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Green's function of compressible Navier-Stokes around a hyperbolic contact discontinuity

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1. Introduction

In [1, 2, 3], one introduced Laplace transform in the time variable to obtain the solution of an inial-boundary value problems in the Laplace transform. We will use this concept to investigate a related problem for a contact discontinuity.

The one-dimensional compressible Navier-Stokes equation for ideal gas in the Lagrangian coordinate is

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = \mu(u_x/v)_x, \\ E_t + (up)_x = \mu(uu_x/v)_x + \kappa(T_x/v)_x, \end{cases} \tag{1.1}$$

where

$$\begin{cases} v \text{ (volume per unit mass)}, \\ u \text{ (fluid velocity)}, \\ T \text{ (thermal temperature)}, \\ p \equiv T/v \text{ (pressure)}, \\ E \equiv \frac{1}{2}u^2 + T/(\gamma-1) \text{ (energy)}, \\ \mu > 0 \text{ (dissipation constant)}, \\ \kappa > 0 \text{ (heat conductivity)}, \\ \gamma \geq 1 \text{ (gas constant for a polytropic ideal gas)}, \\ \mathsf{C}(\mathsf{U}) \equiv \sqrt{\gamma p/v}, \text{ (sound speed at rest)}, \mathsf{U} \equiv (v,u,T). \end{cases}$$

In this paper, the constants (γ, μ, κ) are assumed

$$(\gamma, \mu, \kappa) = (5/3, 1, 1) \tag{1.2}$$

when one considers the Laplace wave trains for the ease of expression. The main interest of this paper is to compute the Laplace transform of Green's function in t-variable for the linearized equation around a hyperbolic contact wave for the polytropic ideal gas in the Lagrangian coordinate

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \\ E_t + (up)_x = 0. \end{cases} \tag{1.3}$$

By a Galellian translation, one can assume the hyperbolic contact wave for (1.3) is a stationary contact wave:

$$\mathbb{W}(x,t) \equiv \begin{pmatrix} v \\ u \\ E \end{pmatrix}(x,t) = H(x)\begin{pmatrix} v_+ \\ 0 \\ v_+/(\gamma-1) \end{pmatrix} + (1-H(x))\begin{pmatrix} v_- \\ 0 \\ v_-/(\gamma-1) \end{pmatrix} \tag{1.4}$$

where $H(x)$ is the Heaviside function:

$$H(x) \equiv \begin{cases} 1 \text{ for } x > 0, \\ 0 \text{ else } . \end{cases}$$

The two end states $(\mathsf{U}_-, \mathsf{U}_+)$ of the hyperbolic contact discontinuity given in (1.4) is used to denote the hyperbolic wave itself, i.e.

$$\mathsf{U}_\pm \equiv (v_\pm, 0, v_\pm/(\gamma-1))^T.$$

It is conventional to develop an analysis in terms of vectors and matrices so that the developed methodology can be applied to various problems. Thus, one rewrites both systems in (1.1) and (1.3) as follows

$$\partial_t \mathsf{U} + \partial_x \mathsf{F}(\mathsf{U}) = \partial_x \mathsf{B}(\mathsf{U}) \partial_x \mathsf{U}, \tag{1.5}$$

$$\partial_t \mathsf{W} + \partial_x \mathsf{F}(\mathsf{W}) = \vec{0}, \tag{1.6}$$

where

$$\begin{cases} \mathsf{V} \equiv (v, u, E)^T, \\ \mathsf{F}(\mathsf{V}) \equiv (-u, p, up)^T, \\ \mathsf{B}(\mathsf{V}) \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\mu}{v} & 0 \\ 0 & \frac{(-\kappa(\gamma-1)+\mu)u}{v} & \frac{\kappa}{v} \end{pmatrix}. \end{cases}$$

2. Preliminaries

2.1. **Hyperbolic system.**

For the system (1.6), the matrix $\mathsf{F}'(\mathsf{W})$, $(\mathsf{W} = (v, u, E)^T)$, possesses three real eigenvalues,

$$\xi_1(\mathsf{W}) < \xi_2(\mathsf{W}) < \xi_3(\mathsf{W}),$$

$$\{\xi_1, \xi_2, \xi_3\} \equiv \{u - \sqrt{\gamma p/v}, u, u + \sqrt{\gamma p/v}\}.$$

For the system (1.6), the 2nd characteristic field $\xi_2(\mathsf{W})$ is linearly degenerated. A hyperbolic contact discontinuity $(\mathsf{U}_-, \mathsf{U}_+)$ is a two-valued function with pressure and velocity remain constant cross the discontinuity, i.e.

