Algebraic independence of the values of a certain map defined on the set of orbits of the action of Klein four-group

慶應義塾大学理工学部　田中 孝明（Taka-aki Tanaka）
Faculty of Science and Technology, Keio Univ.

1 Introduction

Let $\{R_k\}_{k\geq 1}$ be a linear recurrence of positive integers satisfying

$$R_{k+n} = c_1 R_{k+n-1} + \cdots + c_n R_k \quad (k \geq 1), \tag{1}$$

where $n \geq 2$ and $c_1, \ldots, c_n$ are nonnegative integers with $c_n \neq 0$. The author [9] studied the two-variable function $E(x, q)$ defined by

$$E(x,q) = \sum_{k=1}^{\infty} \prod_{l=1}^{k} \frac{xq^{R_l}}{1-q^{R_l}} = \sum_{k=1}^{\infty} \frac{x^k q^{R_1+R_2+\cdots+R_k}}{(1-q^{R_1})(1-q^{R_2})\cdots(1-q^{R_k})},$$

which may be regarded as an analogue of q-exponential function

$$E_q(x) = 1 + \sum_{k=1}^{\infty} \frac{x^k q^{1+2+\cdots+k}}{(1-q)(1-q^2)\cdots(1-q^k)}$$

(cf. Gasper and Rahman [2]), if we replace k in the exponent of q in $E_q(x)$ with $\{R_k\}_{k\geq 1}$ defined above.

Let

$$\Phi(X) = X^n - c_1 X^{n-1} - \cdots - c_n \tag{2}$$

and let $\overline{\mathbb{Q}}^{\times}$ be the set of nonzero algebraic numbers. The author proved the following

Theorem 0 (Corollary 4 of [9]). *Let $\{R_k\}_{k\geq 1}$ be a linear recurrence satisfying* (1). *Suppose that $\Phi(\pm 1) \neq 0$ and the ratio of any pair of distinct roots of $\Phi(X)$ is not a root of unity. Assume that $\{R_k\}_{k\geq 1}$ is not a geometric progression. Then the values*

$$E(x,q) \quad (x, q \in \overline{\mathbb{Q}}^{\times},\ |q| < 1)$$

are algebraically dependent if and only if there exist some distinct pairs (x_1, q_1) and (x_2, q_2) of nonzero algebraic numbers x_1, x_2, q_1, and q_2 with $|q_1|, |q_2| < 1$ such that $x_1 = x_2$ and $q_1^{N_k} = q_2^{N_k}$ for some $k \geq 1$, where $N_k = \mathrm{g.c.d.}(R_k, R_{k+1}, \ldots, R_{k+n-1})$.

In particular, if $N_k = 1$ for any $k \geq 1$, then the values $E(x, q)$ are algebraically independent for any distinct pairs (x, q) of nonzero algebraic numbers x and q with $|q| < 1$.

Example 0. Let $\{F_k\}_{k\geq 1}$ be the sequence of Fibonacci numbers defined by $F_1 = 1,\ F_2 = 1$, and $F_{k+2} = F_{k+1} + F_k$ $(k \geq 1)$. Since $\{F_k\}_{k\geq 1}$ satisfies the conditions in Theorem 0, the infinite set of the values

$$\left\{ \sum_{k=1}^{\infty} \frac{x^k q^{F_1+F_2+\cdots+F_k}}{(1-q^{F_1})(1-q^{F_2})\cdots(1-q^{F_k})} \;\middle|\; x, q \in \overline{\mathbb{Q}}^{\times},\ |q| < 1 \right\}$$

is algebraically independent.

The two-variable function $E(x,q)$ converges on the domain

$$(\mathbb{C} \times \{|q| < 1\}) \cup (\{|x| < 1\} \times \{|q| > 1\}) := \{(x,q) \in \mathbb{C}^2 \mid |q| < 1 \vee (|x| < 1 \wedge |q| > 1)\},$$

whereas a 'balanced' analogue

$$\Theta(x,q) = \sum_{k=1}^{\infty} \prod_{l=1}^{k} \frac{xq^{R_l}}{1-q^{2R_l}} = \sum_{k=1}^{\infty} \frac{x^k q^{R_1+R_2+\cdots+R_k}}{(1-q^{2R_1})(1-q^{2R_2})\cdots(1-q^{2R_k})}$$

converges on the wider domain

$$\mathbb{C} \times \{|q| \neq 1\} := \{(x,q) \in \mathbb{C}^2 \mid |q| \neq 1\}.$$

Indeed, if $q \neq 0$, $\Theta(x,q)$ is invariant under the map

$$\sigma_1 \ :\ (x,q) \longmapsto (-x, q^{-1}),$$

namely

$$\Theta(\sigma_1(x,q)) = \sum_{k=1}^{\infty} \frac{(-x)^k q^{-R_1-R_2-\cdots-R_k}}{(1-q^{-2R_1})(1-q^{-2R_2})\cdots(1-q^{-2R_k})} = \Theta(x,q)$$

and so $\Theta(x,q)$ converges on $\mathbb{C} \times \{|q| \neq 1\}$ by the similar reason to the convergence of $E(x,q)$.

Moreover, if $\{R_k\}_{k\geq 1}$ is a sequence of odd integers, then $\Theta(x,q)$ is invariant also under the maps

$$\begin{aligned} \sigma_2 \ &:\ (x,q) \longmapsto (-x, -q), \\ \sigma_3 \ &:\ (x,q) \longmapsto (x, -q^{-1}). \end{aligned}$$

Since $\sigma_1 \circ \sigma_1 = \sigma_2 \circ \sigma_2 = \mathrm{id}$ and $\sigma_1 \circ \sigma_2 = \sigma_2 \circ \sigma_1 = \sigma_3$, we see that $G_4 = \{\mathrm{id},\ \sigma_1,\ \sigma_2,\ \sigma_3\}$ is Klein four-group. Therefore, $\Theta(x,q)$ can be regarded as a map defined on the set of orbits $(\mathbb{C} \times \{|q| \neq 0,1\})/G_4$, where $\mathbb{C} \times \{|q| \neq 0,1\} = \{(x,q) \in \mathbb{C}^2 \mid |q| \neq 0,1\}$, namely the map

$$\widetilde{\Theta} \ :\ (\mathbb{C} \times \{|q| \neq 0,1\})/G_4 \longrightarrow \Theta(\mathbb{C} \times \{|q| \neq 0,1\})$$

