ON THE HEAT KERNEL ON FORMS ON THE HEISENBERG GROUP

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ABSTRACT. In this note we give an expression of the heat kernel on forms on the Heisenberg group. This expression is obtained by using the fundamental solutions of a series of systems of ordinary differential equations of second order. This study generalizes previous results in the case of one-forms on the Heisenberg group.

1. INTRODUCTION

There are many studies about both, the Laplacian Δ_0 and sub-Laplacian Δ_{sub} acting on functions on the Heisenberg group $\mathbb{H}_{2n+1}$. In this note we study the heat kernel of the Hodge Laplacian acting on forms on $\mathbb{H}_{2n+1}$ (see also [1, 2]). Our aim is to obtain an exact expression of this heat kernel. We expect that this result will enable us to study the Laplacian on the Heisenberg group under an adiabatic type limit similar to the analysis performed by M. Rumin in [3] and [4].

It is natural to consider a one-form as a $(2n + 1)$-vector as in the paper [2] by D. Müller, M.M. Peloso, and F. Ricci. In Section 4 we show that the problem of finding the heat kernel on one-forms is reduced to the solution of a system of ordinary differential equations of size $(2n + 1) \times (2n + 1)$. We give an expression of the heat kernel by using operators defined in terms of the Laplacian Δ_0. More precisely, we use the fundamental solution of an ordinary differential equation of second order with Δ_0 as a parameter (see Theorem 4.1). We show that this mechanism can be applied to calculate the heat kernel on forms. That is, the heat kernel on forms is obtained inductively by using the fundamental solutions of ordinary differential equations of second order. This observation is the main result of the present work (see Theorem 7.1).

The structure of the paper is as follows: we introduce some notations in Section 2. In Section 3 we give an expression of the Heisenberg Laplacian on p-forms of any degree (Theorem 3.3). We provide an exact formula of the heat kernel on one-forms with a rough sketch of a proof in Section 4. In Section 5 some properties of the operators are given which appear in Theorem 3.3 and are necessary to obtain a system of equations in Theorem 6.1. In Section 6 we make an *Ansatz* for the solution of the heat equation in the form given in Section 3. By using results from Section 5 we derive a system of operator-valued ODEs for the coefficients in this *Ansatz*. After introducing new variables $\{w_j(t),\ z_j(t)\}$ Theorem 7.1 then states that

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the problem is reduced to solving certain systems of operator-valued first order ordinary differential equations inductively. In Section 8 we give an expression of $e^{-t\tilde{N}_j}$ (see Definition 7.2) which is an essential part of Theorem 7.1. In the last section we obtain an exact expression of $X_1(t)$, the starting term of the series of the inductive equations in Theorem 7.1. From the solutions of the equations in this theorem we can in principle derive an exact form of the heat kernel of the Laplacian on p-forms. This generalizes the expression in Theorem 4.1 in the case $p = 1$.

2. Heisenberg Laplacian on p-forms

Let $\mathbb{H}_{2n+1}$ denote the $(2n+1)$-dimensional Heisenberg group with coordinates

$$(x_1, \cdots, x_n, y_1, \cdots, y_n, z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \cong \mathbb{H}_{2n+1}$$

and a frame of left-invariant vector fields

$$\mathcal{X} := [X_1, \cdots, X_n, Y_1, \cdots, Y_n, Z]$$

defined by

$$X_i = \frac{\partial}{\partial x_i} - \frac{y_i}{2}\frac{\partial}{\partial z}, \quad Y_i = \frac{\partial}{\partial y_i} + \frac{x_i}{2}\frac{\partial}{\partial z}, \quad Z = \frac{\partial}{\partial z}, \qquad (i = 1, \cdots, n).$$

The dual basis is given by $\Theta := [\theta_{X_1}, \cdots, \theta_{X_n}, \theta_{Y_1}, \cdots, \theta_{Y_n}, \theta_Z]$ where

$$\theta_{X_i} := dx_i, \quad \theta_{Y_i} := dy_i, \quad \theta_Z = dz + \frac{1}{2}\sum_{i=1}^n \Big(y_i dx_i - x_i dy_i\Big).$$