$$\mathsf{W}(x,t) = \begin{cases} \mathsf{U}_- \text{ for } x < ut, \\ \mathsf{U}_+ \text{ for } x > ut. \end{cases}$$

$$\begin{cases} u_- = u_+, \\ T_-/v_- = T_+/v_+. \end{cases}$$

For any given $p > 0$, one denotes the set of all end states $\mathsf{U}_\pm$ of *stationary* hyperbolic contact discontinuities with pressure p for (1.3) by $\mathscr{C}_p$:

$$\mathscr{C}_p \equiv \{\mathscr{G}(v;p) | \mathscr{G}(v;p) \equiv (v, 0, pv/(\gamma-1))^t, \ v > 0\},$$

and the eigenvalues ξ_i, $i = 1, 2, 3$, become

$$\{\xi_1, \xi_2, \xi_3\} \equiv \left\{-\sqrt{\gamma p/v}, \ 0, \ \sqrt{\gamma p/v}\right\}, \tag{2.1}$$

and the Jacobian $\mathsf{F}'(\mathsf{U})$ can be diagonalized as follows

$$\begin{pmatrix} 0 & -1 & 0 \\ -\frac{1}{v} & 0 & \frac{(\gamma-1)}{v} \\ 0 & p & 0 \end{pmatrix} = \mathsf{R} \cdot \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix} \cdot \mathsf{L}, \tag{2.2}$$

where the matrices L and R; and the right eigenvectors $\boldsymbol{r}_i$ and left eigenvectors $\boldsymbol{l}_i$ are given as follows

$$
\begin{cases}
\begin{pmatrix} \boldsymbol{r}_1 & \boldsymbol{r}_2 & \boldsymbol{r}_3 \end{pmatrix} \equiv \mathsf{R}(\mathsf{U}) \equiv \begin{pmatrix} -\frac{1}{p} & \frac{\gamma-1}{p} & -\frac{1}{p} \\ -\frac{\sqrt{\gamma}}{\sqrt{pv}} & 0 & \frac{\sqrt{\gamma}}{\sqrt{pv}} \\ 1 & 1 & 1 \end{pmatrix}, \\
\begin{pmatrix} \boldsymbol{l}_1 \\ \boldsymbol{l}_2 \\ \boldsymbol{l}_3 \end{pmatrix} \equiv \mathsf{L}(\mathsf{U}) \equiv \begin{pmatrix} -\frac{p}{2\gamma} & -\frac{\sqrt{pv}}{2\sqrt{\gamma}} & \frac{\gamma-1}{2\gamma} \\ \frac{p}{\gamma} & 0 & \frac{1}{\gamma} \\ -\frac{p}{2\gamma} & \frac{\sqrt{pv}}{2\sqrt{\gamma}} & \frac{\gamma-1}{2\gamma} \end{pmatrix}.
\end{cases} \tag{2.3}
$$

2.2. **Laplace transform of the Green's function around a constant state: System of ODE.**
The linearized equation around U_0 is

$$\Big(\partial_t + \mathsf{F}'(\mathsf{U}_0)\partial_x - \mathsf{B}(\mathsf{U}_0)\partial_x^2\Big)\mathsf{V} = 0. \tag{2.4}$$

The Green's $\mathbb{G}(x,t)$ for (2.4) is a matrix-valued solution of the initial value problem:

$$
\begin{cases}
\Big(\partial_t + \mathsf{F}'(\mathsf{U}_0)\partial_x - \mathsf{B}(\mathsf{U}_0)\partial_x^2\Big)\mathbb{G}(x,t) = 0, \\
\mathbb{G}(x,0) = \delta(x)\mathsf{I}.
\end{cases} \tag{2.5}
$$

Now, we introduce the Laplace transform with respect to the t-variable:

$$\mathbb{L}[h](x,s) \equiv \int_0^\infty e^{-st}h(x,t)dt, \ \text{(Laplace transform in } t).$$

Under the Laplace transform, (2.5) becomes the system of ODE:

$$(s + \mathsf{F}'(\mathsf{U}_0)\partial_x - \mathsf{B}(\mathsf{U}_0)\partial_x^2)\mathbb{L}[\mathbb{G}](x,s) = \delta(x)\mathsf{I}. \tag{2.6}$$

Stable manifold and Unstable manifold: Laplace wave numbers
The characteristic polynomial of the above ODE in λ is

$$\mathscr{P}_{\mathsf{U}_0}(s; -i\lambda) \text{ for } Re(s) > 0.$$

The roots of this characteristic polynomial are $\pm\lambda_1$ and $\pm\lambda_2$ and the roots λ_i satisfy

$$\mathrm{Re}(\lambda_k(s)) < 0 \text{ for } \mathrm{Re}(s) > 0, \tag{2.7}$$

as well as the asymptotic around $s = 0$

$$\lambda_1(s) = O(1)s, \ \lambda_2(s) = O(1)\sqrt{s}. \tag{2.8}$$

The stable manifold and unstable manifold of the ODE in (2.6) are the span of vectors:

$$
\begin{cases}
\text{Stable manifold} = span\{e^{\lambda_1 x}\mathbb{E}_1^+(s), e^{\lambda_2 x}\mathbb{E}_2^+(s)\} \text{ for } x > 0 \\
\text{Unstable manifold} = span\{e^{-\lambda_1 x}\mathbb{E}_1^-(s), e^{-\lambda_2 x}\mathbb{E}_2^-(s)\} \text{ for } x < 0.
\end{cases}
$$