given by

$$\text{the orbit of } (x,q) \longmapsto \Theta(x,q)$$

is well-defined. Hence the restriction to algebraic points

$$\widetilde{\Theta} : \left((\mathbb{C} \times \{|q| \neq 0,1\}) \cap \left(\overline{\mathbb{Q}}^{\times}\right)^2 \right) \Big/ G_4 \longrightarrow \Theta\left((\mathbb{C} \times \{|q| \neq 0,1\}) \cap \left(\overline{\mathbb{Q}}^{\times}\right)^2 \right),$$

or equivalently

$$\widetilde{\Theta} : \left(\overline{\mathbb{Q}}^{\times} \times (\overline{\mathbb{Q}}^{\times} \setminus \{|q| = 1\})\right) \Big/ G_4 \longrightarrow \Theta\left(\overline{\mathbb{Q}}^{\times} \times (\overline{\mathbb{Q}}^{\times} \setminus \{|q| = 1\})\right)$$

is also well-defined, where the second $\overline{\mathbb{Q}}^{\times}$ denotes the multiplicative group of nonzero algebraic numbers while the first $\overline{\mathbb{Q}}^{\times}$ simply denotes the set of nonzero algebraic numbers. In this paper we prove the following

Theorem 1. *Let $\{R_k\}_{k\geq 1}$ be a linear recurrence satisfying (1). Suppose that $\Phi(\pm 1) \neq 0$ and the ratio of any pair of distinct roots of $\Phi(X)$ is not a root of unity. Assume that* g.c.d.$(R_k, R_{k+1}, \ldots, R_{k+n-1}) = 1$ *for any $k \geq 1$. Assume further that $\Phi(2) < 0$ and that $\{R_k\}_{k\geq 1}$ is a sequence of odd integers. Then the infinite set of the values*

$$\widetilde{\Theta}\left(\left(\overline{\mathbb{Q}}^{\times} \times (\overline{\mathbb{Q}}^{\times} \setminus \{|q| = 1\})\right) \Big/ G_4\right)$$

is algebraically independent.

Remark 1. The condition that g.c.d.$(R_k, R_{k+1}, \ldots, R_{k+n-1}) = 1$ for any $k \geq 1$ implies that the sequence $\{R_k\}_{k\geq 1}$ is not a geometric progression.

Corollary 1. *Let $\{R_k\}_{k\geq 1}$ be as in Theorem 1. Then the infinite set consisting of the distinct values of*

$$\left\{ \sum_{k=1}^{\infty} \frac{x^k q^{R_1+R_2+\cdots+R_k}}{(1-q^{2R_1})(1-q^{2R_2})\cdots(1-q^{2R_k})} \;\middle|\; x, q \in \overline{\mathbb{Q}}^{\times},\ |q| \neq 1 \right\}$$

is algebraically independent.

Example 1. Let $\{P_k\}_{k\geq 1}$ be the sequence defined either by $P_1 = P_2 = 1$ and $P_{k+2} = 2P_{k+1} + P_k$ $(k \geq 1)$ or by $P_1 = P_2 = P_3 = 1$ and $P_{k+3} = P_{k+2} + P_{k+1} + 3P_k$ $(k \geq 1)$. Since $\{P_k\}_{k\geq 1}$ satisfies all the conditions of Theorem 1, the infinite set consisting of the distinct values of

$$\left\{ \sum_{k=1}^{\infty} \frac{x^k q^{P_1+P_2+\cdots+P_k}}{(1-q^{2P_1})(1-q^{2P_2})\cdots(1-q^{2P_k})} \;\middle|\; x, q \in \overline{\mathbb{Q}}^{\times},\ |q| \neq 1 \right\}$$

is algebraically independent.

If $\{R_k\}_{k\geq 1}$ is a sequence of odd integers, then

$$\Theta_+(x, q) := \sum_{k=1}^{\infty} \prod_{l=1}^{k} \frac{x q^{R_l}}{1 + q^{2R_l}} = \sum_{k=1}^{\infty} \frac{x^k q^{R_1+R_2+\cdots+R_k}}{(1+q^{2R_1})(1+q^{2R_2})\cdots(1+q^{2R_k})}$$

is invariant under the maps

$$\begin{aligned} \tau_1 &: (x, q) \longmapsto (x, q^{-1}), \\ \tau_2 &: (x, q) \longmapsto (-x, -q), \\ \tau_3 &: (x, q) \longmapsto (-x, -q^{-1}). \end{aligned}$$

Since $\tau_1 \circ \tau_1 = \tau_2 \circ \tau_2 = \text{id}$ and $\tau_1 \circ \tau_2 = \tau_2 \circ \tau_1 = \tau_3$, we see that $G_4' = \{\text{id}, \tau_1, \tau_2, \tau_3\}$ is also Klein four-group. Hence the map

$$\widetilde{\Theta}_+ \ : \ (\mathbb{C} \times \{|q| \neq 0, 1\})/G_4' \longrightarrow \Theta_+(\mathbb{C} \times \{|q| \neq 0, 1\})$$

given by

$$\text{the orbit of } (x, q) \longmapsto \Theta_+(x, q)$$

is well-defined. We also have the following

Theorem 2. *Let $\{R_k\}_{k\geq 1}$ be as in Theorem 1. Then the infinite set of the values*

$$\widetilde{\Theta}_+ \left(\left(\overline{\mathbb{Q}}^\times \times (\overline{\mathbb{Q}}^\times \setminus \{|q| = 1\}) \right) \Big/ G_4' \right)$$

is algebraically independent.

Example 2. Let $\{P_k\}_{k\geq 1}$ be one of the linear recurrences defined in Example 1. Since $\{P_k\}_{k\geq 1}$ satisfies all the conditions of Theorem 1, the infinite set consisting of the distinct values of

$$\left\{ \sum_{k=1}^{\infty} \frac{x^k q^{P_1+P_2+\cdots+P_k}}{(1+q^{2P_1})(1+q^{2P_2})\cdots(1+q^{2P_k})} \ \middle| \ x, q \in \overline{\mathbb{Q}}^\times, \ |q| \neq 1 \right\}$$

is algebraically independent.

