ON ERROR TERM ESTIMATES À LA WALFISZ FOR MEAN VALUES OF ARITHMETIC FUNCTIONS

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1. Introduction

Let $\varphi(n)$ be the Euler totient function. In 1963, Walfisz [11] proved

$$\sum_{n\le x}\frac{\varphi(n)}{n}=\frac{6}{\pi^2}x+O((\log x)^{\frac{2}{3}}(\log\log x)^{\frac{4}{3}}),\tag{1}$$

which improved the trivial error term estimate $O(\log x)$ due to Mertens [6]. Recently, H. Q. Liu [5] improved Walfisz's error term estimate to $O((\log x)^{\frac{2}{3}}(\log\log x)^{\frac{1}{3}})$, i.e. Liu succeeded in removing one $\log\log x$ factor. He found that Walfisz's result was obtained through Vinogradov's classical combinatorial decomposition which produces not so efficient summation range for Type II sums and that Vaughan's identity for the Möbius function can produce a $\log\log x$ improvement. It is rather surprising that no one pointed out this improvement before Liu.

Walfisz's result (1) was generalized by Balakrishnan and Pétermann [1, 7] to a wide class of arithmetic functions which behave similarly to the Euler totient function. (Note that Theorem 2 of [1] was withdrawn and a corrected version was given in [7].) Their result can be decomposed into two parts. The first part of their result gives an expansion for the mean value of arithmetic functions:

Theorem A ([1, Theorem 1]). *Let α be a complex number and*

$$f(s)=\sum_{n=1}^{\infty}\frac{b(n)}{n^s}$$

be a Dirichlet series absolutely convergent for $\sigma>1-\lambda$ with some real number $\lambda>0$. Define arithmetic functions $a(n)$ and $v(n)$ by

$$\sum_{n=1}^{\infty}\frac{a(n)}{n^s}=\zeta(s)\zeta(s+1)^{\alpha}f(s+1)\quad\textit{and}\quad\sum_{n=1}^{\infty}\frac{v(n)}{n^s}=\zeta(s)^{\alpha}f(s)$$

for $\sigma>1$, where we take the branch of $\zeta(s+1)^{\alpha}$ by $\arg\zeta(s+1)=0$ on the positive real line. Then we have

$$\sum_{n\le x}a(n)=\zeta(2)^{\alpha}f(2)x+\sum_{r=0}^{[\mathrm{Re}\,\alpha]}A_r(\log x)^{\alpha-r}-\sum_{n\le y}\frac{v(n)}{n}\psi\left(\frac{x}{n}\right)+o(1)$$

as $x\to\infty$, where the coefficients (A_r) are computable from the Laurent expansion of $\zeta(s)^{\alpha}f(s)$ at $s=1$, $y=x\exp(-(\log x)^{\frac{1}{6}})$, $\psi(x)=\{x\}-\frac{1}{2}$ and the Landau symbol $o(1)$ depends on the assumptions of this theorem.

Thus the error term estimate now amounts to the estimate of the sum

$$\sum_{n\le y}\frac{v(n)}{n}\psi\left(\frac{x}{n}\right),$$

which is the second part of the result of Balakrishnan and Pétermann. The error term estimate of Balakrishnan and Pétermann [7, Theorem 1] is just of the type (1) so it is weaker than Liu's result [5]. Thus, it is a natural problem to improve the result of Balakrishnan and Pétermann up to the level of Liu's result. It turned out that we cannot apply Liu's approach straightforwardly to the general setting since there seems to be no simple Vaughan-type identity for general arithmetic functions. The author [8, 9], instead, used some finer Vinogradov-type combinatorial decomposition to obtain the following result:

Theorem B ([9, Theorem 5])**.** *Let $v(n)$ be a complex-valued multiplicative function such that there exists a real number $C\ge 2$ satisfying the following three conditions*:

(V1) $$|v(p)|\le C \text{ for every prime number } p,$$

(V2) $$\sum_{n\le x}|v(n)|^2\le Cx(\log x)^C \quad (x\ge 4),$$

(V3) $$\sum_{p_n\le x}|v(p_{n+1})-v(p_n)|\le C(\log x)^C \quad (x\ge 4),$$

where p_n is the n-th prime number. Assume that a real number $\kappa\ge 0$ satisfies

(V) $$\sum_{n\le x}\frac{|v(n)|}{n}\ll(\log x)^\kappa$$

for $x\ge 4$. Then for $x\ge 4$ and $\theta>0$, we have

$$\sum_{n\le y}\frac{v(n)}{n}\psi\left(\frac{x}{n}\right)\ll(\log x)^{\frac{2\kappa}{3}}(\log\log x)^{\frac{\kappa}{3}},$$

where $y\le xe^{-(\log x)^\theta}$ and the implicit constant depends only on θ, C and the implicit constant in the above condition (**V**).

