\mathrm{homo}} \cong \mathrm{PDiv}(S_{\mathbf{0}}).$$

This follows from the idea of M. Nagata on homogeneous localization (e.g., [3]).

By Lemma 4.3 we must have the isomorphism

$$\mathrm{Cl}(S) \cong \mathrm{Cl}(S_{\mathbf{0}}).$$

The remainder of the sketch of the proof of Theorem 4.2 is omitted.

5 Toric quotients

In this section let (R, G) be a regular action of a connected algebraic group G on a Krull domain R containing K as a subring.

Using Nagata's pseudo-geometric rings ([5]) and Rosenlicht's theorem on $\mathrm{U}_K(R')$ of affine normal domains R', we can generalize the result of [4] without the assumption of finite generations of R as follows.

Theorem 5.1 (cf. [10]) *Let f be a nonzero element of $Q(R)$. If Rf is invariant under the action of G, then Kf is G-invariant and, moreover if $\mathfrak{P} \cap R^G \neq \{0\}$ for any $\mathfrak{P} \in \mathrm{Ht}_1(R)$ such that $\mathrm{v}_{R,\mathfrak{P}}(f) < 0$, then*

$$G \ni \sigma \mapsto \frac{\sigma(f)}{f} \in \mathrm{U}(K)$$

is a rational character of G.

By this theorem, for a nonzero $f \in R$ satisfying that Rf is G-invariant, the symbol $\delta_{f,G}$ is denoted to the homomorphism

$$\delta_{f,G} : G \ni \sigma \mapsto \frac{\sigma(f)}{f} \in \mathrm{U}(K).$$

Lemma 5.2 *We have:*

(i) If the set $\bigcup_{\mathfrak{p}\in\Lambda} \mathrm{Over}_{\mathfrak{p}}(R)$ *consists of principal ideals, then it is a finite set, where* $\Lambda := \{\mathfrak{p} \in \mathrm{Ht}_1(R^G) \mid |\mathrm{Over}_{\mathfrak{p}}(R)| \geq 2\}$.

(ii) If the set $\mathrm{Ht}_1^{(2)}(R, R^G)$ *consists of principal ideals, then it is a finite set.*

This finiteness follows from Theorem 2.1 and $\mathrm{rank}(\mathfrak{X}(G)) < \infty$.

Assumption 5.3 *Suppose that the both sets of Lemma 5.2 consist of principal ideals of R.*

By this there exist non-associated prime elements $f_1, \ldots, f_m$ of R such that

$$|\{Rf_1, \ldots, Rf_m\} \cap \mathrm{Over}_{\mathfrak{p}}(R)| = |\mathrm{Over}_{\mathfrak{p}}(R)| - 1$$

for every $\mathfrak{p} \in \mathrm{Ht}_1(R^G)$ and

$$\{Rf_1, \ldots, Rf_m\} \backslash (\bigcup_{\mathfrak{p}\in\mathrm{Ht}_1(R^G)} \mathrm{Over}_{\mathfrak{p}}(R)) = \mathrm{Ht}_1^{(2)}(R, R^G).$$

According to Theorem 5.1, the homomorphisms $\delta_{f_i,G}$ are rational characters of G. Let H be the stabilizer

$$\mathrm{Stab}(G : f_1, \ldots, f_m) = \bigcap_{i=1}^{m} G_{f_i} = \bigcap_{i=1}^{m} \mathrm{Ker}(\delta_{f_i,G})$$

of G at the set $\{f_1, \ldots, f_m\}$.

From the choice of f_i and Theorem 2.1, we must have

$$R_{\sum_i a_i\delta_{f_i,G}} = R^G \prod_i f_i^{a_i} \tag{5.1}$$

for any integer $a_i \geq 0$ $(1 \leq i \leq m)$ and put

$$R^{\boldsymbol{f}} = \sum_{a_1,\ldots,a_m\in\boldsymbol{Z}} R_{\sum_i a_i\delta_{f_i,G}} \subset R$$

which is a K-subalgebra of R^H. Clearly $R^H = R^{\boldsymbol{f}}$ in the case where the ground field K is of characteristic $p = 0$. The equalities (5.1) imply that the subgroup $\langle\delta_{f_1,G}, \ldots, \delta_{f_m,G}\rangle$ of $\mathfrak{X}(G)$ is free of rank m. On the other hand

$$R^{\boldsymbol{f}} = \mathcal{Q}(R^{\boldsymbol{f}}) \cap R$$

and hence the K-subalgebra $R^{\boldsymbol{f}}$ is a Krull domain with the $\boldsymbol{Z}^m$-graded structure defined by the homogeneous part

$$R^{\boldsymbol{f}}_{\boldsymbol{a}} = R_{\sum_i a_i\delta_{f_i,G}}$$

of degree $\boldsymbol{a} = (a_1, \ldots, a_m) \in \boldsymbol{Z}^m$. Consequently, from (5.1) we infer that, for $S = R^{\boldsymbol{f}}$ and $S_0 = R^G$, S is half primary $\boldsymbol{Z}^m$-free with respect to $\{f_1, \ldots, f_m\}$.

