THREE-TERM MACHIN-TYPE FORMULAE

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Abstract. We shall show that there exist only finitely many nondegenerate three-term Machin-type formulae and give explicit upper bounds for the sizes of variables.

1. Introduction

The Machin's formula

$$4\arctan\frac{1}{5}-\arctan\frac{1}{239}=\frac{\pi}{4}, \tag{1}$$

is well known and have been used to calculate approximate values of π. Analogous formulae $\arctan(1/2)+\arctan(1/3)=\pi/4$, $2\arctan(1/2)-\arctan(1/7)=\pi/4$ and $2\arctan(1/3)-\arctan(1/7)=\pi/4$, which are also well known, were attributed to Euler, Hutton and Hermann, respectively. But according to Tweddle [11], these formulae also seem to have been found by Machin.

Several three-term formulae such as $8\arctan(1/10)-\arctan(1/239)-4\arctan(1/515)=\pi/4$ due to Simson in 1723 (see [11]) and $12\arctan(1/18)+8\arctan(1/57)-5\arctan(1/239)=\frac{\pi}{4}$ due to Gauss in 1863 also have been known.

More generally, an n-terms Machin-type formula is defined to be an identity of the form

$$y_1\arctan\frac{1}{x_1}+y_2\arctan\frac{1}{x_2}+\cdots+y_n\arctan\frac{1}{x_n}=\frac{r\pi}{4} \tag{2}$$

with integers $x_1,x_2,\ldots,x_n,y_1,y_2,\ldots,y_n$ and $r\neq 0$.

Theoretical studies of Machin-type formulae have begun with a series of works of Størmer', who proved that the four formulae mentioned above are all two-term ones in 1895 [8] and gave a necessary and sufficient condition for given integers $x_1,x_2,\ldots,x_n>1$ to have a Machin-type formula (2) and 102 three-term ones in 1896 [9]. Størmer asked for other three-term Machin-type formulae and questioned whether there exist infinitely many ones or not. Up to now the only known other nontrivial (i.e. not derived from the three formulae given above) three-term formulae are $5\arctan(1/2)+2\arctan(1/53)+\arctan(1/4443)=3\pi/4$, $5\arctan(1/3)-2\arctan(1/53)-\arctan(1/4443)=\pi/2$ and $5\arctan(1/7)+4\arctan(1/53)+2\arctan(1/4443)=\pi/4$. [12] attributes these formulae to Wrench

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[14] although these formulae cannot be found there. We note that the second and the third formulae follow from the first formula using $\arctan(1/2)+\arctan(1/3) = \pi/4$ and $2\arctan(1/2) - \arctan(1/7) = \pi/4$ respectively.

The purpose of this paper is to answer to Størmer's other question in negative. We shall show that there exist only finitely many three-term Machin-type formulae which does not arise from a linear combinations of three two-term formulae.

Størmer's criterion is essentially as follows: For given integers $x_1, x_2, \ldots, x_n > 1$, (2) holds for some integers $y_1, y_2, \ldots, y_n$ and $r \neq 0$ if and only if there exist integers $s_{i,j} (i = 1, 2, \ldots, n, j = 1, 2, \ldots, n-1)$ and Gaussian integers $\eta_1, \eta_2, \ldots, \eta_{n-1}$ such that

$$\left[\frac{x_i + \sqrt{-1}}{x_i - \sqrt{-1}}\right] = \left[\frac{\eta_1}{\bar{\eta}_1}\right]^{\pm s_{i,1}} \left[\frac{\eta_2}{\bar{\eta}_2}\right]^{\pm s_{i,2}} \cdots \left[\frac{\eta_{n-1}}{\bar{\eta}_{n-1}}\right]^{\pm s_{i,n-1}}. \tag{3}$$

for $i = 1, 2, \ldots, n$.