During the preparation of the preprint [8], the author found that Drappeau and Topacogullari [2] recently dealt with a similar problem on general combinatorial decompositions in a different context, the generalized Titchmarsh divisor problem. While the author's method [8] is based on Vinogradov's decomposition, Drappeau and Topacogullari prepared Vaughan's identity for generalized divisor function $\tau_\alpha(n)$ with rational α and applied the Lagrange interpolation to extend the result to general α. This method of Drappeau and Topacogullari is sufficient to deal with the setting in Theorem A, but at least for Theorem B, the author's method is available slightly wider class of arithmetic functions $v(n)$. In particular, in the method of Drappeau and Topacogullari, we should approximate the arithmetic function by generalized divisor function, which is not possible in general for $v(n)$ given as in Theorem B.

Recently, Drappeau and Topacogullari gave still another approach in the second version of their preprint [2]. Their new approach is based on Linnik's identity [4] and well-factorable property of smooth numbers. Although their new approach still relies on the approximation via the generalized divisor function, this new approach now shares similar flavor with the author's method. In particular, the decomposition into rough and smooth numbers in [9, (4.45)] may correspond to the decomposition $F(s) = F(s, y)G(s, y)$ in the proof of the Linnik–Drappeau–Topacogullari identity [2, Theorem 3.3]. However, one significant difference is that in the author's argument, the sum over smooth numbers is estimated trivially by using well-known estimates on smooth numbers. Indeed, the well-factorability of smooth numbers has significant effects on the author's argument. This improves the author's argument when it is applied for the problems in which the size of cancellation is important. (We cannot see this improvement in Theorem B.) Also, we can simplify the author's argument: There is no need to take so long summation over ν as in [9, (4.49)]. The author is now preparing the second version of the preprint [8] in which these improvements and simplifications will be implemented. So we refer rather [9] not [8] to cite the author's original approach.

The author mentioned in the preprint [8] without details that we can apply the method of Drappeau and Topacogullari to our problem with slightly restricted range of $v(n)$. In this note, we sketch this alternative proof of the special case of Theorem B. The special case of Theorem B in this note can be stated as follows. For a complex number α, let $\mathcal{V}(\alpha)$ be a class of complex-valued arithmetic functions $v(n)$ for which there exists a complex valued arithmetic function $b(n)$ such that

$$v(n) = \sum_{dm=n} b(d)\tau_\alpha(m) \quad \text{and} \quad D := \sum_{d=1}^{\infty} \frac{|b(d)|}{d} < +\infty. \tag{2}$$

Note that the function $v(n)$ in Theorem A belongs to $\mathcal{V}(\alpha)$.

Theorem 1. *Let α be a complex number and $v(n)$ be an arithmetic function from the class $\mathcal{V}(\alpha)$. Then for $x \geq 4$ and $\theta > 0$, we have*

$$\sum_{n \leq y} \frac{v(n)}{n} \psi\left(\frac{x}{n}\right) \ll (\log x)^{\frac{2|\alpha|}{3}} (\log\log x)^{\frac{|\alpha|}{3}},$$

where $y \leq x e^{-(\log x)^\theta}$ and the implicit constant depends on θ, α and D given in (2).

By using the argument used in the proof of Theorem 6 in [9, p. 103] (cf. Lemma 2.2 of [2]), we can reduce the proof of Theorem 1 to the case $v(n) = \tau_\alpha(n)$. Furthermore, by using the arguments in [9, p. 99–101], we can reduce the problem to the proof of the following lemma (where we use the notation $e(x) = \exp(2\pi i x)$):

Lemma 1. *For any real number $A \geq 1$, there exists a real number $B = B(A) \geq 1$ such that for any complex number α and any real numbers $P, P', Q \geq 4$ with $P \leq P' \leq 2P$, we have*

$$\sum_{P < n \leq P'} \tau_\alpha(n) e\left(\frac{Q}{n}\right) \ll \left(P(\log Q)^{-A} + P^{\frac{3}{2}} Q^{-\frac{1}{2}}\right) (\log Q)^{|\alpha|^2 + |\alpha| + 9}$$

provided

$$P \geq \exp(B(\log Q)^{\frac{2}{3}} (\log\log Q)^{\frac{1}{3}}),$$

where the implicit constant depends only on A and α.