Therefore we have

$$dz = \theta_Z - \frac{1}{2}\sum_{i=1}^n \Big(y_i\theta_{X_i} - x_i\theta_{Y_i}\Big). \tag{2.1}$$

Let $k \in \{0, \cdots, 2n+1\}$ and consider the *star-operator*:

$$* : \Omega^k\big(\mathbb{H}_{2n+1}\big) \to \Omega^{2n+1-k}\big(\mathbb{H}_{2n+1}\big).$$

More precisely, with a permutation $\sigma = (i_1, \cdots, i_{2n+1})$ of $(1, \cdots, 2n+1)$ we find:

$$*\big(\theta_{i_1} \wedge \cdots \wedge \theta_{i_k}\big) = \operatorname{sgn}(\sigma)\theta_{i_{k+1}} \wedge \cdots \wedge \theta_{i_{2n+1}}. \tag{2.2}$$

We denote the *de Rham complex* as follows:

$$0 \longrightarrow \Omega^0(\mathbb{H}_{2n+1}) \xrightarrow{d^0} \Omega^1(\mathbb{H}_{2n+1}) \xrightarrow{d^1} \cdots \xrightarrow{d^{2n}} \Omega^{2n+1}(\mathbb{H}_{2n+1}) \longrightarrow 0.$$

With $k = 0, \cdots, 2n$ and in a standard way we define

$$\delta^k : \Omega^{k+1}(\mathbb{H}_{2n+1}) \longrightarrow \Omega^k(\mathbb{H}^{2n+1})$$

through the relation

$$\big(\delta^k\omega_1, \omega_2\big) = \big(\omega_1, d^k\omega_2\big).$$

Then it is well-known that

$$\delta^k = (-1)^{(2n+1-k)(k+1)} * d^{2n-k} * .$$

Recall that the *Hodge Laplacian* $\Delta_k : \Omega^k(\mathbb{H}_{2n+1}) \to \Omega^k(\mathbb{H}_{2n+1})$ is defined by

$$\Delta_k = \delta^k d^k + d^{k-1}\delta^{k-1}.$$

In the following we use the notation

$$W_j = X_j, \quad W_{n+j} = Y_j, \quad W_{2n+1} = Z \qquad (j = 1, \cdots, n),$$
$$\theta^j = \theta_{X_j}, \quad \theta^{n+j} = \theta_{Y_j}, \quad \theta^{2n+1} = \theta_Z \qquad (j = 1, \cdots, n).$$

The Riemannian metric g on $T\mathbb{H}_{2n+1}$ and $T^*\mathbb{H}_{2n+1}$ is defined by assuming that $\mathcal{X}$ and Θ form an orthonormal basis, respectively. Let ∇ be the Riemannian connection, that is ∇ is the unique connection which satifies

$$\nabla g = 0, \quad T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = 0.$$

Definition 2.1. *Let $c^k_{i,j}$ for $i, j, k = 1, \cdots, 2n+1$ denote the coefficients of ∇ defined by*

$$\nabla_{W_i} W_j = \sum_{k=1}^{2n+1} c^k_{i,j} W_k.$$

Lemma 2.1. *The coefficients $c^k_{i,j}$ are constant with the values:*

$$c^{2n+1}_{i,i+n} = c^i_{i+n,2n+1} = c^i_{2n+1,i+n} = \frac{1}{2},$$
$$c^{2n+1}_{i+n,i} = c^{i+n}_{i,2n+1} = c^{i+n}_{2n+1,i} = -\frac{1}{2}.$$

All other coefficients $c^k_{i,j}$ are zero.