Here,

$$\mathbb{E}_i^{\pm}(s) \in ker\Big(s\mathsf{I} \pm \lambda_i \mathsf{F}'(\mathsf{U}_0) - \lambda_i^2 \mathsf{B}(\mathsf{U}_0)\Big).$$

The stable manifold represents wave motions travelling towards the right, and the unstable manifold represents wave motions towards the left.

The vectors $\mathbb{E}_j^{\pm}(s)$ are normalized to satisfy

$$\begin{cases} \mathbb{E}_1^+(0) = \boldsymbol{r}_3(\mathsf{U}_0), \\ \mathbb{E}_2^+(0) = \boldsymbol{r}_2(\mathsf{U}_0), \\ \mathbb{E}_1^-(0) = \boldsymbol{r}_1(\mathsf{U}_0), \\ \mathbb{E}_2^-(0) = \boldsymbol{r}_2(\mathsf{U}_0). \end{cases} \tag{2.9}$$

Definition 2.1 (Laplace wave number and wave train)**.** *For (2.4), the notions of Laplace wave numbers, Laplace wave trains, Laplace spectral vectors are defined as follows:*
Let $\lambda = \pm\lambda_k(s)$, $k = 1, 2$, be the implicit functions given by $\mathscr{P}_{\mathsf{U}_0}(s, -i\lambda) = 0$ with the property (2.7)

$$\begin{cases} \lambda_k(s): & \textit{Laplace wave number,} \\ e^{\lambda_k(s)x}: & \textit{Laplace wave train travelling towards the right,} \\ e^{-\lambda_k(s)x}: & \textit{Laplace wave train travelling towards the left,} \\ \mathbb{E}_k^+(s): & \textit{Laplace wave vector towards the right,} \\ \mathbb{E}_k^-(s): & \textit{Laplace wave vector towards the left.} \end{cases} \tag{2.10}$$

The stable and unstable manifolds of ODE (2.6) can be express in terms of the Laplace wave trains and Laplace wave vectors; and one can express the solution of the ODE for $\mathbb{L}[\mathbb{G}](x,s)$ in terms of the Laplace wave train and Laplace wave vectors:

$$\mathbb{L}[\mathbb{G}](x,s) = \delta(x)\mathsf{J}_0(s) + H(x)(e^{\lambda_1 x}\mathbb{E}_1^+, e^{\lambda_2 x}\mathbb{E}_2^+)\mathsf{J}_+(s) + H(-x)(e^{-\lambda_1 x}\mathbb{E}_1^-, e^{-\lambda_2 x}\mathbb{E}_2^-)\mathsf{J}_-(s), \tag{2.11}$$

where $\mathsf{J}_\pm$ are 2×3 matrices, J_0 is a 3×3 matrix.

We assume that (1.2), i.e. $(\gamma, \mu, \kappa) = (5/3, 1, 1)$ to make the calculations easier.

3. Laplace wave numbers $\lambda_j(s)$ and Laplace wave vector

The Laplace wave number λ_i and the Laplace wave vector are defined by (2.10), which is a system of ODE to find a solution of the form $e^{\lambda x}\mathbb{E}$:

$$(s + \mathsf{F}'(\mathsf{U}_0)\partial_x - \mathsf{B}(\mathsf{U}_0)\partial_x^2)e^{\lambda x}\mathbb{E} = \vec{0}. \tag{3.1}$$

The solution λ is the root of the characteristic polynomial $p(\lambda)$:

$$p(\lambda) \equiv det(s\mathsf{I} + \mathsf{F}'(\mathsf{U}_0)\lambda - \mathsf{B}(\mathsf{U}_0)\lambda^2) = \frac{2\lambda^4(s+1)}{3v_0^2} - \frac{5\lambda^2 s(s+1)}{3v_0} + s^3.$$

$$\begin{cases} \lambda_1 = -\sqrt{6}s\sqrt{\dfrac{v_0}{\sqrt{s^2+26s+25}+5s+5}}, \\ \lambda_2 = -\dfrac{1}{2}\sqrt{s}\sqrt{\dfrac{\left(\sqrt{s^2+26s+25}+5s+5\right)v_0}{s+1}}. \end{cases} \tag{3.2}$$