2. Notation and Conventions

For a complex number α, we define the generalized divisor function $\tau_\alpha(n)$ by

$$\zeta(s)^\alpha = \sum_{n=1}^{\infty} \frac{\tau_\alpha(n)}{n^s} \quad (\sigma > 1)$$

where the branch of $\zeta(s)^\alpha$ is taken by $\arg \zeta(s) = 0$ for $s > 1$.

For a positive integer $n > 1$, we let $p_{\max}(n)$ and $p_{\min}(n)$ be the largest and smallest prime factor of n, respectively, and $P_{\max}(n)$ and $P_{\min}(n)$ be the largest and smallest prime power dividing n, respectively. As a convention, we let $p_{\max}(1) = P_{\max}(1) = 1$ and $p_{\min}(1) = P_{\min}(1) = +\infty$.

If Theorem or Lemma is stated with the phrase "where the implicit constant depends only on $a, b, c, \ldots$", then every implicit constant in the corresponding proof may also depend on $a, b, c, \ldots$ even without special mentions.

3. Preliminary lemmas and basic exponential sums

In this section, we prove some preliminary lemmas. Most of those are just taken from [9] or some immediate consequence of lemmas in [9]. We start with some basic estimates on generalized divisor function.

Lemma 2. *For any complex number α and $x \geq 2$,*

$$\sum_{n \leq x} \frac{|\tau_\alpha(n)|}{n} \ll (\log x)^{|\alpha|}, \quad \sum_{n \leq x} |\tau_\alpha(n)|^2 \ll x(\log x)^{|\alpha|^2},$$

where the implicit constant depends on α.

Proof. See [9, Lemma 4.16]. □

We next recall some basic estimates for our current type of exponential sums:

Lemma 3. *Let $P, P', Q \geq 4$ be real numbers with $P \leq P' \leq 2P$. Then*

$$\sum_{P < n \leq P'} e\left(\frac{Q}{n}\right) \ll P \exp\left(-\gamma \frac{(\log P)^3}{(\log Q)^2}\right) + P^2 Q^{-1},$$

where the implicit constant and the constant $\gamma > 0$ are absolute.

Proof. This follows by Vinogradov's mean value theorem. See [9, Lemma 4.3]. □

Actually, for the positive integral order divisor function $\tau_k(n)$, our current problem is rather easy. It suffices to take convolutions of Lemma 3. This case will be the basis of the subsequent Type I sum estimates.

Lemma 4. *For any real number $A \geq 1$ and any positive integer k, there exists a real number $B = B(A, k) \geq 1$ such that for any real numbers $P, P', Q \geq 4$ with $P \leq P' \leq 2P$, we have*

$$\sum_{P < n \leq P'} \tau_k(n) e\left(\frac{Q}{n}\right) \ll \left(P(\log Q)^{-A} + P^2 Q^{-1}\right) (\log Q)^{k-1}$$

provided

$$P > \exp(B(\log Q)^{\frac{2}{3}} (\log \log Q)^{\frac{1}{3}}),$$

where the implicit constant depends on A and k.

Proof. We may assume $P \leq Q$ since if otherwise the assertion is reduced to the trivial estimate. Let $B(A, k) := (k+1)(A/\gamma)^{\frac{1}{3}}$ with γ in Lemma 3. We have

$$\sum_{P < n \leq P'} \tau_k(n) e\left(\frac{Q}{n}\right) = \sum_{P < d_1 \cdots d_k \leq P'} e\left(\frac{Q}{d_1 \cdots d_k}\right) = \sum_{\nu=1}^{k} S_\nu,$$

where

$$S_\nu := \sum_{\substack{P<d_1\cdots d_k\le P' \\ d_1,\ldots,d_{\nu-1}\le P^{\frac{1}{k+1}},\ d_\nu>P^{\frac{1}{k+1}}}} e\left(\frac{Q}{d_1\cdots d_k}\right).$$