2 Lemmas

Let $F(z_1, \ldots, z_n)$ and $F[[z_1, \ldots, z_n]]$ denote the field of rational functions and the ring of formal power series in variables $z_1, \ldots, z_n$ with coefficients in a field F, respectively, and $F^\times$ the multiplicative group of nonzero elements of F. Let $\Omega = (\omega_{ij})$ be an $n \times n$ matrix with nonnegative integer entries. Then the maximum ρ of the absolute values of the eigenvalues of Ω is itself an eigenvalue (cf. Gantmacher [1, p. 66, Theorem 3]). If $\boldsymbol{z} = (z_1, \ldots, z_n)$ is a point of $\mathbb{C}^n$, we define a transformation $\Omega : \mathbb{C}^n \to \mathbb{C}^n$ by

$$\Omega \boldsymbol{z} = \left(\prod_{j=1}^{n} z_j^{\omega_{1j}}, \prod_{j=1}^{n} z_j^{\omega_{2j}}, \ldots, \prod_{j=1}^{n} z_j^{\omega_{nj}} \right). \tag{3}$$

We suppose that Ω and an algebraic point $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$, where α_i are nonzero algebraic numbers, have the following four properties:

(I) Ω is nonsingular and none of its eigenvalues is a root of unity, so that in particular $\rho > 1$.

(II) Every entry of the matrix Ω^k is $O(\rho^k)$ as k tends to infinity.

(III) If we put $\Omega^k \boldsymbol{\alpha} = (\alpha_1^{(k)}, \ldots, \alpha_n^{(k)})$, then

$$\log |\alpha_i^{(k)}| \leq -c\rho^k \quad (1 \leq i \leq n)$$

for all sufficiently large k, where c is a positive constant.

(IV) For any nonzero $f(\boldsymbol{z}) \in \mathbb{C}[[z_1,\ldots,z_n]]$ which converges in some neighborhood of the origin, there are infinitely many positive integers k such that $f(\Omega^k \boldsymbol{\alpha}) \neq 0$.

Lemma 1 (Lemma 4 and Proof of Theorem 2 in [6]). *Suppose that $\Phi(\pm 1) \neq 0$ and the ratio of any pair of distinct roots of $\Phi(X)$ is not a root of unity, where $\Phi(X)$ is the polynomial defined by* (2)*. Let*

$$\Omega = \begin{pmatrix} c_1 & 1 & 0 & \ldots & 0 \\ c_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ c_n & 0 & \ldots & \ldots & 0 \end{pmatrix} \tag{4}$$

and let $\beta_1,\ldots,\beta_s$ be multiplicatively independent algebraic numbers with $0 < |\beta_j| < 1$ $(1 \leq j \leq s)$. Let p be a positive integer and put $\Omega' = \mathrm{diag}(\underbrace{\Omega^p,\ldots,\Omega^p}_{s})$. Then the matrix Ω' and the point $\boldsymbol{\beta} = (\underbrace{1,\ldots,1}_{n-1},\beta_1,\ldots\ldots,\underbrace{1,\ldots,1}_{n-1},\beta_s)$ have the properties (I)$-$(IV).

Lemma 2 (Kubota [3], see also Nishioka [5]). *Let K be an algebraic number field. Suppose that $f_1(\boldsymbol{z}),\ldots,f_m(\boldsymbol{z}) \in K[[z_1,\ldots,z_n]]$ converge in an n-polydisc U around the origin and satisfy the functional equations*

$$f_i(\boldsymbol{z}) = a_i(\boldsymbol{z}) f_i(\Omega \boldsymbol{z}) + b_i(\boldsymbol{z}) \quad (1 \leq i \leq m),$$

where $a_i(\boldsymbol{z}), b_i(\boldsymbol{z}) \in K(z_1,\ldots,z_n)$ and $a_i(\boldsymbol{z})$ $(1 \leq i \leq m)$ are defined and nonzero at the origin. Assume that the $n \times n$ matrix Ω and a point $\boldsymbol{\alpha} \in U$ whose components are nonzero algebraic numbers have the properties (I)$-$(IV) *and that $a_i(\boldsymbol{z})$ $(1 \leq i \leq m)$ are defined and nonzero at $\Omega^k \boldsymbol{\alpha}$ for any $k \geq 1$. If $f_1(\boldsymbol{z}),\ldots,f_m(\boldsymbol{z})$ are algebraically independent over $K(z_1,\ldots,z_n)$, then the values $f_1(\boldsymbol{\alpha}),\ldots,f_m(\boldsymbol{\alpha})$ are algebraically independent.*

In what follows, C denotes a field of characteristic 0. Let $L = C(z_1,\ldots,z_n)$ and let M be the quotient field of $C[[z_1,\ldots,z_n]]$. Let Ω be an $n \times n$ matrix with nonnegative integer entries having the property (I). We define an endomorphism $\tau : M \to M$ by $f^\tau(\boldsymbol{z}) = f(\Omega \boldsymbol{z})$ $(f(\boldsymbol{z}) \in M)$ and a subgroup H of $L^\times$ by

$$H = \{g^\tau g^{-1} \mid g \in L^\times\}.$$

Lemma 3 (Kubota [3], see also Nishioka [5]). *Let $f_{ij} \in M$ $(i = 1,\ldots,h;\ j = 1,\ldots,m(i))$ satisfy*

$$f_{ij} = a_i f_{ij}^\tau + b_{ij},$$

where $a_i \in L^\times$, $b_{ij} \in L$ $(1 \leq i \leq h,\ 1 \leq j \leq m(i))$, and $a_i a_{i'}^{-1} \notin H$ for any distinct i, i' $(1 \leq i, i' \leq h)$. Suppose for any i $(1 \leq i \leq h)$ there is no element g of L satisfying

$$g = a_i g^\tau + \sum_{j=1}^{m(i)} c_j b_{ij},$$

where $c_1,\ldots,c_{m(i)} \in C$ are not all zero. Then the functions f_{ij} $(i = 1,\ldots,h;\ j = 1,\ldots,m(i))$ are algebraically independent over L.