Writing $m_j = \eta_j \bar{\eta}_j$ for $j = 1, 2, \ldots, n-1$, this condition can be reformulated as follows: there exist nonnegative integers $s_{i,j} (i = 1, 2, \ldots, n, j = 1, 2, \ldots, n)$ with $0 \leq s_{i,n} \leq 1$ such that the equation

$$x_i^2 + 1 = 2^{s_{i,n}} m_1^{s_{i,1}} m_2^{s_{i,2}} \cdots m_{n-1}^{s_{i,n-1}} \tag{4}$$

holds for $i = 1, 2, \ldots n$ and, additionally, $x_i \equiv \pm x_j \pmod{m_k}$ for three indices i, j, k with $x_i^2 + 1 \equiv x_j^2 + 1 \equiv 0 \pmod{m_k}$.

Thus, for given three integers $x_1, x_2, x_3 > 1$, there exist nonzero integers $y_1, y_2, \ldots, y_n$ and r such that a three-term Machin-type formula

$$y_1 \arctan\frac{1}{x_1} + y_2 \arctan\frac{1}{x_2} + y_3 \arctan\frac{1}{x_3} = \frac{r\pi}{4} \tag{5}$$

holds if and only if there exist integers $k_i, l_i (i = 1, 2, 3)$ and Gaussian integers η_1, η_2 such that

$$\left[\frac{x_i + \sqrt{-1}}{x_i - \sqrt{-1}}\right] = \left[\frac{\eta_1}{\bar{\eta}_1}\right]^{\pm k_i} \left[\frac{\eta_2}{\bar{\eta}_2}\right]^{\pm l_i} \tag{6}$$

holds for $i = 1, 2, 3$ or, equivalently, writing $m_j = \eta_j \bar{\eta}_j$ for $j = 1, 2, \ldots, n-1$ and choosing $v_i \in \{0, 1\}$ appropriately, the equation

$$x_i^2 + 1 = 2^{v_i} m_1^{k_i} m_2^{l_i} \tag{7}$$

holds for $i = 1, 2, 3$ and, additionally, $x_i \equiv \pm x_{i'} \pmod{m_j}$ for two indices i, i' with $x_i^2 + 1 \equiv x_{i'}^2 + 1 \equiv 0 \pmod{m_j}$. Furthermore, (6) implies (5) with $y_1 = \pm k_2 l_3 \pm k_3 l_2, y_2 = \pm k_3 l_1 \pm k_1 l_3$ and $y_3 = \pm k_1 l_2 \pm k_2 l_1$ with appropriate choices of signs.

Now we shall state our result in more detail.

Theorem 1.1. *Assume that* $x_1, x_2, x_3, y_1, y_2, y_3$ *and* r *are nonzero integers with* $x_1, x_2, x_3 > 1$ *and* $\{x_1, x_2, x_3\} \neq \{2, 3, 7\}$ *satisfying* (5) *and* $m_1, m_2, s_i, k_i, l_i (i = 1, 2, 3)$ *are corresponding integers with* $m_2 > m_1 > 0$ *satisfying* (7).

I. *If* $x_i^2+1 \geq m_2$ *for* $i = 1, 2, 3$*, then* $m_1 < m_2 < 5.19 \cdot 10^{39}$*,* $x_i < \exp(9.726 \cdot 10^{11})$ *and* $|y_i| < 2KL < 5.656 \cdot 10^{19}$*.*

II. *If* $x_i^2+1 < m_2$ *for some* i*, then* $m_1 < 4.14 \cdot 10^{81}, m_2 < \exp(9964497.86) < 2.7261 \times 10^{4327526}$*,* $x_i < \exp(3.18 \cdot 10^{20})$ *and* $y_i < 2.7 \cdot 10^{31}$*.*

We use a lower bound for linear forms in three logarithms in order to obtain upper bounds for exponents k_i's and l_i's in terms of m_1, m_2.

These upper bounds themselves do not give finiteness of m_1 and m_2. However, noting that $r \neq 0$, which gives $|\sum_i y_i \arctan(1/x_i)| \geq \pi/4$, the first case can be easily settled using these upper bounds. In order to settle the second case, we additionally need an upper bound m_2 in terms of m_1. This can be done using a lower bound for a quantity of the form $y \arctan(1/x) - r\pi/2$, which gives a linear form of two logarithms.