Indeed, if there is a remaining term corresponding to the tuple $(d_1,\ldots,d_k)$, then this tuple should satisfy $d_1,\ldots,d_k \le P^{\frac{1}{k+1}}$ but this provides a contradiction $P < d_1\cdots d_k \le P^{\frac{k}{k+1}}$. For each ν, we can rewrite S_ν as

$$(3)\quad S_\nu = \sum_{d\le P'} a_\nu(d) \sum_{\max(P/d,P^{\frac{1}{k+1}})<m\le P'/d} e\left(\frac{Q/d}{m}\right),\quad a_\nu(d) := \sum_{\substack{d_1\cdots d_{\nu-1}d_{\nu+1}\cdots d_k=d \\ d_1,\ldots,d_{\nu-1}\le P^{\frac{1}{k+1}}}} 1.$$

By applying Lemma 3 to the inner sum of (3) and recalling our choice of $B(A,k)$,

$$S_\nu \ll (P(\log Q)^{-A} + P^2Q^{-1}) \sum_{d\le 2P} \frac{a_\nu(d)}{d} \ll (P(\log Q)^{-A} + P^2Q^{-1})(\log Q)^{k-1}.$$

By summing up this estimate over ν, we arrive at the lemma. □

We also recall the following Type II sum estimate.

Lemma 5 (Type II sum estimate)**.** *For any real number $A \ge 1$, there exists a real number $B = B(A) \ge 1$ such that for any sequences of complex numbers*

$$\mathcal{A} = (\alpha_u)_{u=1}^\infty,\quad \mathcal{B} = (\beta_v)_{v=1}^\infty$$

and any real numbers $P, P', U, U', V, V', Q \ge 4$ with

$$P \le P' \le 2P,\quad U \le U' \le 2U,\quad V \le V' \le 2V',$$

we have

$$\sum_{\substack{P<uv\le P' \\ U<u\le U' \\ V<v\le V'}} \alpha_u\beta_v e\left(\frac{Q}{uv}\right) \ll \left(P^{\frac{1}{2}}(\log Q)^{-A} + PQ^{-\frac{1}{2}}\right)\|\mathcal{A}\|\|\mathcal{B}\|(\log Q)^{\frac{1}{2}}$$

provided $U, V \ge \exp(B(\log Q)^{\frac{2}{3}}(\log\log Q)^{\frac{1}{3}})$, where

$$\|\mathcal{A}\|^2 = \sum_{U<u\le U'} |\alpha_u|^2,\quad \|\mathcal{B}\|^2 = \sum_{V<v\le V'} |\beta_v|^2$$

and the implicit constant depends only on A.

Proof. This is a simple consequence of Lemma 3. See [9, Lemma 4.5]. □

4. Exponential sums over smooth numbers

In this section, we estimate exponential sums over smooth numbers. This is the arguments not included in the author's approach [9].

We first recall a technique on the separation of variables in bilinear forms. We use the form of the separation of variables given in Iwaniec–Kowalski [3, Lemma 13.11]. We start with a simple Fourier analytic lemma.

Lemma 6. *For any $\Delta \geq 2$, there exists a complex-valued function $w(t)$ such that*

$$\int_{-\infty}^{\infty} |w(t)| dt \ll \log \Delta, \tag{4}$$

where the implicit constant is absolute and

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} w(t) \left(\frac{x}{y}\right)^{it} dt = \begin{cases} 0 & (\textit{if } x \geq y), \\ 1 & (\textit{if } \frac{1}{\Delta} y \leq x \leq e^{-\frac{1}{\Delta}} y) \end{cases} \tag{5}$$

for any positive real numbers x, y.

Proof. Let us define $f(x)$ for $x \in \mathbb{R}$ by

$$f(\xi) := \begin{cases} 1 & (\text{if } -\log \Delta \leq \xi \leq -\frac{1}{\Delta}), \\ -\Delta \xi & (\text{if } -\frac{1}{\Delta} \leq \xi \leq 0), \\ \Delta(\xi + \log \Delta + \frac{1}{\Delta}) & (\text{if } -\log \Delta - \frac{1}{\Delta} \leq \xi \leq -\log \Delta), \\ 0 & (\text{if } \xi \leq -\log \Delta - \frac{1}{\Delta} \text{ or } 0 \leq \xi). \end{cases}$$

(Note that $\frac{1}{\Delta} \leq \log \Delta$ since $\Delta \geq 2$.) We take $w(t)$ by

$$w(t) := \hat{f}\left(\frac{t}{2\pi}\right) = \int_{-\infty}^{\infty} f(\xi) e^{-i\xi t} d\xi.$$