Here, the asymptotic of λ_1 and λ_2 are:

$$\begin{cases} \lambda_1(s) = -\sqrt{\frac{3}{5}}s\sqrt{v_0} + O(1)s^2, & \text{at } s = 0, \\ \lambda_2(s) = -\sqrt{\frac{5}{2}}\sqrt{s}\sqrt{v_0}, \end{cases}$$
$$\begin{cases} \lambda_1(s) = -\sqrt{s}\frac{\sqrt{v_0}}{2}\left(\frac{O(1)}{s} + 2\right), & \text{at } s = \infty. \\ \lambda_2(s) = -\sqrt{s}\sqrt{\frac{3}{2}}\sqrt{v_0}\left(\frac{O(1)}{s} + 1\right), \end{cases} \tag{3.3}$$

The right eigenvectors of $\mathsf{F}'(\mathsf{U}_0)$ are

$$\begin{cases} \left(\boldsymbol{r}_1\boldsymbol{r}_2\boldsymbol{r}_3\right) \equiv \begin{pmatrix} -1 & \frac{2}{3} & -1 \\ -\frac{\sqrt{\frac{5}{3}}}{\sqrt{v_0}} & 0 & \frac{\sqrt{\frac{5}{3}}}{\sqrt{v_0}} \\ 1 & 1 & 1 \end{pmatrix}, \\ \mathsf{F}'(\mathsf{U}_0) = \begin{pmatrix} -1 & \frac{2}{3} & -1 \\ -\frac{\sqrt{\frac{5}{3}}}{\sqrt{v_0}} & 0 & \frac{\sqrt{\frac{5}{3}}}{\sqrt{v_0}} \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -\mathsf{C}_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathsf{C}_0 \end{pmatrix} \begin{pmatrix} -1 & \frac{2}{3} & -1 \\ -\frac{\sqrt{\frac{5}{3}}}{\sqrt{v_0}} & 0 & \frac{\sqrt{\frac{5}{3}}}{\sqrt{v_0}} \\ 1 & 1 & 1 \end{pmatrix}^{-1}; \end{cases}$$

and the normalized Laplace wave vectors are

$$\begin{cases} \mathbb{E}_1^+(s) \equiv \begin{pmatrix} -1 \\ \frac{50\sqrt{\frac{5}{3}}}{(50-19s)\sqrt{v_0}} \\ \frac{s+50}{50-19s} \end{pmatrix}, \ \mathbb{E}_2^+(s) \equiv \begin{pmatrix} \frac{2}{3} \\ -\frac{10\sqrt{10}\sqrt{s}}{(75-9s)\sqrt{v_0}} \\ \frac{12s+25}{25-3s} \end{pmatrix}, \\ \mathbb{E}_1^-(s) \equiv \begin{pmatrix} -1 \\ \frac{50\sqrt{\frac{5}{3}}}{(19s-50)\sqrt{v_0}} \\ \frac{s+50}{50-19s} \end{pmatrix}, \ \mathbb{E}_2^-(s) \equiv \begin{pmatrix} \frac{2}{3} \\ \frac{10\sqrt{10}\sqrt{s}}{(75-9s)\sqrt{v_0}} \\ \frac{12s+25}{25-3s} \end{pmatrix}. \end{cases} \tag{3.4}$$

4. Jumps and the Algebraic relationship

One continues to determine the matrices $\mathsf{J}_0(s)$ and $\mathsf{J}_\pm(s)$ given in (2.11). From (2.11) and the property that the $\delta(x)$-function singularity only arouses in $(1,1)$-entry of the Green's function so that

$$\mathsf{J}_0 = \begin{pmatrix} \mathsf{J}_0^1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.1}$$

Next, one sets

$$G_j(x,s) \equiv \mathbb{L}[\mathbb{G}] \cdot (\delta_j^1, \delta_j^2, \delta_j^3)^T;$$

and $[G_k]$ and $[\partial_x G_k]$:

(4.2)

$$\begin{cases} [G_k] \equiv G_k(0+,s) - G_k(0-,s) = \Big((\mathbb{E}_1^+, \mathbb{E}_2^+)\mathsf{J}_+ - ((\mathbb{E}_1^-, \mathbb{E}_2^-)\mathsf{J}_-\Big)(\delta_k^1, \delta_k^2, \delta_k^3)^T, \\ [\partial_x G_k] \equiv \partial_x G_k(0+,s) - \partial_x G_k^i(0-,s) = \Big(\lambda_1(\mathbb{E}_1^+, \lambda_2\mathbb{E}_2^+)\mathsf{J}_+ + (\lambda_1(\mathbb{E}_1^-, \lambda_2\mathbb{E}_2^-)\mathsf{J}_-\Big)(\delta_k^1, \delta_k^2, \delta_k^3)^T. \end{cases}$$