(4) can be seen as a special case of the generalized Ramanujan-Nagell equation

$$x^2 + Ax + B = p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n}, \tag{8}$$

where A and B are given integers with $A^2 - 4B \neq 0$ and $p_1, p_2, \ldots, p_n$ are given primes. Evertse [2] proved that (8) has at most $3 \cdot 7^{4n+6}$ solutions. In the case $n = 2$, the author [15] reduced Evertse's bound $3 \cdot 7^{14}$ to 63.

On the other hand, our result does not give an upper bound for numbers of solutions

$$x^2 + 1 = 2^s p_1^k p_2^l \tag{9}$$

since the case $r = 0$ is not considered. Indeed, Størmer [9] implicitly pointed out that, if $x^2 + 1 = ay$, then

$$\arctan \frac{1}{az - x} - \arctan \frac{1}{az + a - x} = \arctan \frac{1}{az(z+1) - (2z+1)x + y}. \tag{10}$$

Størmer [10] showed that (9) has at most one solution with each fixed combination of parities of s_i, k_i, l_i with zero and nonzero-even distinguished. Although there exist 18 combinations $(0 \mid 1, 0 \mid 1 \mid 2, 0 \mid 1 \mid 2)$, all-even combinations can clearly be excluded and therefore (9) has at most 14 solutions totally.

2. Preliminaries

In this section, we introduce some notation and some basic facts.

For integers N composed of prime factors $\equiv 1 \pmod 4$, we define $\widehat{\log} N = \log N$ if $N \geq 13$ and $\widehat{\log} 5 = 4 \arctan(1/2)$. If we decompose $N = \eta\bar{\eta}$ in Gaussian integers, then $\log(\eta/\bar{\eta}) \leq (\widehat{\log} N)/2$. We write $\gamma(N) = \widehat{\log} N / \log N$. $\gamma(5) = 1.1523\cdots$ and $\gamma(N) = 1$ for $N \geq 13$.

Moreover, we define $\widetilde{\log} N$ by $\widetilde{\log} N = \max\{\log N, (1/2.648) + \max 4 \arg(\eta/\bar{\eta}) / \log N\}$, where the inner maximum is taken over all decompositions $N = \eta\bar{\eta}$ with $|\arg \eta| <$

$\pi/4$. We write $\delta(N) = \widetilde{\log} N/\log N$. We see that $\delta(N) = 1$ when $N > 22685$ and there exist exactly 401 integers N such that $\delta(N) > 1$.

For any gaussian integer η, we have an associate η' of η such that $-\pi/4 < \arg\eta' < \pi/4$ and therefore $-\pi/2 < \arg\eta'/\bar{\eta'} < \pi/2$.

We call a formula (2) to be degenerate if

$$\sum_{i\in S} y_i' \arctan\frac{1}{x_i} = \frac{r'\pi}{4} \tag{11}$$

for some proper subset S of $\{1, 2, \ldots, n\}$ and integers $y_i'(i \in S)$ and r' which may be zero but not all zero.

From Størmer's result in [8] on two-term Machin-type formulae, the degenerate case only occurs in $\{x_1, x_2, x_3\} = \{2, 3, 7\}$.

3. A lower bound for linear forms of three logarithms

Our argument depends on a lower bound for linear forms of three logarithms. Results in Mignotte's *a kit on linear forms in three logarithms*[6] are rather technical but still worthful to use for the purpose of improving our upper bounds. Proposition 5.2 of [6] applied to the Gaussian rationals gives the following result.