Note that $f(\xi)$ is integrable over $\mathbb{R}$, continuous and has bounded variation. Thus, we can apply the Fourier inversion formula to get

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} w(t) \left(\frac{x}{y}\right)^{it} dt = \int_{-\infty}^{\infty} \hat{f}(t) e\left(t \log \frac{x}{y}\right) dt = f\left(\log \frac{x}{y}\right). \tag{6}$$

Since

$$-\log \Delta \leq \log \frac{x}{y} \leq -\frac{1}{\Delta} \iff \frac{1}{\Delta} y \leq x \leq e^{-\frac{1}{\Delta}} y \quad \text{and} \quad \log \frac{x}{y} \geq 0 \iff x \geq y,$$

the inversion formula (6) implies (5). Thus, it suffices to show (4).

By integration by parts, we can easily find

$$w(t) = -\frac{\Delta(\Delta^{it} - 1)(1 - e^{-\frac{it}{\Delta}})}{t^2}$$

so we obtain the estimate

$$w(t) \ll \min\left(\log \Delta, \frac{1}{|t|}, \frac{\Delta}{|t|^2}\right).$$

Therefore,

$$\int_{-\infty}^{\infty} |w(t)| dt \ll 1 + \int_{\frac{1}{\log \Delta}}^{\Delta} \frac{dt}{t} + \Delta \int_{\Delta}^{\infty} \frac{dt}{t^2} \ll \log \Delta + \log \log \Delta \ll \log \Delta.$$

This proves (4) and completes the proof. □

Then, we employ the separation of variables via the following lemma:

Lemma 7. *Let (α_u), (β_v), $(h(u, v))$ be sequences of complex numbers with*

$$\sum_{u,v} |h(u, v)| < +\infty$$

and (x_u), (y_v) *be sequences of positive real numbers satisfying*

$$\frac{1}{\Delta} \le \frac{x_u}{y_v} \quad \text{and} \quad x_u < y_v \implies \frac{x_u}{y_v} \le e^{-\frac{1}{\Delta}}$$

for some real number $\Delta \ge 2$. *Then, we have*

$$\sum_{\substack{u,v \\ x_u < y_v}} \alpha_u \beta_v h(u,v) \ll (\log \Delta) \sup_{t \in \mathbb{R}} \left| \sum_{u,v} \alpha_u x_u^{it} \beta_v y_v^{-it} h(u,v) \right|,$$

where the implicit constant is absolute.

Proof. For any u and v, Lemma 6 and the assumption implies

$$\int_{-\infty}^{\infty} w(t) \left(\frac{x_u}{y_v} \right)^{it} dt = \begin{cases} 0 & (\text{if } x_u \ge y_v), \\ 1 & (\text{if } x_u < y_v). \end{cases}$$

Therefore,

$$\begin{aligned} \sum_{\substack{u,v \\ x_u < y_v}} \alpha_u \beta_v h(u,v) &= \sum_{u,v} \alpha_u \beta_v h(u,v) \int_{-\infty}^{\infty} w(t) \left(\frac{x_u}{y_v} \right)^{it} dt \\ &= \int_{-\infty}^{\infty} w(t) \sum_{u,v} \alpha_u x_u^{it} \beta_v y_v^{-it} h(u,v) dt \\ &\ll \left(\int_{-\infty}^{\infty} |w(t)| dt \right) \sup_{t \in \mathbb{R}} \left| \sum_{u,v} \alpha_u x_u^{it} \beta_v y_v^{-it} h(u,v) \right|. \end{aligned}$$

By (4), we obtain the lemma. □

We shall decompose exponential sums over smooth numbers into some trivial sum and Type II sums. Note that similar trivial sum has been already seen in Vinogradov's decomposition [10, p. 187–188]. We first give a lemma for the Type II sum part.

Lemma 8. *Let* $f(n)$ *be a multiplicative function and* $h(n)$ *be an arithmetic function. Let* P, P', w *be real numbers with* $4 \le P \le P' \le 2P$ *and* $2 \le w \le P$. *Then,*

$$\sum_{\substack{P < n \le P' \\ P_{\max}(n) \le w}} f(n) h(n) \ll (\log P) \sup \left| \sum_{\substack{P < uv \le P' \\ P^{\frac{1}{2}} w^{-\frac{1}{2}} < u \le P^{\frac{1}{2}} w^{\frac{1}{2}}}} \alpha_u \beta_v h(uv) \right|,$$

where the supremum is taken over real coefficients (α_u) *and* (β_u) *satisfying*

$$|\alpha_u| \le |f(u)| \quad \text{and} \quad |\beta_v| \le |f(v)|$$

and the implicit constant is absolute.