Let $\{R_k\}_{k\geq 1}$ be a linear recurrence satisfying (1) and define a monomial

$$M(\boldsymbol{z}) = z_1^{R_n} \cdots z_n^{R_1}, \tag{5}$$

which is denoted similarly to (3) by

$$M(\boldsymbol{z}) = (R_n, \ldots, R_1)\boldsymbol{z}. \tag{6}$$

Let Ω be the matrix defined by (4). It follows from (1), (3), and (6) that

$$M(\Omega^k \boldsymbol{z}) = z_1^{R_{k+n}} \cdots z_n^{R_{k+1}} \quad (k \geq 0). \tag{7}$$

Lemma 4 (Theorem 2 of [7])**.** *Suppose that $\{R_k\}_{k\geq 1}$ is not a geometric progression. Assume that $\Phi(\pm 1) \neq 0$ and the ratio of any pair of distinct roots of $\Phi(X)$ is not a root of unity. Let $\overline{C}$ be an algebraically closed field of characteristic* 0*. Suppose that $F(\boldsymbol{z})$ is an element of the quotient field of $\overline{C}[[z_1, \ldots, z_n]]$ satisfying the functional equation of the form*

$$F(\boldsymbol{z}) = \left(\prod_{k=u}^{p+u-1} Q_k(M(\Omega^k \boldsymbol{z})) \right) F(\Omega^p \boldsymbol{z}),$$

where Ω is defined by (4)*, $p > 0$, $u \geq 0$ are integers, and $Q_k(X) \in \overline{C}(X)$ $(u \leq k \leq p+u-1)$ are defined and nonzero at $X = 0$. If $F(\boldsymbol{z}) \in \overline{C}(z_1, \ldots, z_n)$, then $F(\boldsymbol{z}) \in \overline{C}$ and $Q_k(X) \in \overline{C}^{\times}$ $(u \leq k \leq p+u-1)$.*

We adopt the usual vector notation, that is, if $I = (i_1, \ldots, i_n) \in \mathbb{Z}_{\geq 0}^n$ with $\mathbb{Z}_{\geq 0}$ the set of nonnegative integers, we write $\boldsymbol{z}^I = z_1^{i_1} \cdots z_n^{i_n}$. We denote by $C[z_1, \ldots, z_n]$ the ring of polynomials in variables $z_1, \ldots, z_n$ with coefficients in C.

Lemma 5 (Lemma 3.2.3 in Nishioka [5])**.** *If $A, B \in C[z_1, \ldots, z_n]$ are coprime, then* g.c.d.$(A^\tau, B^\tau) = \boldsymbol{z}^I$*, where $I \in \mathbb{Z}_{\geq 0}^n$.*

Lemma 6 (Lemma 12 of [7])**.** *Let Ω be an $n \times n$ matrix with nonnegative integer entries which has the property* (I)*. Let $R(\boldsymbol{z})$ be a nonzero polynomial in $C[z_1, \ldots, z_n]$. If $R(\Omega \boldsymbol{z})$ divides $R(\boldsymbol{z})\boldsymbol{z}^I$, where $I \in \mathbb{Z}_{\geq 0}^n$, then $R(\boldsymbol{z})$ is a monomial in $z_1, \ldots, z_n$.*

Lemma 7 (Lemma 6 of [8])**.** *Let $P(\boldsymbol{z})$ be a nonconstant polynomial in $\boldsymbol{z} = (z_1, \ldots, z_n)$ with $n \geq 2$. Let Ω be an $n \times n$ matrix with positive integer entries which has the property* (I)*. Then*

$$\deg_{\boldsymbol{z}} P(\Omega \boldsymbol{z}) > \deg_{\boldsymbol{z}} P(\boldsymbol{z}).$$

3 Proof of the main theorem

We prove only Theorem 1, since Theorem 2 is proved in the same way.

Proof of Theorem 1. A complete set of representatives of the orbits $\left(\overline{\mathbb{Q}}^{\times} \times (\overline{\mathbb{Q}}^{\times} \setminus \{|q| = 1\})\right) \Big/ G_4$ is given by

$$\left\{ (x, q) \in \left(\overline{\mathbb{Q}}^{\times}\right)^2 \;\middle|\; |q| < 1,\ 0 \leq \operatorname{Arg} q < \pi \right\} =: \Lambda$$

since, under the action of the Klein four-group G_4, the second component q is transformed either to $q,\ q^{-1},\ -q$, or $-q^{-1}$. Hence it is enough to prove that the values

$$\eta_i := \Theta(x_i, q_i) = \sum_{k=1}^{\infty} \prod_{l=1}^{k} \frac{x_i q_i^{R_l}}{1 - q_i^{2R_l}} \qquad (i = 1, \ldots, r)$$

are algebraically independent for any finite number of distinct pairs (x_1, q_1), $(x_2, q_2), \ldots, (x_r, q_r)$ belonging to Λ.

Assume that the values $\eta_1, \ldots, \eta_r$ are algebraically dependent. There exist multiplicatively independent algebraic numbers $\beta_1, \ldots, \beta_s$ with $0 < |\beta_j| < 1$ $(1 \le j \le s)$ and a primitive N-th root of unity ζ such that

$$q_i = \zeta^{m_i} \prod_{j=1}^{s} \beta_j^{e_{ij}} \quad (1 \le i \le r), \tag{8}$$

where $m_1, \ldots, m_s$ are integers with $0 \le m_i \le N-1$ and e_{ij} $(1 \le i \le r,\ 1 \le j \le s)$ are nonnegative integers (cf. Loxton and van der Poorten [4], Nishioka [5]). We can choose a positive integer p and a sufficiently large integer u, which will be determined later, such that $R_{k+p} \equiv R_k \pmod N$ for any $k \ge u+1$. Let $y_{j\lambda}$ $(1 \le j \le s,\ 1 \le \lambda \le n)$ be variables and let $\boldsymbol{y}_j = (y_{j1}, \ldots, y_{jn})$ $(1 \le j \le s)$, $\boldsymbol{y} = (\boldsymbol{y}_1, \ldots, \boldsymbol{y}_s)$. Define

$$f_i(\boldsymbol{y}) = \sum_{k=u}^{\infty} \prod_{l=u}^{k} \frac{x_i \zeta^{m_i R_{l+1}} \prod_{j=1}^{s} M(\Omega^l \boldsymbol{y}_j)^{e_{ij}}}{1 - \left(\zeta^{m_i R_{l+1}} \prod_{j=1}^{s} M(\Omega^l \boldsymbol{y}_j)^{e_{ij}}\right)^2} \qquad (1 \le i \le r),$$

where $M(\boldsymbol{z})$ and Ω are defined by (5) and (4), respectively. Letting

$$\boldsymbol{\beta} = (\underbrace{1, \ldots, 1}_{n-1}, \beta_1, \ldots\ldots, \underbrace{1, \ldots, 1}_{n-1}, \beta_s),$$

we see by (7) and (8) that

$$f_i(\boldsymbol{\beta}) = \sum_{k=u}^{\infty} \prod_{l=u}^{k} \frac{x_i q_i^{R_{l+1}}}{1 - q_i^{2R_{l+1}}} = \sum_{k=u+1}^{\infty} \prod_{l=u+1}^{k} \frac{x_i q_i^{R_l}}{1 - q_i^{2R_l}}$$

and so

$$\eta_i = \left(\prod_{k=1}^{u} \frac{x_i q_i^{R_k}}{1 - q_i^{2R_k}} \right) f_i(\boldsymbol{\beta}) + \sum_{k=1}^{u} \prod_{l=1}^{k} \frac{x_i q_i^{R_l}}{1 - q_i^{2R_l}}.$$