Lemma 3.1. *Let α_1, α_2 and α_3 be three Gaussian rationals $\neq 1$ with absolute value one and assume that the three numbers $\alpha_1, \alpha_2, \alpha_3$ are multiplicatively independent or two of these numbers are multiplicatively independent and the third one is a root of unity, i.e. -1 or $\pm\sqrt{-1}$. Let b_1, b_2 and b_3 be three coprime positive rational integers and*

$$\Lambda = b_2 \log\alpha_2 - b_1\log\alpha_1 - b_3\log\alpha_3, \tag{12}$$

where the logarithm of each α_i can be arbitrarily determined as long as

$$b_2 |\log\alpha_2| = b_1 |\log\alpha_1| + b_3 |\log\alpha_3| \pm |\Lambda|. \tag{13}$$

We put $d_1 = \gcd(b_1, b_2), d_2 = \gcd(b_2, b_3), b_2 = d_1 b_2' = d_3 b_2''$. Let $w_i = |\log\alpha_i| = |\arg\alpha_i|$ for each $i = 1, 2, 3$, a_1, a_2 and a_3 be real numbers such that $a_i \geq \max\{4, 5.296 w_i + 2h(\alpha_i)\}$ for each $i = 1, 2, 3$ and $\Omega = a_1 a_2 a_3 \geq 100$. Furthermore, put

$$b' = \left(\frac{b_1'}{a_2} + \frac{b_2'}{a_1}\right)\left(\frac{b_3''}{a_2} + \frac{b_2''}{a_3}\right) \tag{14}$$

and $\log B = \max\{0.882 + \log b', 10\}$.

Then, either one of the following holds.

A. *The estimate*

$$\log|\Lambda| > -790.95\Omega\log^2 B \tag{15}$$

holds.

B. *There exist two nonzero rational integers* r_0 *and* s_0 *such that* $r_0 b_2 = s_0 b_1$ *with* $|r_0| \leq 5.61 a_2 \log^{1/3} B$ *and* $|s_0| \leq 5.61 a_1 \log^{1/3} B$.

C. *There exist four rational integers* r_1, s_1, t_1 *and* t_2 *with* $r_1 s_1 \neq 0$ *such that*

$$(t_1 b_1 + r_1 b_3) s_1 = r_1 b_2 t_2, \gcd(r_1, t_1) = \gcd(s_1, t_2) = 1 \tag{16}$$

and

$$|r_1 s_1| \leq 5.61 \delta a_3 \log^{1/3} B, |s_1 t_1| \leq 5.61 \delta a_1 \log^{1/3} B, |r_1 t_2| \leq 5.61 \delta a_2 \log^{1/3} B, \tag{17}$$

where $\delta = \gcd(r_1, s_1)$. *Moreover, when* $t_1 = 0$ *we can take* $r_1 = 1$ *and then* $t_2 = 0$ *we can take* $s_1 = 1$.

This result is nonsymmetric for three logarithms and, in order to make each b_i positive, we should arrange the order of logarithms. Thus, the application of this result requires a fair amount of computations with many branches of cases.

For convenience, we write h_i for $h(\alpha_i)$. For our purpose, we apply Lemma 3.1 to linear forms of two logarithms and $\pi\sqrt{i}/2 = \log \sqrt{-1}$. In this special case, we may assume that (i) $\log \alpha_2 = \pi/2$ or (ii) $\log \alpha_3 = \pi/2$ by exchanging (α_1, b_1) and (α_3, b_3). Thus, there exist six cases: A. i, A. ii, B. i, B. ii, C. i, C. ii.

In Case A, (15) gives a desired lower bounds. In cases B and C, we can reduce Λ into a linear form of two logarithms and apply results of [3]. Here, we shall discuss only in the case C. i.

We put $r_1 = \delta r_0, s_1 = \delta s_0$, which immediately yields that $\gcd(r_0, s_0) = 1$. Dividing (16) by δ, we have

$$s_0 t_1 b_1 + r_0 t_2 b_2 + \delta r_0 s_0 b_3 = 0. \tag{18}$$

From this, we see that r_0 divides b_1 and s_0 divides b_2. Put $b_1 = r_0 u_1, b_2 = s_0 u_2$. Dividing (18) by $r_0 s_0$, we have

$$t_1 u_1 + t_2 u_2 + \delta b_3 = 0. \tag{19}$$

Now we obtain

$$\delta\Lambda = u_2 \log \alpha_5 - u_1 \log \alpha_6, \tag{20}$$

where $\alpha_5 = \alpha_2^{s_1} \alpha_3^{t_2}, \alpha_6 = \alpha_1^{r_1} \alpha_3^{-t_1}$. Moreover,

$$|s_0 t_1| \leq 5.61 a_1 \log^{1/3} B, |r_0 t_2| \leq 5.61 a_2 \log^{1/3} B, |\delta r_0 s_0| \leq 5.61 a_3 \log^{1/3} B. \tag{21}$$