Proof. For any positive integer n with

$$P < n \le P', \quad P_{\max}(n) \le w,$$

we have a unique decomposition

$$n = uv \text{ with } P^{\frac{1}{2}} w^{-\frac{1}{2}} < u \le P^{\frac{1}{2}} w^{-\frac{1}{2}} P_{\max}(u) \text{ and } P_{\max}(u) < P_{\min}(v).$$

Therefore, by using the multiplicativity of $v(n)$ and setting

$$\gamma_u := \begin{cases} f(u) & (\text{if } P^{\frac{1}{2}}w^{-\frac{1}{2}} < u \le P^{\frac{1}{2}}w^{-\frac{1}{2}}P_{\max}(u) \text{ and } P_{\max}(u) \le w), \\ 0 & (\text{otherwise}), \end{cases}$$

$$\delta_v := \begin{cases} f(v) & (\text{if } P_{\max}(v) \le w), \\ 0 & (\text{otherwise}), \end{cases}$$

we find that γ_u is supported only for $P^{\frac{1}{2}}w^{-\frac{1}{2}} < u \le P^{\frac{1}{2}}w^{\frac{1}{2}}$ and

$$\sum_{\substack{P<n\le P' \\ P_{\max}(n)\le w}} f(n)h(n) = \sum_{\substack{P<uv\le P' \\ P^{\frac{1}{2}}w^{-\frac{1}{2}}<u\le P^{\frac{1}{2}}w^{\frac{1}{2}} \\ P_{\max}(u)<P_{\min}(v)}} \gamma_u\delta_v h(uv).$$

Note that in the above sum, we have $P_{\max}(u)/P_{\min}(v) \ge 1/P$ and

$$P_{\max}(u) < P_{\min}(v) \quad \Longrightarrow \quad \frac{P_{\max}(u)}{P_{\min}(v)} \le 1 - \frac{1}{P_{\max}(v)} \le e^{-\frac{1}{P}}.$$

By applying Lemma 7 with $\alpha_u := \gamma_u$, $\beta_v := \delta_v$, $x_u := P_{\max}(u)$, $y_v := P_{\min}(v)$ and

$$h(u,v) := \begin{cases} h(uv) & (\text{if } P < uv \le P'), \\ 0 & (\text{otherwise}), \end{cases}$$

we arrive at the lemma. □

We can now estimate exponential sums over smooth numbers.

Lemma 9. *Let $v(n)$ be a complex-valued multiplicative function satisfying*

$$\sum_{n\le x} |v(n)|^2 \ll x(\log x)^C \quad \text{for } x \ge 2 \tag{7}$$

with some real constant $C > 0$. Then, for any real number $A \ge 1$, there exists a real number $B = B(A) \ge 1$ such that for any real numbers P, P', Q, z with

$$P, P', Q \ge 4, \quad P \le P' \le 2P, \quad z \le (2P)^{\frac{1}{3}},$$

we have

$$\sum_{\substack{P<n\le P' \\ p_{\min}(n)\le z}} v(n)e\left(\frac{Q}{n}\right) \ll \left(P(\log Q)^{-A} + P^{\frac{3}{2}}Q^{-\frac{1}{2}} + Pz^{-\frac{1}{4}}\right)(\log Q)^{C+\frac{5}{2}}$$

provided

$$P \ge \exp(B(\log Q)^{\frac{2}{3}}(\log\log Q)^{\frac{1}{3}}),$$

where the implicit constant depends only on A and α.

Proof. We may assume that $P \le Q$ and Q is sufficiently large. Let $B_1(A)$ be the constant in Lemma 5 and for the current lemma, let $B(A) := 4B_1(A)$.

We decompose the sum as

$$\sum_{\substack{P<n\le P' \\ p_{\min}(n)\le z}} v(n)e\left(\frac{Q}{n}\right) = \sum_{P_{\max}(n)\le z} + \sum_{P_{\max}(n)>z} = \sum\nolimits_{\mathrm{I}} + \sum\nolimits_{\mathrm{II}}, \quad \text{say}.$$

and consider the last two sums separately.