Since $\eta_1, \ldots, \eta_r$ are algebraically dependent, so are $f_i(\boldsymbol{\beta})$ $(1 \le i \le r)$. Let

$$\Omega' = \operatorname{diag}(\underbrace{\Omega^p, \ldots, \Omega^p}_{s}).$$

Then each $f_i(\boldsymbol{y})$ satisfies the functional equation

$$f_i(\boldsymbol{y}) \quad = \quad \left(\prod_{k=u}^{p+u-1} \frac{x_i \zeta^{m_i R_{k+1}} \prod_{j=1}^{s} M(\Omega^k \boldsymbol{y}_j)^{e_{ij}}}{1 - \left(\zeta^{m_i R_{k+1}} \prod_{j=1}^{s} M(\Omega^k \boldsymbol{y}_j)^{e_{ij}}\right)^2} \right) f_i(\Omega' \boldsymbol{y})$$

$$+\sum_{k=u}^{p+u-1}\prod_{l=u}^{k}\frac{x_i\zeta^{m_iR_{l+1}}\prod_{j=1}^{s}M(\Omega^l\boldsymbol{y}_j)^{e_{ij}}}{1-\left(\zeta^{m_iR_{l+1}}\prod_{j=1}^{s}M(\Omega^l\boldsymbol{y}_j)^{e_{ij}}\right)^2},$$

where $\Omega'\boldsymbol{y}=(\Omega^p\boldsymbol{y}_1,\dots,\Omega^p\boldsymbol{y}_s)$. Let $D=|\det(\Omega-E)|=|\Phi(1)|$, where E is the identity matrix. Then D is a positive integer, since $\Phi(1)\neq 0$. Let $y'_{j\lambda}=y_{j\lambda}^{1/D}$ $(1\le j\le s,\ 1\le\lambda\le n)$, $\boldsymbol{y}'_j=(y'_{j1},\dots,y'_{jn})$ $(1\le j\le s)$, and $\boldsymbol{y}'=(\boldsymbol{y}'_1,\dots,\boldsymbol{y}'_s)$. Noting that $\prod_{j=1}^{s}M((\Omega-E)^{-1}\Omega^u\boldsymbol{y}_j)^{e_{ij}}=\prod_{j=1}^{s}M(D(\Omega-E)^{-1}\Omega^u\boldsymbol{y}'_j)^{e_{ij}}\in\overline{\mathbb{Q}}(\boldsymbol{y}')$, we define

$$\begin{aligned}g_i(\boldsymbol{y}')&=\left(\prod_{j=1}^{s}M((\Omega-E)^{-1}\Omega^u\boldsymbol{y}_j)^{e_{ij}}\right)f_i(\boldsymbol{y})-T_i(\boldsymbol{y}')\\&=\left(\prod_{j=1}^{s}M(D(\Omega-E)^{-1}\Omega^u\boldsymbol{y}'_j)^{e_{ij}}\right)f_i(\boldsymbol{y}')-T_i(\boldsymbol{y}')\qquad(1\le i\le r),\end{aligned}$$

where

$$\begin{aligned}f_i(\boldsymbol{y}')&=\sum_{k=u}^{\infty}\prod_{l=u}^{k}\frac{x_i\zeta^{m_iR_{l+1}}\prod_{j=1}^{s}M(\Omega^l\boldsymbol{y}'_j)^{D\,e_{ij}}}{1-\left(\zeta^{m_iR_{l+1}}\prod_{j=1}^{s}M(\Omega^l\boldsymbol{y}'_j)^{D\,e_{ij}}\right)^2}\in\overline{\mathbb{Q}}[[\boldsymbol{y}']],\\T_i(\boldsymbol{y}')&=\left(\prod_{j=1}^{s}M(D(\Omega-E)^{-1}\Omega^u\boldsymbol{y}'_j)^{e_{ij}}\right)\sum_{k=u}^{k_1}\prod_{l=u}^{k}\frac{x_i\zeta^{m_iR_{l+1}}\prod_{j=1}^{s}M(\Omega^l\boldsymbol{y}'_j)^{D\,e_{ij}}}{1-\left(\zeta^{m_iR_{l+1}}\prod_{j=1}^{s}M(\Omega^l\boldsymbol{y}'_j)^{D\,e_{ij}}\right)^2}\\&\in\overline{\mathbb{Q}}(\boldsymbol{y}'),\end{aligned}$$

and k_1 is such a large integer that $g_i(\boldsymbol{y}')\in\overline{\mathbb{Q}}[[\boldsymbol{y}']]$ $(1\le i\le r)$. Since $M(D(\Omega-E)^{-1}\Omega^u\boldsymbol{y}'_j)\prod_{k=u}^{p+u-1}M(\Omega^k\boldsymbol{y}'_j)^D=M(D(\Omega-E)^{-1}\Omega^{u+p}\boldsymbol{y}'_j)$, each $g_i(\boldsymbol{y}')$ satisfies the functional equation

$$\begin{aligned}g_i(\boldsymbol{y}')&=\left(\prod_{k=u}^{p+u-1}\frac{x_i\zeta^{m_iR_{k+1}}}{1-\left(\zeta^{m_iR_{k+1}}\prod_{j=1}^{s}M(\Omega^k\boldsymbol{y}'_j)^{D\,e_{ij}}\right)^2}\right)g_i(\Omega'\boldsymbol{y}')\\&\quad+\left(\prod_{j=1}^{s}M(D(\Omega-E)^{-1}\Omega^u\boldsymbol{y}'_j)^{e_{ij}}\right)\sum_{k=u}^{p+u-1}\prod_{l=u}^{k}\frac{x_i\zeta^{m_iR_{l+1}}\prod_{j=1}^{s}M(\Omega^l\boldsymbol{y}'_j)^{D\,e_{ij}}}{1-\left(\zeta^{m_iR_{l+1}}\prod_{j=1}^{s}M(\Omega^l\boldsymbol{y}'_j)^{D\,e_{ij}}\right)^2}\\&\quad+\left(\prod_{k=u}^{p+u-1}\frac{x_i\zeta^{m_iR_{k+1}}}{1-\left(\zeta^{m_iR_{k+1}}\prod_{j=1}^{s}M(\Omega^k\boldsymbol{y}'_j)^{D\,e_{ij}}\right)^2}\right)T_i(\Omega'\boldsymbol{y}')-T_i(\boldsymbol{y}'),\end{aligned}$$

where $\Omega'\boldsymbol{y}'=(\Omega^p\boldsymbol{y}'_1,\dots,\Omega^p\boldsymbol{y}'_s)$. Since $f_i(\boldsymbol{\beta})$ $(1\le i\le r)$ are algebraically dependent, so are $g_i(\boldsymbol{\beta}')$ $(1\le i\le r)$, where

$$\boldsymbol{\beta}'=(\underbrace{1,\dots,1}_{n-1},\beta_1^{1/D},\dots\dots,\underbrace{1,\dots,1}_{n-1},\beta_s^{1/D}).$$