Taking

$$a_5 = \max\left\{|t_2|\, h_3, |t_2|\, w_3 + \frac{|s_1|\, \pi}{2}\right\}, a_6 = \max\left\{|r_1|\, h_1 + |t_1|\, h_3, |r_1|\, w_1 + |t_1|\, w_3\right\}$$

and

$$b'' = \frac{|u_1|}{a_5} + \frac{|u_2|}{a_6} \leq \frac{b_1}{|s_0|\, a_5} + \frac{b_2}{|s_0|\, a_6},$$

Corollaire 1 of [3] gives

$$\log |\delta\Lambda| \geq -30.9 \max\{\log^2 b'', 441\} a_5 a_6. \tag{22}$$

Table 1. Constants in (24)

Case	C	μ_1	μ_2	μ	ν_1	ν_2	β	τ
A, i	$28962 f_1(m_1, m_2)$	1	1	2	1/2	1/2	$\frac{1}{\log m_1} + \frac{1}{5.296\pi}$	2.351
A, ii	$28962 f_1(m_1, m_2)$	1	1	2	0	1/2	$\sqrt{\frac{1}{2.648\pi} + \frac{1}{\log m_1}}$	2.393
B, i	$460.63 f_3(m_1, m_2)$	1	1	7/3	1/2	1/2	$\frac{1}{\log m_1} + \frac{1}{5.296\pi}$	4.574
					0	0	2	3.967
B, ii	$127.408 f_4(m_1, m_2)$	1	1	7/3	0	1/2	$\sqrt{\frac{1}{2.648\pi} + \frac{1}{\log m_1}}$	4.902
					0	1	126.844	2.838
C, i	$6631 g_5(m_1, m_2)$	1	2	8/3	1/2	3/2	$\frac{1}{\log m_1} + \frac{1}{5.296\pi}$	4.529
					0	1	$\pi/2 + 2$	4.025
C, ii	$27574 \delta(m_1)\delta(m_2)$	1	1	8/3	0	1/2	$\sqrt{\frac{1}{2.648\pi} + \frac{1}{\log m_1}}$	4.475
					0	0	2	4.006

4. Upper bounds for exponents

In this section, we shall prove upper bounds for exponents in (6) or, equivalently, (7).

Lemma 4.1. *Let η_1 and η_2 be Gaussian integers with $-\pi/2 < \arg \eta_i/\bar{\eta}_i < \pi/2$ and $m_i = \eta_i\bar{\eta}_i > 1$ for $i = 1, 2$ with $m_2 > m_1$ both odd.*

We set

$$f_1(m_1, m_2) = \left(1 + \frac{5.296\pi}{\log m_1}\right)\left(1 + \frac{5.296\pi}{\log m_2}\right),$$

$$f_3(m_1, m_2) = \max\{\delta(m_1), \gamma(m_1)\delta(m_2)\},$$

$$\begin{aligned} &f_4(m_1, m_2) \\ =&\frac{1}{2}\left(\left(1 + \frac{5.296\theta_1}{\log m_1}\right)\left(1 + \frac{10.98\theta_2}{\log m_2}\right) + \left(1 + \frac{10.98\theta_1}{\log m_1}\right)\left(1 + \frac{5.296\theta_2}{\log m_2}\right)\right), \end{aligned}$$

$$f_5(m_1, m_2) = 1 + \frac{2.648\pi(\widehat{\log} m_1 + \log m_2)}{\widehat{\log} m_1 \log m_2}$$