For the sum $\sum_{\mathrm{I}}$, we use Lemma 8 with $w = z$. Note that the condition $p_{\max}(n) \le z$ can be translated to the multiplicative function $f(n)$ of Lemma 8. Then, we get

$$\sum\nolimits_{\mathrm{I}} \ll (\log Q) \sup \left| \sum_{\substack{P<uv\le P' \\ P^{\frac{1}{2}}z^{-\frac{1}{2}}<u\le P^{\frac{1}{2}}z^{\frac{1}{2}}}} \alpha_u \beta_v e\left(\frac{Q}{uv}\right) \right| \tag{8}$$

with the supremum given as in Lemma 8. The double sum on the right hand side can be decomposed dyadically into at most $O(\log Q)$ double sums of the form

$$\sum_{\substack{P<uv\le P' \\ U<u\le U' \\ V<v\le V'}} \alpha_u \beta_v e\left(\frac{Q}{uv}\right)$$

where $U \le U' \le 2U$, $V \le V' \le 2V$, $UV \le 2P$ and $U, V > P^{\frac{1}{2}}z^{-\frac{1}{2}} > 2^{-\frac{1}{6}}P^{\frac{1}{3}} > P^{\frac{1}{4}}$ for large P since $v = n/u > P/u \ge P^{\frac{1}{2}}z^{-\frac{1}{2}}$ in (8) and we are assuming $z \le (2P)^{\frac{1}{3}}$. Thus, by using the estimate $\|\mathcal{A}\|\|\mathcal{B}\| \ll P^{\frac{1}{2}}(\log Q)^C$ and our choice of $B(A)$ in Lemma 5,

$$\sum\nolimits_{\mathrm{I}} \ll \left(P(\log Q)^{-A} + P^{\frac{3}{2}}Q^{-\frac{1}{2}}\right)(\log Q)^{C+\frac{5}{2}}.$$

This completes the estimate for $\sum_{\mathrm{I}}$.

For the sum $\sum_{\mathrm{II}}$, we use a trivial estimate. By the Cauchy–Schwarz inequality,

$$\sum\nolimits_{\mathrm{II}} \ll \left(\sum_{P<n\le P'} |v(n)|^2\right)^{\frac{1}{2}} \left(\sum_{\substack{P<n\le P' \\ p_{\max}(n)\le z \\ P_{\max}(n)>z}} 1\right)^{\frac{1}{2}} \tag{9}$$

For the former factor, we can apply (7). For the latter factor, we use the following observation: For any positive integer n with $p_{\max}(n) \le z$ and $P_{\max}(n) > z$, there exists a prime power $p^\nu > z$ with $\nu \ge 2$ dividing n. Indeed, supposing $P_{\max}(n) = p^\nu$, we have $p \le p_{\max}(n) \le z$ so the exponent ν should be at least 2 and obviously $p^\nu > z$. Therefore,

$$\sum_{\substack{P<n\le P' \\ p_{\max}(n)\le z \\ P_{\max}(n)>z}} 1 \le \sum_{\substack{z<p^\nu\le 2P \\ \nu\ge 2}} \sum_{\substack{P<n\le P' \\ p^\nu | n}} 1 \le P \sum_{\substack{z<p^\nu\le 2P \\ \nu\ge 2}} \frac{1}{p^\nu}.$$

Since

$$\sum_{\substack{p^\nu\le x \\ \nu\ge 2}} 1 \ll x^{\frac{1}{2}} + \sum_{\nu=3}^{O(\log x)} \sum_{p^\nu\le x} 1 \ll x^{\frac{1}{2}} + x^{\frac{1}{3}}\log x \ll x^{\frac{1}{2}},$$

by partial summation, we have

$$\sum_{\substack{P<n\le P' \\ p_{\max}(n)\le z \\ P_{\max}(n)>z}} 1 \ll Pz^{-\frac{1}{2}}.$$

Therefore, by (9),

$$\sum\nolimits_{\mathrm{II}} \ll (P(\log P)^C)^{\frac{1}{2}}(Pz^{-\frac{1}{2}})^{\frac{1}{2}} \ll Pz^{-\frac{1}{4}}(\log Q)^C.$$

This completes the proof. □

5. Exponential sums with generalized divisor functions

The next lemma gives a variant of Linnik's identity which is used by Drappeau and Topacogullari [2].