By Lemma 1, the matrix Ω' and $\boldsymbol{\beta}'$ have the properties (I)$-$(IV). By Lemma 2, the functions $g_i(\boldsymbol{y}')$ $(1\le i\le r)$ are algebraically dependent over $\overline{\mathbb{Q}}(\boldsymbol{y}')$.

In order to apply Lemma 3, we assert that

$$
\begin{aligned}
Q_{ii'}(\boldsymbol{y}') &= \prod_{k=u}^{p+u-1} \frac{x_i \zeta^{m_i R_{k+1}} \left(1 - \left(\zeta^{m_{i'} R_{k+1}} \prod_{j=1}^{s} M(\Omega^k \boldsymbol{y}'_j)^{De_{i'j}}\right)^2\right)}{x_{i'} \zeta^{m_{i'} R_{k+1}} \left(1 - \left(\zeta^{m_i R_{k+1}} \prod_{j=1}^{s} M(\Omega^k \boldsymbol{y}'_j)^{De_{ij}}\right)^2\right)} \\
\in H &= \left\{ \frac{h(\Omega' \boldsymbol{y}')}{h(\boldsymbol{y}')} \;\middle|\; h(\boldsymbol{y}') \in \overline{\mathbb{Q}}(\boldsymbol{y}') \setminus \{0\} \right\}
\end{aligned}
$$

if and only if $m_i = m_{i'}$, $(e_{i1}, \dots, e_{is}) = (e_{i'1}, \dots, e_{i's})$, and $x_i^p = x_{i'}^p$. It is clear that, if $m_i = m_{i'}$, $(e_{i1}, \dots, e_{is}) = (e_{i'1}, \dots, e_{i's})$, and $x_{i'}^p = x_i^p$, then $Q_{ii'}(\boldsymbol{y}') = 1 \in H$. Conversely, suppose that $Q_{ii'}(\boldsymbol{y}') \in H$. Then there exits an $F(\boldsymbol{y}') \in \overline{\mathbb{Q}}(\boldsymbol{y}') \setminus \{0\}$ satisfying

$$
F(\boldsymbol{y}') = \left(\prod_{k=u}^{p+u-1} \frac{x_{i'} \zeta^{m_{i'} R_{k+1}} \left(1 - \left(\zeta^{m_i R_{k+1}} \prod_{j=1}^{s} M(\Omega^k \boldsymbol{y}'_j)^{De_{ij}}\right)^2\right)}{x_i \zeta^{m_i R_{k+1}} \left(1 - \left(\zeta^{m_{i'} R_{k+1}} \prod_{j=1}^{s} M(\Omega^k \boldsymbol{y}'_j)^{De_{i'j}}\right)^2\right)} \right) F(\Omega' \boldsymbol{y}'). \tag{9}
$$

Let P be a positive integer divisible by D and let

$$
\boldsymbol{y}'_j = (y'_{j1}, \dots, y'_{jn}) = (z_1^{P^j/D}, \dots, z_n^{P^j/D}) \quad (1 \le j \le s).
$$

We choose a sufficiently large P such that the following two properties are both satisfied:

(a) If $(e_{i1}, \dots, e_{is}) \neq (e_{i'1}, \dots, e_{i's})$, then $\sum_{j=1}^{s} e_{ij} P^j \neq \sum_{j=1}^{s} e_{i'j} P^j$.

(b) $F^*(\boldsymbol{z}) = F(z_1^{P/D}, \dots, z_n^{P/D}, \dots, z_1^{P^s/D}, \dots, z_n^{P^s/D}) \in \overline{\mathbb{Q}}(z_1, \dots, z_n) \setminus \{0\}$.

Then by (9), $F^*(\boldsymbol{z})$ satisfies the functional equation

$$
F^*(\boldsymbol{z}) = \left(\prod_{k=u}^{p+u-1} \frac{x_{i'} \zeta^{m_{i'} R_{k+1}} \left(1 - \left(\zeta^{m_i R_{k+1}} M(\Omega^k \boldsymbol{z})^{\ell_i}\right)^2\right)}{x_i \zeta^{m_i R_{k+1}} \left(1 - \left(\zeta^{m_{i'} R_{k+1}} M(\Omega^k \boldsymbol{z})^{\ell_{i'}}\right)^2\right)} \right) F^*(\Omega^p \boldsymbol{z}), \tag{10}
$$

where $\ell_i = \sum_{j=1}^{s} e_{ij} P^j$ $(1 \le i \le r)$. Therefore by Lemma 4 we see that

$$
\frac{x_{i'} \zeta^{m_{i'} R_{k+1}} \left(1 - \zeta^{2 m_i R_{k+1}} X^{2\ell_i}\right)}{x_i \zeta^{m_i R_{k+1}} \left(1 - \zeta^{2 m_{i'} R_{k+1}} X^{2\ell_{i'}}\right)} \in \overline{\mathbb{Q}}^{\times}
$$

for any k $(u \le k \le p+u-1)$, where X is a variable, and $F^*(\boldsymbol{z}) \in \overline{\mathbb{Q}}^{\times}$. Hence $\ell_i = \ell_{i'}$ and $\zeta^{2 m_i R_{k+1}} = \zeta^{2 m_{i'} R_{k+1}}$ $(u \le k \le p+u-1)$. Thus $(e_{i1}, \dots, e_{is}) = (e_{i'1}, \dots, e_{i's})$ by the property (a), and $\zeta^{2m_i} = \zeta^{2m_{i'}}$ since g.c.d.$(R_k, R_{k+1}, \dots, R_{k+n-1}) = 1$ for any $k \ge 1$. Hence $q_i^2 = q_{i'}^2$ by (8) and so $q_i = q_{i'}$ since $0 \le \operatorname{Arg} q_i < \pi$ $(1 \le i \le r)$. Then $m_i = m_{i'}$, and the functional equation (10) becomes $x_i^p F^*(\boldsymbol{z}) = x_{i'}^p F^*(\Omega^p \boldsymbol{z})$. Since $F^*(\boldsymbol{z}) \in \overline{\mathbb{Q}}^{\times}$, we have $x_i^p = x_{i'}^p$, and the assertion is proved.