and

$$g_5(m_1, m_2) = f_5(m_1, m_2)\gamma(m_1)\delta(m_2).$$

If x, e_1, e_2 are nonnegative integers such that

$$\left[\frac{x + \sqrt{-1}}{x - \sqrt{-1}}\right] = \left[\frac{\eta_1}{\bar{\eta}_1}\right]^{\pm e_1}\left[\frac{\eta_2}{\bar{\eta}_2}\right]^{\pm e_2}, \tag{23}$$

then we have

$$e_1 \log m_1 + e_2 \log m_2 < 2\tau C \log^{\mu_1} m_1 \log^{\mu_2} m_2 \log^{\mu} Y \tag{24}$$

with $(C, \mu_1, \mu_2, \mu, \nu_1, \nu_2, \beta, \tau)$ taken from one of ten rows in Table 1 and $Y = 2C\beta \log^{\nu_1} m_1 \log^{\nu_2} m_2$.

Proof. From the result of [4], $x^2+1=m^t$ with $x>0, t>1$ has no solution. Théorème 8 of [10] shows that $x^2+1=2m^t$, then t must be a power of two. By Ljunggren's result [5], the only integer solution of $x^2+1=2m^4$ with $x, m>1$ is $(x, m)=(239, 13)$ (Easier proofs of Ljunggren's result have been obtained by Steiner and Tzanakis [7] and Wolfskill [13]). Thus, we may assume that $e_1e_2\neq 0$ since $e_i=0$ implies that $e_{3-i}=1, 2$ or 4. Furthermore, we may assume that $m_1^{e_1}m_2^{e_2}>10^{20}$.

We can decompose $m_i=\eta_i\bar{\eta}_i$ in a way such that $-\pi/4<\arg\eta'<\pi/4$. We put $\xi_i=\eta_i/\bar{\eta}_i$ and write $\theta_i=|\arg\xi_i|=|\log\xi_i|$, so that $\theta_i<\pi/2$.

Now $\Lambda=\log[(x+\sqrt{-1})/(x-\sqrt{-1})]$ can be represented as a linear form of three logarithms

$$\Lambda=\pm e_1\log\xi_1\pm e_2\log\xi_2\pm\frac{e_3\pi\sqrt{-1}}{2} \tag{25}$$

for an appropriate integer $e_3\geq 0$. Moreover, we can easily see that

$$\log|\Lambda|<-\log x<-\frac{e_1\log m_1+e_2\log m_2}{2}+10^{-9}. \tag{26}$$

Applying Lemma 3.1 and some technical argument in each of six cases, which are too complicated to describe here, we are led to 24. This proves the lemma. □

5. Proof of the Theorem

Let $x_1, x_2, x_3, y_1, y_2, y_3$ and r be integers with $x_1, x_2, x_3>1, r\neq 0$ satisfying (5) and $m_1, m_2, s_i, k_i, l_i (i=1,2,3)$ be corresponding integers, η_1, η_2 be gaussian integers satisfying (6) and (7). We write $K=\max k_i$ and $L=\max l_i$. We may assume that $\{x_1, x_2, x_3\}\neq\{2,3,7\}$. From a note in the preliminaries, this implies that (5) is nondegenerate.

We have two cases: I. $x_1^2+1\geq m_2$ and II. $x_1^2+1<m_2$.

Case I. In this case, $x_i\geq\sqrt{m_2-1}$ for $i=1,2,3$. Since (5) is nondegenerate, $u_i=0$ for at most one index i. In the case there exists such an index i, we may assume that $i=1$. Since $x_2^2+1\equiv x_3^2+1\equiv 0 \pmod{m_2}$, Størmer's criterion implies that $x_2>m_2/2$ or $x_3>m_2/2$.

It immediately follows from (5) with $r\neq 0$ that

$$\frac{|y_1|+|y_2|}{\sqrt{m_2-1}}+\frac{2|y_3|}{m_2}>\frac{\pi}{4}. \tag{27}$$

Since $|y_1|\leq k_2l_3+k_3l_2\leq 2TU$ and so on, we have $m_2<(4(2+10^{-8})KL/\pi)^2<6.49(KL)^2$.