Theorem 5.4 *Under the circumstances as above,* $(R^{\boldsymbol{f}}, \{f_1, \dots, f_m\})$ *is a paralleled linear hull of* R^G.

This theorem follows from Theorem 4.2.

Next, the class group $\mathrm{Cl}(R^{\boldsymbol{f}}) \cong \mathrm{Cl}(R^G)$ shall be studied by the abstract descent method. For this purpose we introduce the notation as bellow: Consider a K-subalegba M of R such that $M \supset \{f_1, \dots, f_m\}$ and $\mathcal{Q}(M) \cap R = M$ which is invariant under the action of G. Since M is a Krull domain, for a subset $\mathcal{D}$ of the divisor group $\mathrm{Div}(M)$ of M, let us define the subset

$$\mathcal{D}_{\boldsymbol{f}(M)} := \{D \in \mathcal{D} \mid \mathrm{supp}_M(D) \cap \{Mf_1, \dots, Mf_m\} = \emptyset\}$$

without prime elements f_i as supports of divisors. The group G acts on $\mathrm{Div}(M)$ naturally. If $\mathcal{D}$ is an G-invariant subset, let $\mathcal{D}^G$ denote the set consisting G-invariant divisors of $\mathcal{D}$ and, for a simplicity, denote $\mathcal{D}^G_{\boldsymbol{f}(M)}$ by the set $\mathcal{D}^G \cap \mathcal{D}_{\boldsymbol{f}(M)}$.

As $R^{\boldsymbol{f}}$ is invariant under the action of G on R, we see $\mathrm{Ht}_1(R^{\boldsymbol{f}})^{\mathrm{homo}} = \mathrm{Ht}_1(R^{\boldsymbol{f}})^G$ and

$$\mathrm{Div}(R^{\boldsymbol{f}})_0^{\mathrm{hom}} = \mathrm{Div}(R^{\boldsymbol{f}})^G_{\boldsymbol{f}(R^{\boldsymbol{f}})}. \tag{5.2}$$

Recalling $\mathcal{Q}(R^{\boldsymbol{f}}) \cap R = R^{\boldsymbol{f}}$, we have

$$\Phi^*_{R,R^{\boldsymbol{f}}} : E^*(R, R^{\boldsymbol{f}}) \to \mathrm{Div}(R^{\boldsymbol{f}})$$

which is an isomorphism, since $\mathrm{Bup}(R, R^{\boldsymbol{f}}) = \{0\}$ follows from Assumption 5.3. For any $\mathfrak{p} \in \mathrm{Ht}_1(R^{\boldsymbol{f}})_0^{\mathrm{hom}}$, $\mathrm{ht}(\mathfrak{p} \cap R^G) = 1$ and $\mathrm{Over}_{\mathfrak{p} \cap R^G}(R^{\boldsymbol{f}}) = \{\mathfrak{p}\}$, which shows the set $\mathrm{Over}_{\mathfrak{p}}(R)$ consists of a unique prime ideal and is G-invariant and $\mathrm{Over}_{\mathfrak{p}}(R) = \mathrm{Over}_{\mathfrak{p} \cap R^G}(R)$. Thus we have the commutative diagram

$$\begin{array}{ccc}
\mathrm{Div}(R)^G_{\boldsymbol{f}(R)} \cap E^*(R, R^{\boldsymbol{f}}) & \xrightarrow{\subset} & E^*(R, R^{\boldsymbol{f}}) \\
\big\downarrow & & \cong \big\downarrow \Phi^*_{R,R^{\boldsymbol{f}}} \\
\mathrm{Div}(R^{\boldsymbol{f}})^G_{\boldsymbol{f}(R^{\boldsymbol{f}})} & \xrightarrow[\subset]{} & \mathrm{Div}(R^{\boldsymbol{f}})
\end{array}$$

and $\mathrm{Div}(R)^G_{\boldsymbol{f}(R)} \cap E^*(R, R^{\boldsymbol{f}}) \cong \mathrm{Div}(R^{\boldsymbol{f}})^G_{\boldsymbol{f}(R^{\boldsymbol{f}})}$. Putting

$$L(R, R^{\boldsymbol{f}})_{\boldsymbol{f}} := \{g \in L(R, R^{\boldsymbol{f}}) \mid \mathrm{div}_R(g) \in \mathrm{Div}(R)_{\boldsymbol{f}(R)}\},$$

we have the exact sequence

$$0 \to L(R, R^{\boldsymbol{f}})_{\boldsymbol{f}} / (\mathrm{U}(R) \cap L(R, R^{\boldsymbol{f}})_{\boldsymbol{f}}) \to \mathrm{Div}(R)^G_{\boldsymbol{f}(R)} \cap E^*(R, R^{\boldsymbol{f}}) \to \mathrm{Cl}(R).$$