For each k, the function $G_k(x,s)$ is corresponding to the equation

$$\left(\begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} + \partial_x \begin{pmatrix} 0 & -1 & 0 \\ -\frac{p_0}{v_0} & -\frac{\mu}{v_0}\partial_x & \frac{\gamma-1}{v_0} \\ 0 & p_0 & -\frac{(\gamma-1)\kappa}{v_0}\partial_x \end{pmatrix}\right) G_k = \delta(x) \begin{pmatrix} \delta_k^1 \\ \delta_k^2 \\ \delta_k^3 \end{pmatrix}. \tag{4.3}$$

For $k=1$, by balancing the $\delta(x)$-functions in the conservation law of mass and $\delta'(x)$-function in the momentum flux of (4.3) it results in

$$s\mathsf{J}_0^1 - [G_1^2] = 1, \tag{4.4}$$

$$-p_0\mathsf{J}_0^1 - \mu[G_1^2] = 0. \tag{4.5}$$

Then, the balance of the δ-function in the momentum equation becomes

$$-p_0[G_1^1] + (\gamma-1)[G_1^3] - \mu[G_1^2] = 0. \tag{4.6}$$

Next, the continuity in energy and conservation law of energy yield that

$$[G_1^3] = 0, \tag{4.7}$$

$$p_0[G_1^2] - (\gamma-1)\kappa[\partial_x G_1^3]/v_0 = 0. \tag{4.8}$$

For $k = 2, 3$, the equations for $[G_k^i]$ and $[\partial_x G_k^i]$ are

$$\begin{cases} [G_k^2] = [G_k^3] = 0, & \text{(continuities in velocity and energy)} \\ -p_0[G_k^1]/v_0 + (\gamma-1)[G_k^3]/v_0 - \mu[\partial_x G_k^2]/v_0 = \delta_k^2, & \text{(momentum flux)} \\ p_0[G_k^2] - (\gamma-1)\kappa[\partial_x G_k^3]/v_0 = \delta_k^3, & \text{(energy flux).} \end{cases} \tag{4.9}$$

Thus,

$$\mathsf{J}_0^1(s) = \mu/(\mu s + p_0),$$

and one has the following 12 jump conditions:

$$\begin{cases} [G_1^2] = -p_0/(\mu s + p_0), \\ [G_1^3] = 0, \\ [\partial_x G_1^2] = -[G_1^1]p_0/\mu, \\ (\gamma-1)\kappa[\partial_x G_1^3] - p_0^2 v_0/(\mu s + p_0) = 0, \end{cases} \quad \begin{cases} [G_2^2] = 0, \\ [G_2^3] = 0, \\ [\partial_x G_2^3] = 0, \\ [\partial_x G_2^2] = -([G_2^1]p_0 + v_0)/\mu, \end{cases} \quad \begin{cases} [G_3^2] = 0, \\ [G_3^3] = 0, \\ [\partial_x G_3^2] = -[G_3^1]p_0/\mu, \\ -(\gamma-1)\kappa[\partial_x G_3^3] = v_0. \end{cases} \tag{4.10}$$

This gives 12 jump conditions on G_k, $k = 1, 2, 3$. Substitute these 12 jump conditions into (2.11), then it gives 12 equations on matrices J_- and J_+, where each $\mathsf{J}_\pm$ is a 2×3 matrix. One can solves $\mathsf{J}_\pm$ uniquely. It yields that