Lemma 10 (Drappeau–Topacogullari–Linnk identity)**.** *Let α be a complex number, $z \geq 2$ be a real number and $K \geq 1$ be a positive integer. Then for any positive integer $n \leq z^{K+1}$, we have*

$$\tau_\alpha(n) = \sum_{k=0}^{K} c_k \sum_{\substack{dm=n \\ p_{\max}(d)\leq z}} \tau_{\alpha-k}(d)\tau_k(m), \quad \textit{where} \quad c_k = \sum_{\ell=0}^{K} (-1)^{\ell-k}\binom{\alpha}{\ell}\binom{\ell}{k}.$$

Proof. See [2, Theorem 3.3] □

We finally arrive at the proof of Lemma 1.

Proof of Lemma 1. We may assume that $P \leq Q$ and Q is sufficiently large. Let $B_1(A,k)$ be the constant in Lemma 4 and $B_2(A)$ be the constant in Lemma 9. Let

$$B(A) \coloneqq \max(2B_1(A,0),\ldots,2B_1(A,5),2B_2(A)).$$

We apply Lemma 10 with $z \coloneqq (2P)^{\frac{1}{6}}$ and $K = 5$. Then, since $P < n \leq P'$ implies $n \leq z^6$, we can apply Lemma 10 for every term on the left-hand side to obtain

$$\sum_{P<n\leq P'} \tau_\alpha(n) e\left(\frac{Q}{n}\right) \ll \sup_{0\leq k\leq 5} |S_k|,$$

where

$$S_k = \sum_{\substack{P<dm\leq P' \\ p_{\max}(d)\leq z}} \tau_{\alpha-k}(d)\tau_k(m) e\left(\frac{Q}{dm}\right).$$

We estimate each S_k separately.

We first introduce the decomposition

$$S_k = \sum_{\substack{P<dm\leq P' \\ p_{\max}(d)\leq z \\ m>P^{\frac{1}{2}}}} + \sum_{\substack{P<dm\leq P' \\ p_{\max}(d)\leq z \\ m\leq P^{\frac{1}{2}}}} = S_{k,1} + S_{k,2}.$$

For the sum $S_{k,1}$, we apply Lemma 4. By definition,

$$S_{k,1} = \sum_{\substack{d\leq P'/P^{\frac{1}{2}} \\ p_{\max}(d)\leq z}} \tau_{\alpha-k}(d) \sum_{\max(P/d,P^{\frac{1}{2}})<m\leq P'/d} \tau_k(m) e\left(\frac{Q/d}{m}\right)$$

By our choice of $B(A)$ and by Lemma 4, this gives

$$\begin{aligned} S_{k,1} &\ll (P(\log Q)^{-A} + P^2Q^{-1})(\log Q)^{k-1} \sum_{d\leq 2P} \frac{|\tau_{\alpha-k}(d)|}{d} \\ &\ll (P(\log Q)^{-A} + P^2Q^{-1})(\log Q)^{|\alpha|+9}, \end{aligned}$$

where we used Lemma 2.

For the sum $S_{k,2}$, we apply Lemma 9. By definition,

$$S_{k,2} = \sum_{m \le P^{\frac{1}{2}}} \tau_k(m) \sum_{\substack{P/m < d \le P'/m \\ p_{\max}(d) \le z}} \tau_{\alpha-k}(d) e\left(\frac{Q/m}{d}\right).$$

Since in the inner sum on the right-hand side, we have $P/m \ge P^{\frac{1}{2}}$ and

$$z = (2P)^{\frac{1}{6}} \le (2P/m)^{\frac{1}{3}},$$

by using Lemma 2, we can apply Lemma 9 to obtain

$$\begin{aligned} S_{k,2} &\ll \left(P(\log Q)^{-A} + P^{\frac{3}{2}} Q^{-\frac{1}{2}} + P z^{-\frac{1}{4}}\right) (\log Q)^{|\alpha|^2 + \frac{5}{2}} \sum_{m \le 2P} \frac{\tau_k(m)}{m} \\ &\ll \left(P(\log Q)^{-A} + P^{\frac{3}{2}} Q^{-\frac{1}{2}}\right) (\log Q)^{|\alpha|^2 + 9}. \end{aligned}$$

By combining the above estimates, we arrive at the lemma. □

Acknowledgement

The author would like to thank Prof. Hidehiko Mishou and Prof. Masatoshi Suzuki for kindly giving the author an opportunity to give a talk in RIMS workshop "Analytic Number Theory and Related Topics". This work was supported by Grant-in-Aid for JSPS Research Fellow (Grant Number: JP16J00906).

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