Moreover putting

$$L(R^{\boldsymbol{f}})_{\boldsymbol{f}} := \{h \in \mathrm{U}(\mathcal{Q}(R^{\boldsymbol{f}})) \mid \mathrm{div}_{R^{\boldsymbol{f}}}(h) \in \mathrm{Div}(R^{\boldsymbol{f}})^G_{\boldsymbol{f}(R^{\boldsymbol{f}})}\},$$

by Lemma 4.3 and (5.2) we have the exact sequence

$$0 \to L(R^{\boldsymbol{f}})_{\boldsymbol{f}}/(\mathrm{U}(R^{\boldsymbol{f}}) \cap L(R^{\boldsymbol{f}})_{\boldsymbol{f}}) \to \mathrm{Div}(R^{\boldsymbol{f}})^G_{\boldsymbol{f}(R^{\boldsymbol{f}})} \to \mathrm{Cl}(R^{\boldsymbol{f}}) \to 0$$

and $L(R^{\boldsymbol{f}})_{\boldsymbol{f}}/(\mathrm{U}(R^{\boldsymbol{f}} \cap L(R^{\boldsymbol{f}})_{\boldsymbol{f}}) \cong \mathrm{U}(\mathcal{Q}(R^G))/\mathrm{U}(R^G)$ whose isomorphism demoted to $\tilde{\Phi}^*_{R^{\boldsymbol{f}},R^G}$.

Consequently under the circumstances as above, we see

Theorem 5.5 *If R is factorial , then*

$$\mathrm{Cl}(R^G) \cong \mathrm{Cl}(R^{\boldsymbol{f}}) \cong \frac{L(R,R^{\boldsymbol{f}})_{\boldsymbol{f}}/(\mathrm{U}(R) \cap L(R,R^{\boldsymbol{f}})_{\boldsymbol{f}})}{L(R^{\boldsymbol{f}})_{\boldsymbol{f}}/(\mathrm{U}(R^{\boldsymbol{f}}) \cap L(R^{\boldsymbol{f}})_{\boldsymbol{f}})}$$

$$= \frac{L(R,R^{\boldsymbol{f}})_{\boldsymbol{f}}/(\mathrm{U}(R) \cap L(R,R^{\boldsymbol{f}})_{\boldsymbol{f}})}{\tilde{\Phi}^{*-1}_{R^{\boldsymbol{f}},R^G}(\mathrm{U}(\mathcal{Q}(R^G))/\mathrm{U}(R^G))}.$$

For any $g \in L(R,R^{\boldsymbol{f}})_{\boldsymbol{f}}$, as $\mathrm{div}_R(g)$ is G-invariant and

$$\mathrm{supp}_R(\mathrm{div}_R(g)) \subset \{\mathfrak{P} \in \mathrm{Ht}^1(R) \mid \mathfrak{P} \cap R^G \neq \{0\}\},$$

the subspace Kg is G-invariant and $\delta_{g,G} \in \mathfrak{X}(G)$. Suppose that

$$\mathrm{U}(R) \cap L(R,R^{\boldsymbol{f}})_{\boldsymbol{f}} \subset R^{\boldsymbol{f}}. \tag{5.3}$$

Then $\mathrm{Cl}(R^G) \cong L(R,R^{\boldsymbol{f}})_{\boldsymbol{f}}/L(R^{\boldsymbol{f}})_{\boldsymbol{f}}$. Put

$$\mathfrak{X}(H)_{R,\boldsymbol{f}} := \{\delta_{g,G}|_H \mid g \in L(R,R^{\boldsymbol{f}})_{\boldsymbol{f}}\}.$$

In case of $p = 0$ we see $R^H = R^{\boldsymbol{f}}$ and obtain

Corollary 5.6 *Suppose that R is factorial and the condition (5.3) holds. If $p = 0$, then*

$$\mathrm{Cl}(R^G) \cong \mathfrak{X}(H)_{R,\boldsymbol{f}}.$$

Moreover by [6, 8, 12] we have

Corollary 5.7 *Suppose that R is affine factorial K-domain with trivial units. Let (R,G) be a stable regular action of an algebraic torus G (i.e.,* $\mathrm{Spec}(R)$ *contains a non-empty open subset consisting of closed G-orbits, see [12]). If $p = 0$, then $\mathrm{Cl}(R^G) \cong \mathfrak{X}(H/\mathfrak{R}(R,H))$.*

In this case, the extension $R^H \to R^{\mathfrak{R}(R,H)}$ is divisorially unramified and $R^{\mathfrak{R}(R,H)}$ is factorial. Thus this follows from Corollary 5.6 for $R = R^{\mathfrak{R}(R,H)}$.

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