$$G_1(x,s) = \delta(x)\begin{pmatrix} \frac{1}{1+s} \\ 0 \\ 0 \end{pmatrix} + H(x)\Bigg(\begin{pmatrix} \frac{\sqrt{\frac{3}{2}}\left(\sqrt{s^2+26s+25}+s+1\right)\sqrt{\frac{v_0}{\sqrt{s^2+26s+25}+5s+5}}}{2(s+1)\sqrt{s^2+26s+25}} \\ -\frac{\sqrt{s^2+26s+25}+s+1}{4(s+1)\sqrt{s^2+26s+25}} \\ -\frac{3\sqrt{\frac{3}{2}}\sqrt{\frac{v_0}{\sqrt{s^2+26s+25}+5s+5}}}{\sqrt{s^2+26s+25}} \end{pmatrix} e^{\lambda_1 x} + \begin{pmatrix} \frac{\left(-\sqrt{s^2+26s+25}+s+25\right)\sqrt{\frac{\left(\sqrt{s^2+26s+25}+5s+5\right)v_0}{s+1}}}{8\sqrt{s}(s+1)(s+25)} \\ \frac{\sqrt{s^2+26s+25}-s-25}{4(s+1)(s+25)} \\ \frac{3\sqrt{\frac{\left(\sqrt{s^2+26s+25}+5s+5\right)v_0}{s+1}}}{4\sqrt{s}\sqrt{s^2+26s+25}} \end{pmatrix} e^{\lambda_2 x}\Bigg) + H(-x)\Bigg(\begin{pmatrix} \frac{\sqrt{\frac{3}{2}}\left(\sqrt{s^2+26s+25}+s+1\right)\sqrt{\frac{v_0}{\sqrt{s^2+26s+25}+5s+5}}}{2(s+1)\sqrt{s^2+26s+25}} \\ \frac{\sqrt{s^2+26s+25}+s+1}{4(s+1)\sqrt{s^2+26s+25}} \\ -\frac{3\sqrt{\frac{3}{2}}\sqrt{\frac{v_0}{\sqrt{s^2+26s+25}+5s+5}}}{\sqrt{s^2+26s+25}} \end{pmatrix} e^{-\lambda_1 x} + \begin{pmatrix} \frac{\left(\sqrt{s^2+26s+25}-s-1\right)\sqrt{\frac{\left(\sqrt{s^2+26s+25}+5s+5\right)v_0}{s+1}}}{8\sqrt{s}(s+1)\sqrt{s^2+26s+25}} \\ \frac{\sqrt{s^2+26s+25}-s-1}{4(s+1)\sqrt{s^2+26s+25}} \\ \frac{\left(\sqrt{s^2+26s+25}-s-1\right)\left(\sqrt{s^2+26s+25}+s+1\right)\sqrt{\frac{\left(\sqrt{s^2+26s+25}+5s+5\right)v_0}{s+1}}}{32\sqrt{s}(s+1)\sqrt{s^2+26s+25}} \end{pmatrix} e^{-\lambda_2 x}\Bigg), \tag{4.11}$$

$$
\begin{aligned}
(4.12)\quad & G_2(x,s) \\
&= H(x)\left(\begin{pmatrix} -\frac{\left(\sqrt{s^2+26s+25}+s+1\right)v_0}{4(s+1)\sqrt{s^2+26s+25}} \\ \frac{\left(\sqrt{s^2+26s+25}+s+1\right)v_0}{4\sqrt{6}(s+1)\sqrt{s^2+26s+25}\sqrt{\frac{v_0}{\sqrt{s^2+26s+25}+5s+5}}} \\ \frac{3v_0}{2\sqrt{s^2+26s+25}} \end{pmatrix} e^{\lambda_1 x} + \begin{pmatrix} \frac{\left(\sqrt{s^2+26s+25}-s-25\right)v_0}{4(s+1)(s+25)} \\ \frac{\sqrt{s}\left(-\sqrt{s^2+26s+25}+s+25\right)v_0}{2(s+1)(s+25)\sqrt{\frac{\left(\sqrt{s^2+26s+25}+5s+5\right)v_0}{s+1}}} \\ -\frac{3v_0}{2\sqrt{s^2+26s+25}} \end{pmatrix} e^{\lambda_2 x}\right) \\
&+H(-x)\left(\begin{pmatrix} \frac{\left(\sqrt{s^2+26s+25}+s+1\right)v_0}{4(s+1)\sqrt{s^2+26s+25}} \\ \frac{\left(\sqrt{s^2+26s+25}+s+1\right)v_0}{4\sqrt{6}(s+1)\sqrt{s^2+26s+25}\sqrt{\frac{v_0}{\sqrt{s^2+26s+25}+5s+5}}} \\ -\frac{3v_0}{2\sqrt{s^2+26s+25}} \end{pmatrix} e^{-\lambda_1 x} + \begin{pmatrix} \frac{\left(\sqrt{s^2+26s+25}-s-1\right)v_0}{4(s+1)\sqrt{s^2+26s+25}} \\ \frac{\sqrt{s}\left(\sqrt{s^2+26s+25}-s-1\right)v_0}{2(s+1)\sqrt{s^2+26s+25}\sqrt{\frac{\left(\sqrt{s^2+26s+25}+5s+5\right)v_0}{s+1}}} \\ \frac{3v_0}{2\sqrt{s^2+26s+25}} \end{pmatrix} e^{-\lambda_2 x}\right)
\end{aligned}
$$