Now let S be a maximal subset of $\{1, \dots, r\}$ such that $(x_i^p, q_i) = (x_{i'}^p, q_{i'})$ for any $i, i' \in S$, which is equivalent to $x_i^p = x_{i'}^p$, $m_i = m_{i'}$, and $(e_{i1}, \dots, e_{is}) = (e_{i'1}, \dots, e_{i's})$. Fix a $\lambda \in S$ and let $\xi = x_\lambda^p$, $m = m_\lambda$, and $e_j = e_{\lambda j}$ $(1 \le j \le s)$. Then $x_i^p = \xi$, $m_i = m$, and $(e_{i1}, \dots, e_{is}) = (e_1, \dots, e_s)$ for any $i \in S$ and by Lemma 3 there exits a $G(\boldsymbol{y}') \in \overline{\mathbb{Q}}(\boldsymbol{y}')$ satisfying

$$
\begin{aligned}
G(\boldsymbol{y}') &= \xi\left(\prod_{k=u}^{p+u-1}\frac{\zeta^{mR_{k+1}}}{1-\left(\zeta^{mR_{k+1}}\prod_{j=1}^{s}M(\Omega^k\boldsymbol{y}_j')^{D\,e_j}\right)^2}\right)G(\Omega'\boldsymbol{y}')\\
&\quad+\left(\prod_{j=1}^{s}M(D(\Omega-E)^{-1}\Omega^u\boldsymbol{y}_j')^{e_j}\right)\\
&\qquad\times\sum_{k=u}^{p+u-1}\left(\sum_{i\in S}c_ix_i^{k-u+1}\right)\prod_{l=u}^{k}\frac{\zeta^{mR_{l+1}}\prod_{j=1}^{s}M(\Omega^l\boldsymbol{y}_j')^{D\,e_j}}{1-\left(\zeta^{mR_{l+1}}\prod_{j=1}^{s}M(\Omega^l\boldsymbol{y}_j')^{D\,e_j}\right)^2}\\
&\quad+\xi\left(\prod_{k=u}^{p+u-1}\frac{\zeta^{mR_{k+1}}}{1-\left(\zeta^{mR_{k+1}}\prod_{j=1}^{s}M(\Omega^k\boldsymbol{y}_j')^{D\,e_j}\right)^2}\right)\sum_{i\in S}c_iT_i(\Omega'\boldsymbol{y}')-\sum_{i\in S}c_iT_i(\boldsymbol{y}'),
\end{aligned}
$$

where c_i $(i\in S)$ are algebraic numbers not all zero. Then

$$
G^*(\boldsymbol{y}')=\left(\prod_{j=1}^{s}M(D(\Omega-E)^{-1}\Omega^u\boldsymbol{y}_j')^{e_j}\right)^{-2}\left(G(\boldsymbol{y}')+\sum_{i\in S}c_iT_i(\boldsymbol{y}')\right)\in\overline{\mathbb{Q}}(\boldsymbol{y}')
$$

satisfies the functional equation

$$
\begin{aligned}
G^*(\boldsymbol{y}') &= \xi\left(\prod_{k=u}^{p+u-1}\frac{\zeta^{mR_{k+1}}\prod_{j=1}^{s}M(\Omega^k\boldsymbol{y}_j')^{2D\,e_j}}{1-\left(\zeta^{mR_{k+1}}\prod_{j=1}^{s}M(\Omega^k\boldsymbol{y}_j')^{D\,e_j}\right)^2}\right)G^*(\Omega'\boldsymbol{y}')\\
&\quad+\frac{1}{\prod_{j=1}^{s}M(D(\Omega-E)^{-1}\Omega^u\boldsymbol{y}_j')^{e_j}}\\
&\qquad\times\sum_{k=u}^{p+u-1}\left(\sum_{i\in S}c_ix_i^{k-u+1}\right)\prod_{l=u}^{k}\frac{\zeta^{mR_{l+1}}\prod_{j=1}^{s}M(\Omega^l\boldsymbol{y}_j')^{D\,e_j}}{1-\left(\zeta^{mR_{l+1}}\prod_{j=1}^{s}M(\Omega^l\boldsymbol{y}_j')^{D\,e_j}\right)^2}. \qquad (11)
\end{aligned}
$$

Let P be a positive integer and let $\boldsymbol{y}_j'=(y_{j1}',\dots,y_{jn}')=(z_1^{P^j},\dots,z_n^{P^j})$ $(1\le j\le s)$. We choose a sufficiently large P such that

$$
H(\boldsymbol{z})=G^*(z_1^P,\dots,z_n^P,\dots,z_1^{P^s},\dots,z_n^{P^s})\in\overline{\mathbb{Q}}(z_1,\dots,z_n).
$$

Then by (11), $H(\boldsymbol{z})$ satisfies the functional equation

$$
\begin{aligned}
H(\boldsymbol{z}) &= \xi\left(\prod_{k=u}^{p+u-1}\frac{\zeta^{mR_{k+1}}M(\Omega^k\boldsymbol{z})^{2D\ell}}{1-(\zeta^{mR_{k+1}}M(\Omega^k\boldsymbol{z})^{D\ell})^2}\right)H(\Omega^p\boldsymbol{z})\\
&\quad+\frac{1}{M(D(\Omega-E)^{-1}\Omega^u\boldsymbol{z})^{\ell}}\sum_{k=u}^{p+u-1}\left(\sum_{i\in S}c_ix_i^{k-u+1}\right)\prod_{l=u}^{k}\frac{\zeta^{mR_{l+1}}M(\Omega^l\boldsymbol{z})^{D\ell}}{1-(\zeta^{mR_{l+1}}M(\Omega^l\boldsymbol{z})^{D\ell})^2},
\end{aligned}
$$