By the previous lemma we have the following formula for the connection:

Corollary 2.2.

$$\nabla_{W_j} W_{j+n} = \frac{1}{2} W_{2n+1}, \quad \nabla_{W_{j+n}} W_j = -\frac{1}{2} W_{2n+1},$$
$$\nabla_{W_j} W_{2n+1} = \nabla_{W_{2n+1}} W_j = -\frac{1}{2} W_{j+n}, \quad \nabla_{W_{j+n}} W_{2n+1} = \nabla_{W_{2n+1}} W_{j+n} = \frac{1}{2} W_j.$$

3. Representation of Δ

We use the following notations:

$$a^*_\alpha \omega = e(\theta^\alpha)\omega = \theta^\alpha \wedge \omega,$$
$$(a_\alpha \omega)(Y_1, \cdots, Y_{p-1}) = (\imath(W_\alpha)\omega)(Y_1, \cdots, Y_{p-1}) = \omega(W_\alpha, Y_1, \cdots, Y_{p-1}).$$

Then we can write

$$d = \sum_{\alpha=1}^{2n+1} e(\theta^\alpha) \nabla_{W_\alpha}, \quad \delta = -\sum_{\alpha=1}^{2n+1} \imath(W_\alpha) \nabla_{W_\alpha}.$$

The following proposition collects some fundamental equations for a_α, a^*_β.

Proposition 3.1. *The operators $\{a_\alpha, a^*_\beta\}_{1 \le \alpha, \beta \le 2n+1}$ satisfy the relations:*

$$a_\alpha a_\beta + a_\beta a_\alpha = 0,$$
$$a^*_\alpha a^*_\beta + a^*_\beta a^*_\alpha = 0,$$
$$a_\alpha a^*_\beta + a^*_\beta a_\alpha = \delta_{\alpha\beta}.$$

According to Corollary 2.2 we have the formula below for ∇_{W_α}:

$$\nabla_{W_\alpha} = W_\alpha - G_\alpha, \qquad (\alpha = 1, 2, \cdots, 2n+1), \tag{3.1}$$

where for $j = 1, \cdots, n$:

$$\begin{aligned} G_j &= \frac{1}{2}\left(a^*_{j+n}a_{2n+1} - a^*_{2n+1}a_{j+n}\right), \\ G_{j+n} &= -\frac{1}{2}\left(a^*_j a_{2n+1} - a^*_{2n+1}a_j\right), \\ G_{2n+1} &= -\frac{1}{2}\sum_{j=1}^{n}\left(a^*_j a_{j+n} - a^*_{j+n}a_j\right). \end{aligned}$$

Definition 3.1. *Throughout the paper the operators below will play a role:*

$$d_H = \sum_{j=1}^{2n}\left(a^*_j W_j\right), \quad \delta_H = -\sum_{j=1}^{2n}\left(a_j W_j\right), \quad \Delta_H = d_H\delta_H + \delta_H d_H,$$

$$L = -\sum_{j=1}^{n}\left(a^*_j a^*_{j+n}\right), \quad \Lambda = \sum_{k=1}^{n}\left(a_k a_{k+n}\right), \quad Q = \sum_{\alpha=1}^{2n} a^*_\alpha a_\alpha.$$

For example, $a^*_j W_j$ acts on a p-form $\phi = f\theta^{\alpha_1}\wedge\theta^{\alpha_2}\wedge\cdots\theta^{\alpha_p}$ in the following way:

$$(a^*_j W_j)(\phi) = (W_j f)\theta_j \wedge \theta^{\alpha_1}\wedge\theta^{\alpha_2}\wedge\cdots\theta^{\alpha_p}.$$

By (3.1) we have:

Proposition 3.2.

$$\begin{aligned} d =& d_H + a^*_{2n+1}Z + La_{2n+1}, \\ \delta =& \delta_H - a_{2n+1}Z + a^*_{2n+1}\Lambda. \end{aligned}$$

Finally, we represent the *Hodge Laplacian* Δ_k for any k in a closed form. Put

$$P := \Delta_H - Z^2, \quad A := [\delta_H, L], \quad B := [\Lambda, d_H].$$

Theorem 3.3. *For all $k \in \{0, \cdots, 2n+1\}$:*

$$\Delta = \Delta_k = P + a^*_{2n+1}B + Aa_{2n+1} + (n-Q)a^*_{2n+1}a_{2n+1} + L\Lambda.$$