$$
\begin{aligned}
(4.13)\quad & G_3(x,s) \\
&= H(x)\left(\begin{pmatrix} -\frac{\sqrt{6}\sqrt{\frac{v_0}{\sqrt{s^2+26s+25}+5s+5}}}{\sqrt{s^2+26s+25}} \\ \frac{1}{\sqrt{s^2+26s+25}} \\ \frac{\sqrt{\frac{3}{2}}\left(\sqrt{s^2+26s+25}-s-1\right)\sqrt{\frac{v_0}{\sqrt{s^2+26s+25}+5s+5}}}{2\sqrt{s^2+26s+25}} \end{pmatrix} e^{\lambda_1 x} + \begin{pmatrix} \frac{\sqrt{\frac{\left(\sqrt{s^2+26s+25}+5s+5\right)v_0}{s+1}}}{2\sqrt{s}\sqrt{s^2+26s+25}} \\ -\frac{1}{\sqrt{s^2+26s+25}} \\ \frac{\left(\sqrt{s^2+26s+25}+s+1\right)\sqrt{\frac{\left(\sqrt{s^2+26s+25}+5s+5\right)v_0}{s+1}}}{8\sqrt{s}\sqrt{s^2+26s+25}} \end{pmatrix} e^{\lambda_2 x}\right) \\
&\quad + H(-x)\left(\begin{pmatrix} -\frac{\sqrt{6}\sqrt{\frac{v_0}{\sqrt{s^2+26s+25}+5s+5}}}{\sqrt{s^2+26s+25}}, -\frac{1}{\sqrt{s^2+26s+25}} \\ \frac{\sqrt{6}(s+1)\left(\sqrt{s^2+26s+25}-s+5\right)\left(\frac{v_0}{\sqrt{s^2+26s+25}+5s+5}\right)^{3/2}}{\sqrt{s^2+26s+25}v_0} \end{pmatrix} e^{-\lambda_1 x}\right. \\
&\qquad \left. + \begin{pmatrix} \frac{\sqrt{\frac{\left(\sqrt{s^2+26s+25}+5s+5\right)v_0}{s+1}}}{2\sqrt{s}\sqrt{s^2+26s+25}} \\ \frac{1}{\sqrt{s^2+26s+25}} \\ \frac{\left(\sqrt{s^2+26s+25}+s+1\right)\sqrt{\frac{\left(\sqrt{s^2+26s+25}+5s+5\right)v_0}{s+1}}}{8\sqrt{s}\sqrt{s^2+26s+25}} \end{pmatrix} e^{-\lambda_2 x}\right);
\end{aligned}
$$

and the Green's function $\mathbb{L}[\mathbb{G}](x,s)$ is expressed in terms of G_k, $k=1,2,3$:

$$
(4.14)\qquad \Big(\mathbb{L}[\mathbb{G}_{ij}](x,s)\Big)_{3\times 3} = \mathbb{L}[\mathbb{G}](x,s) = \Big(G_1 G_2 G_3\Big).
$$

This also yields that

(4.15)

$$
\begin{cases}
e^{\lambda_1 x} = \\
\dfrac{\sqrt{\frac{2}{3}}\sqrt{\frac{v_0}{\sqrt{s^2+26s+25}+5s+5}}\left(3\mathbb{L}[\mathbb{G}_{22}](s+1)\left(\sqrt{s^2+26s+25}+s+5\right)+2\mathbb{L}[\mathbb{G}_{33}]s\left(-\sqrt{s^2+26s+25}+s+1\right)\right)}{(s+3)v_0}, \\
e^{\lambda_2 x} = \dfrac{2\sqrt{s}\left(\mathbb{L}[\mathbb{G}_{22}]\left(-\sqrt{s^2+26s+25}+s+1\right)+\mathbb{L}[\mathbb{G}_{33}]\left(\sqrt{s^2+26s+25}+s+5\right)\right)}{(s+3)\sqrt{\frac{\left(\sqrt{s^2+26s+25}+5s+5\right)v_0}{s+1}}}.
\end{cases}
$$

This also gives
(4.16)

$$\begin{cases} \lambda_1 e^{\lambda_1 x} = \\ \dfrac{\sqrt{\frac{2}{3}}\sqrt{\frac{v_0}{\sqrt{s^2+26s+25}+5s+5}}\left(3\mathbb{L}[\partial_x\mathbb{G}_{22}](s+1)\left(\sqrt{s^2+26s+25}+s+5\right)\right.}{(s+3)v_0} \\ \qquad + \dfrac{\sqrt{\frac{2}{3}}\sqrt{\frac{v_0}{\sqrt{s^2+26s+25}+5s+5}}\left(2\mathbb{L}[\partial_x\mathbb{G}_{33}]s\left(-\sqrt{s^2+26s+25}+s+1\right)\right)}{(s+3)v_0}, \\ \lambda_2 e^{\lambda_2 x} = \dfrac{2\sqrt{s}\left(\mathbb{L}[\partial_x\mathbb{G}_{22}]\left(-\sqrt{s^2+26s+25}+s+1\right)+\mathbb{L}[\partial_x\mathbb{G}_{33}]\left(\sqrt{s^2+26s+25}+s+5\right)\right)}{(s+3)\sqrt{\frac{\left(\sqrt{s^2+26s+25}+5s+5\right)v_0}{s+1}}}. \end{cases}$$