Combining with Lemma 4.1, we have $m_1<m_2<5.19\cdot 10^{39}$, $|y_i|<2KL<5.656\cdot 10^{19}$ and $\log x_i<k_i\log m_1+l_i\log m_2<9.726\cdot 10^{11}$, that is, $x_i<\exp(9.726\cdot 10^{11})$. This shows the Theorem in Case I.

Case II. We may assume that $x_1^2+1<m_2$. We must have $l_1=0$ and $x_1^2+1=2m_1^{k_1}<m_2$.

Combining diophantine results mentioned in the proof of Lemma 4.1 allow us to assume that $k_1=1$ or 2. Since (5) is nondegenerate, $l_i\neq 0$ for another index $i>1$. Thus, $l_2,l_3>0$ and $x_2^2+1, x_3^2+1>m_2$.

Now we clearly have

$$y_1\arctan\frac{1}{x_1}\pm l_3k_1\arctan\frac{1}{x_2}\pm l_2k_1\arctan\frac{1}{x_3}=\frac{r\pi}{4}. \tag{28}$$

Let

$$\Lambda_1=2y_1\log\frac{x_1+\sqrt{-1}}{x_1-\sqrt{-1}}-r\pi\sqrt{-1}. \tag{29}$$

Then, observing that $|r\pi/4-y_1\arctan(1/x_1)|<(1+10^{-8})k_1(l_2+l_3)/m_2^{1/2}$ with $|y_1|\leq k_2l_3+l_3k_2<2KL$, we have

$$|\Lambda_1|<\frac{4(1+10^{-8})k_1L}{m_2^{1/2}}, \tag{30}$$

while Théorème 3 of [3] gives that

$$-\log|\Lambda_1|<8.87aH_1^2, \tag{31}$$

where $a=\max\{20, 10.98\widehat{\log}\, m_1+(\log m_1)/2\}$ and $H_1=\max\{17, 2.38+\log((r/2a)+(2y_1/68.9))\}$.

We observe that $10.98\widehat{\log}\, N>20$ for any N and therefore $a=10.98\widehat{\log}\, m_1+\frac{\log m_1}{2}$. Moreover, we have

$$\left|\frac{r\pi}{4}\right|\leq\frac{2KL}{x_1}+\frac{2u_1L}{x_2}<\frac{2.3L}{m_1^{1/2}}. \tag{32}$$

and $|r|<3KL/m_1^{1/2}$.

If $KL\geq 4\cdot 10^7$, then, from (37), we obtain $\log m_2<8.87(10.98\gamma(m_1)+0.51)\log m_1\log^2(KL)$. If $m_2\leq e^{187}$, then $m_1<e^{187}<4.14\cdot 10^{81}$ and the Theorem immediately follows. If $m_2>e^{187}$, then Lemma 4.1 yields that $KL<\log^{8.88163} m_2$. Observing that $8.87\cdot 8.88163^2(10.98\gamma(m_1)+0.51)\log m_1>14822.4$, we obtain

$$\log m_2<881.32(10.98\gamma(m_1)+0.51)\log m_1\log\log m_1. \tag{33}$$

Recalling (34), we have

$$\frac{2KL}{\sqrt{m_1-1}}>\frac{\pi}{4}-\frac{2k_1L}{\sqrt{m_2-1}}>\frac{\pi(1-10^{-8})}{4}. \tag{34}$$

Combining this with (39) and Lemma 4.1, we have $m_1<4.14\cdot 10^{81}, m_2<\exp(9964497.86)<2.7261\times 10^{4327526}$, $\log x_i=k_i\log m_1+l_i\log m_2<3.18\cdot 10^{20}$, that is, $x_i<\exp(3.18\cdot 10^{20})$, and $y_i\leq 2KL<2.7\cdot 10^{31}$. This completes the proof of the Theorem.

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