We uniquely can decompose any p-form ϕ as $\phi = u + \theta_Z \wedge v$ where u is a p-form and v is a $(p-1)$-form both not containing θ_Z. According this resolution we have:

Corollary 3.4.

$$\Delta(u + \theta_Z\wedge v) = \{(P + L\Lambda)u + Av\} + \theta_Z \wedge \{(P + L\Lambda + n - Q)v + Bu\}.$$

The heat equation on p-forms

$$\left(\frac{d}{dt} + \Delta\right)(u(t) + \theta_Z\wedge v(t)) = 0, \quad u(0) = U_0, \quad v(0) = V_0$$

is equivalent to a system of equations:

$$\left(\frac{d}{dt} + P + L\Lambda\right) u(t) + Av(t) = 0, \qquad u(0) = U_0,$$
$$\left(\frac{d}{dt} + P + L\Lambda + n - p + 1\right) v(t) + Bu(t) = 0, \qquad v(0) = V_0.$$

In the following it will be convenient to use complex vector fields Z_j, $\overline{Z}_j$ instead of X_j, Y_j:

Definition 3.2. *For $j = 1, \cdots, n$ put*

$$Z_j := \frac{1}{\sqrt{2}}(X_j - iY_j), \quad \overline{Z}_j := \frac{1}{\sqrt{2}}(X_j + iY_j), \quad W_{2n+1} := Z$$
$$\zeta_j := \frac{1}{\sqrt{2}}(\theta_{X_j} + i\theta_{Y_j}), \quad \overline{\zeta}_j := \frac{1}{\sqrt{2}}(\theta_{X_j} - i\theta_{Y_j}), \quad \zeta := \theta_Z,$$
$$b_j^* := e(\zeta_j), \quad b_j := \imath(Z_j), \quad \overline{b}_j^* := e(\overline{\zeta}_j), \quad \overline{b}_j := \imath(\overline{Z}_j).$$

According to our previous notations we obtain for the Laplacian on forms:

$$\Delta = \Delta_H - Z^2 + Aa_{2n+1} + a_{2n+1}^* B + (n - Q)a_{2n+1}^* a_{2n+1} + L\Lambda$$

with

$$\Delta_H = -\sum_{j=1}^n \left(Z_j\overline{Z}_j + \overline{Z}_j Z_j\right) + MZ, \quad M = i\sum_{j=1}^n \left(\overline{b}_j^*\overline{b}_j - b_j{}^* b_j\right),$$
$$A = i\sum_{j=1}^n \left(\overline{b}_j^*\overline{Z}_j - b_j{}^* Z_j\right), \quad B = i\sum_{j=1}^n \left(\overline{b}_j Z_j - b_j\overline{Z}_j\right),$$
$$Q = \sum_{j=1}^n \left(\overline{b}_j^*\overline{b}_j + b_j{}^* b_j\right), \quad L = i\sum_{j=1}^n \overline{b}_j^* b_j{}^*, \quad \Lambda = i\sum_{j=1}^n \overline{b}_j b_j.$$

Remark 3.1. The identity $P = \Delta_0 + MZ$ holds, where

$$\Delta_0 = -\sum_{j=1}^n \left(X_j^2 + Y_j^2\right) - Z^2$$

denotes the Laplacian on 0-forms.

4. The heat kernel on one-forms

Consider the fundamental solution $h = h(t)$ of the following differential operator:

$$\left(\frac{d^2}{dt^2} + n\frac{d}{dt} - \Delta_0\right) h(t) = 0, \quad h(0) = 0, \quad \frac{d}{dt}h(0) = 1.$$

We will see that the heat kernel of the Laplacian on one-forms can be expressed in form of an operator-valued matrix. The function h appears in the definition of the matrix entries. In the present section we give a rough sketch of this fact.