By this one has the asymptotic of the symbols $e^{\lambda_i x}$ satisfy the asymptotic at $s=\infty$

$$\begin{cases} e^{\lambda_1 x} = \left(\dfrac{11\mathbb{L}[\mathbb{G}_{22}]-8\mathbb{L}[\mathbb{G}_{33}]}{\sqrt{s}\sqrt{v_0}} + \dfrac{2\mathbb{L}[\mathbb{G}_{22}]\sqrt{s}}{\sqrt{v_0}}\right)(1+O(1/s)), \\ e^{\lambda_2 x} = \left(\dfrac{2\sqrt{\frac{2}{3}}\mathbb{L}[\mathbb{G}_{33}]\sqrt{s}}{\sqrt{v_0}} - \dfrac{2\sqrt{\frac{2}{3}}(6\mathbb{L}[\mathbb{G}_{22}]-5\mathbb{L}[\mathbb{G}_{33}])}{\sqrt{s}\sqrt{v_0}}\right)(1+O(1/s)). \end{cases} \tag{4.17}$$

$$\begin{cases} e^{\lambda_1 x} = \left(-\dfrac{2(7\partial_x\mathbb{L}[\mathbb{G}_{22}]-4\partial_x\mathbb{L}[\mathbb{G}_{33}])}{sv_0} - \dfrac{2\partial_x\mathbb{L}[\mathbb{G}_{22}]}{v_0}\right)(1+O(1/s)), \\ e^{\lambda_2 x} = \left(\dfrac{8(3\partial_x\mathbb{L}[\mathbb{G}_{22}]-2\partial_x\mathbb{L}[\mathbb{G}_{33}])}{3sv_0} - \dfrac{4\partial_x\mathbb{L}[\mathbb{G}_{33}]}{3v_0}\right)(1+O(1/s)), \end{cases} \tag{4.18}$$

$$\begin{cases} \dfrac{\lambda_1 e^{\lambda_1 x}}{s} = \left(\dfrac{8\mathbb{L}[\mathbb{G}_{33}]-8\mathbb{L}[\mathbb{G}_{22}]}{s} - 2\mathbb{L}[\mathbb{G}_{22}]\right)\big(1+O(1/s)\big), \\ \dfrac{\lambda_2 e^{\lambda_1 x}}{s} = \left(\dfrac{4\sqrt{6}\mathbb{L}[\mathbb{G}_{33}]-13\sqrt{\frac{3}{2}}\mathbb{L}[\mathbb{G}_{22}]}{s} - \sqrt{6}\mathbb{L}[\mathbb{G}_{22}]\right)\big(1+O(1/s)\big), \\ \dfrac{\lambda_2 e^{\lambda_2 x}}{s} = \left(\dfrac{12\mathbb{L}[\mathbb{G}_{22}]-12\mathbb{L}[\mathbb{G}_{33}]}{s} - 2\mathbb{L}[\mathbb{G}_{33}]\right)\big(1+O(1/s)\big), \\ \dfrac{\lambda_1 e^{\lambda_2 x}}{s} = \left(\dfrac{12\sqrt{6}\mathbb{L}[\mathbb{G}_{22}]-7\sqrt{6}\mathbb{L}[\mathbb{G}_{33}]}{3s} - 2\sqrt{\dfrac{2}{3}}\mathbb{L}[\mathbb{G}_{33}]\right)\big(1+O(1/s)\big). \end{cases} \tag{4.19}$$

5. Asymptotic structure of $\mathbb{L}[\mathbb{G}](x,s)$

In this section, we conclude the relevant asymptotic of $\mathbb{L}[\mathbb{G}](x,s)$.
Around $s=0$,

$$\left(\mathbb{L}[\mathbb{G}]_{ij}(x,s) - \frac{\delta_i^1\delta_j^1\delta(x)}{1+s}\right)_{3\times 3} \tag{5.1}$$

$$= H(x)\begin{pmatrix} \frac{1}{100}(-3)\sqrt{\frac{3}{5}}(13s-10)\sqrt{v_0} & \frac{3}{250}(23s-25)v_0 & \frac{1}{50}\sqrt{\frac{3}{5}}(9s-10)\sqrt{v_0} \\ \frac{3}{250}(23s-25) & -\frac{1}{100}\sqrt{\frac{3}{5}}(27s-50)\sqrt{v_0} & \frac{1}{125}(25-13s) \\ \frac{3}{100}\sqrt{\frac{3}{5}}(9s-10)\sqrt{v_0} & \frac{1}{250}(-3)(13s-25)v_0 & -\frac{1}{10}\sqrt{\frac{3}{5}}(s-2)\sqrt{v_0} \end{pmatrix} e^{\lambda_1 x}(1+O(1)s)$$