MIXED HODGE STRUCTURE ON FUNDAMENTAL GROUPS AND SULLIVAN MINIMAL MODELS

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We consider the following "theorem".

"Theorem". *Let M be a compact Kähler manifold and $\pi_1(M,x)$ its fundamental group. There exist mixed Hodge structures on the Malcev completion of $\pi_1(M,x)$.*

There are two ways to construct mixed Hodge structures as this "Theorem". The first way is given by Morgan ([6]) by using the Sullivan 1-minimal model. Consider the Sullivan 1-minimal model $\mathcal{M}^*$ of the de Rham complex $A^*(M)$ of a compact Kähler manifold M. In [6], Morgan constructed a non-unique mixed Hodge structure on $\mathcal{M}^*$. It is known that the dual Lie algebra of $\mathcal{M}^*$ is non-canonically isomorphic to the Malcev Lie algebra of the fundamental group $\pi_1(M,x)$. Hence we obtain a non-canonical mixed Hodge structure on the Malcev completion of $\pi_1(M,x)$.

The second way is given by Hain ([2]) by using iterated integrals. In [2], Hain constructed a mixed Hodge structure on the Malcev completion of $\pi_1(M,x)$ canonically defined by pointed compact kähler manifold (M,x).

We are interested in relation between mixed Hodge structure on Sullivan 1-minimal model $\mathcal{M}^*$ and Hain's mixed Hodge structure on the Malcev completion of $\pi_1(M,x)$. Consider the category $VMHS^u_{\mathbb{R}}(M)$ of unipotent variations of mixed Hodge structures over M and the fiber functor $\epsilon_x : \mathcal{V}MHS^u_{\mathbb{R}}(M) \ni (\mathbf{E},\mathbf{W},\mathbf{F}) \mapsto (\mathbf{E},\mathbf{W},\mathbf{F})_x \in \mathcal{M}HS_{\mathbb{R}}$. For the category $\mathrm{Rep}(\mathbb{R}\widehat{\pi_1(M,x)}, W_*, F^*)$ of mixed Hodge representations of the Malcev completion of $\pi_1(M,x)$ with Hain's mixed Hodge structure associated with (M,x) and the forgetful functor $\tau : \mathrm{Rep}(\mathbb{R}\widehat{\pi_1(M,x)}, W_*, F^*) \to \mathcal{M}HS_{\mathbb{R}}$, in [3], Hain and Zucker proved that the monodromy representation functor defines an equivalence $h_x : VMHS^u_{\mathbb{R}}(M) \to \mathrm{Rep}(\mathbb{R}\widehat{\pi_1(M,x)})$ between tensor categories such that the diagram

$$\begin{array}{ccc} VMHS^u_{\mathbb{R}}(M) & \xrightarrow{\epsilon_x} & \mathcal{M}HS_{\mathbb{R}} \\ \downarrow{\scriptstyle h_x(\cong)} & & \downarrow{\scriptstyle =} \\ (\mathrm{Rep}(\mathbb{R}\widehat{\pi_1(M,x)}), W_*, F^*) & \xrightarrow{\tau} & \mathcal{M}HS_{\mathbb{R}} \end{array}$$

commutes.

Theorem ([5]). *There exists a mixed hodge structure on the Sullivan 1-minimal model $\mathcal{M}^*$ of the de Rham complex $A^*(M)$ of a compact Kähler manifold M such that for the category $\mathrm{Rep}(\mathcal{M}^*, W_*, F^*)$ of mixed Hodge representations of the dual Lie algebra of $\mathcal{M}^*$ corresponding to this mixed hodge structure and the forgetful functor $\sigma : \mathrm{Rep}(\mathcal{M}^*, W_*, F^*) \to \mathcal{M}HS$, we have an equivalence $\Phi_x : \mathrm{Rep}(\mathcal{M}^*, W_*, F^*) \to$*

$VMHS^u_{\mathbb{R}}(M)$ so that the diagram

$$\begin{array}{ccc} \mathrm{Rep}(\mathcal{M}^*, W_*, F^*) & \xrightarrow{\sigma} & \mathcal{M}HS_{\mathbb{R}} \\ \downarrow{\scriptstyle \Phi_x(\cong)} & & \downarrow{\scriptstyle =} \\ VMHS^u_{\mathbb{R}}(M) & \xrightarrow{\epsilon_x} & \mathcal{M}HS_{\mathbb{R}} \end{array}$$

commutes.

By the theory of Tannaka category, $\mathrm{Rep}(\mathbb{R}\widehat{\pi_1(M, x)}), W_*, F^*)$ and $\mathrm{Rep}(\mathcal{M}^*, W_*, F^*)$ with the functors τ and σ can be non-abelian Hodge structures (see [1]). Via Φ_x and h_x, we can say that two non-abelian Hodge structures equivalent.

References

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