where $\ell=\sum_{j=1}^{s}e_jP^j$. Letting $H(\boldsymbol{z})=A(\boldsymbol{z})/B(\boldsymbol{z})$, where $A(\boldsymbol{z})$ and $B(\boldsymbol{z})$ are coprime polynomials in $\overline{\mathbb{Q}}[z_1,\dots,z_n]$ with $B\not\equiv 0$, and letting $M(D(\Omega-E)^{-1}\Omega^u\boldsymbol{z})=M_1(\boldsymbol{z})/M_2(\boldsymbol{z})$, where $M_1(\boldsymbol{z})$ and $M_2(\boldsymbol{z})$ are coprime monomials in $\overline{\mathbb{Q}}[z_1,\dots,z_n]$, we have

$$
A(\boldsymbol{z})B(\Omega^p\boldsymbol{z})M_1(\boldsymbol{z})^{\ell}\prod_{k=u}^{p+u-1}\left(1-\left(\zeta^{mR_{k+1}}M(\Omega^k\boldsymbol{z})^{D\ell}\right)^2\right)
$$

$$
\begin{aligned}
&= \xi A(\Omega^p \boldsymbol{z}) B(\boldsymbol{z}) M_1(\boldsymbol{z})^\ell \prod_{k=u}^{p+u-1} \zeta^{mR_{k+1}} M(\Omega^k \boldsymbol{z})^{2D\ell} \\
&\quad + \sum_{k=u}^{p+u-1} \left(\sum_{i\in S} c_i x_i^{k-u+1} \right) B(\boldsymbol{z}) B(\Omega^p \boldsymbol{z}) M_2(\boldsymbol{z})^\ell \prod_{l=u}^{k} \zeta^{mR_{l+1}} M(\Omega^l \boldsymbol{z})^{D\ell} \\
&\qquad \times \prod_{l'=k+1}^{p+u-1} \left(1 - \left(\zeta^{mR_{l'+1}} M(\Omega^{l'} \boldsymbol{z})^{D\ell} \right)^2 \right). \qquad (12)
\end{aligned}
$$

In what follows, let u be sufficiently large. By the condition $\Phi(2) < 0$, the root ρ of $\Phi(X)$ such that $R_k = b\rho^k + o(\rho^k)$ with $b > 0$ (cf. Remark 4 in [6]) satisfies $\rho > 2$ and hence $R_{k+1} > 2R_k$ for all sufficiently large k. Then the componentwise inequality $(R_n, \dots, R_1) D(\Omega - E)^{-1} \Omega^u = (R_n, \dots, R_1) \Omega^u D(\Omega - E)^{-1} = (R_{u+n}, \dots, R_{u+1}) D(\Omega - E)^{-1} < D(R_{u+n}, \dots, R_{u+1})$ holds and so $z_1 \cdots z_n M_1(\boldsymbol{z})^\ell$ divides $M(\Omega^u \boldsymbol{z})^{D\ell} = M(D\Omega^u \boldsymbol{z})^\ell$. In what follows, p is replaced with its multiple if necessary. We can put the greatest common divisor of $A(\Omega^p \boldsymbol{z})$ and $B(\Omega^p \boldsymbol{z})$ as $\boldsymbol{z}^{I(p)}$, where $I(p)$ is an n-dimensional vector with nonnegative integer components, by Lemma 5. Then $B(\Omega^p \boldsymbol{z})$ divides $B(\boldsymbol{z}) M_1(\boldsymbol{z})^\ell \boldsymbol{z}^{I(p)} \prod_{k=u}^{p+u-1} M(\Omega^k \boldsymbol{z})^{2D\ell}$ by (12). Therefore $B(\boldsymbol{z})$ is a monomial in $z_1, \dots, z_n$ by Lemmas 1 and 6. Since p and u are independent, the right-hand side of (12) is divisible by $z_1 \cdots z_n M_1(\boldsymbol{z})^\ell B(\Omega^p \boldsymbol{z})$ and thus $A(\boldsymbol{z})$ is divisible by $z_1 \cdots z_n$. Since $A(\boldsymbol{z})$ and $B(\boldsymbol{z})$ are coprime, $B(\boldsymbol{z}) \in \overline{\mathbb{Q}}^\times$. If $A(\boldsymbol{z}) \notin \overline{\mathbb{Q}}$ and if p is sufficiently large, then by Lemma 7, $\deg_{\boldsymbol{z}} A(\Omega^p \boldsymbol{z}) > \max\{\deg_{\boldsymbol{z}} A(\boldsymbol{z}), \deg_{\boldsymbol{z}} M_2(\boldsymbol{z})^\ell\}$, which is a contradiction by comparing the total degrees of both sides of (12). Hence $A(\boldsymbol{z}) \in \overline{\mathbb{Q}}$. Then by (12), we see that $\sum_{i\in S} c_i x_i^{k-u+1} = 0$ $(u \le k \le p+u-1)$ and so $\sum_{i\in S} c_i x_i^k = 0$ $(1 \le k \le p)$. Hence $x_i = x_{i'}$ for some distinct $i, i' \in S$ since c_i $(i \in S)$ are not all zero. Then $(x_i, q_i) = (x_{i'}, q_{i'})$, which is a contradiction, and the proof of the theorem is completed. □

References

[1] F. R. Gantmacher, Applications of the Theory of Matrices, vol. II, Interscience, New York, 1959.

[2] G. Gasper and M. Rahman, Basic Hypergeometric Series, Cambridge University Press, Cambridge, 1990.

[3] K. K. Kubota, *On the algebraic independence of holomorphic solutions of certain functional equations and their values*, Math. Ann. **227** (1977), 9–50.

[4] J. H. Loxton and A. J. van der Poorten, *Algebraic independence properties of the Fredholm series*, J. Austral. Math. Soc. Ser. A **26** (1978), 31–45.

[5] K. Nishioka, Mahler functions and transcendence, Lecture Notes in Mathematics No. **1631**, Springer, 1996.

[6] T. Tanaka, *Algebraic independence of the values of power series generated by linear recurrences*, Acta Arith. **74** (1996), 177–190.

[7] T. Tanaka, *Algebraic independence results related to linear recurrences*, Osaka J. Math. **36** (1999), 203–227.

[8] T. Tanaka, *Algebraic independence of the values of Mahler functions associated with a certain continued fraction expansion*, J. Number Theory **105** (2004), 38–48.

[9] T. Tanaka, *Conditions for the algebraic independence of certain series involving continued fractions and generated by linear recurrences*, J. Number Theory **129** (2009), 3081–3093.