Set

$$\mathbf{Z}=\begin{pmatrix} Z_1\\ \vdots\\ Z_n\end{pmatrix},\quad \overline{\mathbf{Z}}=\begin{pmatrix} \overline{Z_1}\\ \vdots\\ \overline{Z_n}\end{pmatrix}.$$

Especially for 1-forms and with respect to the basis

$$\Big[g_j\zeta_j, g_{n+j}\overline{\zeta}_j, g_{2n+1}\zeta :\ j=1,\cdots,n \textit{ where } g_j, g_{n+j}, g_{2n+1}\in C^\infty(\mathbb{H}_{2n+1})\Big]$$

the Laplacian Δ_1 can be written in a $(2n+1)\times(2n+1)$ matrix form:

$$\Delta_1=\begin{pmatrix} P & A\\ B & \Delta_0+n\end{pmatrix}=\begin{pmatrix} \Delta_0-iZ & 0 & -i\mathbf{Z}\\ 0 & \Delta_0+iZ & i\bar{\mathbf{Z}}\\ -i{}^t\bar{\mathbf{Z}} & i{}^t\mathbf{Z} & \Delta_0+n\end{pmatrix}.$$

Definition 4.1. *(1)*

$$C=\sqrt{\Delta_0+\frac{n^2}{4}},\quad \Delta_{sub}=-\sum_{j=1}^n\left(X_j^2+Y_j^2\right).$$

(2)

$$h(t)=e^{-nt/2}C^{-1}\sinh(Ct),\quad \Phi(t)=\int_0^t h(s)ds.$$

Characterization of h and Φ: Consider the second order differential operator:

$$\mathcal{L}=\frac{d^2}{dt^2}+n\frac{d}{dt}-\Delta_0.$$

Then $h(t)$ and $\Phi(t)$ satisfy the following equations:

$$\mathcal{L}h(t)=0,\quad h(0)=0,\quad \dot{h}_0(0)=1,$$
$$\mathcal{L}\Phi(t)=1,\quad \Phi(0)=0,\quad \dot{\Phi}(0)=0.$$

Theorem 4.1. *The heat kernel $E(t)$ on one-forms is given by:*

$$E(t)=\begin{pmatrix} e^{-t(\Delta_0-iZ)} & 0 & 0\\ 0 & e^{-t(\Delta_0+iZ)} & 0\\ 0 & 0 & 0\end{pmatrix}+\begin{pmatrix} \mathbf{Z}\alpha(t){}^t\bar{\mathbf{Z}} & \mathbf{Z}\beta(t){}^t\mathbf{Z} & \mathbf{Z}a(t)\\ \overline{\mathbf{Z}}\beta(t){}^t\bar{\mathbf{Z}} & \overline{\mathbf{Z}\alpha(t)}{}^t\mathbf{Z} & \overline{\mathbf{Z}a(t)}\\ a(t){}^t\bar{\mathbf{Z}} & \overline{a(t)}{}^t\mathbf{Z} & c(t)\end{pmatrix},$$

where

$$\alpha(t)=e^{-t\Delta_0}\Big\{\Phi(t)-2h(t)*e^{-itZ}\Big\},\quad \beta(t)=e^{-t\Delta_0}\Phi(t),$$
$$a(t)=e^{-t\Delta_0}\left(i\frac{d}{dt}+Z\right)\Phi(t),\quad c(t)=e^{-t\Delta_0}\left(\frac{d^2}{dt^2}+Z\right)\Phi(t).$$

Here and in some of the formulas below we denote by $*$ the convolution product, i.e.

$$\big(h(t)*e^{-itZ}\big)(t)=h(t)*e^{-itZ}:=\int_0^t h(t-s)e^{-isZ}ds.$$

Note that there are various ways to define $e^{-t(\Delta_0\pm iZ)}$ which all lead to the same operator.

Outline of proof: Assume that the heat kernel $E(t)$ is of the form

$$E(t) = \begin{pmatrix} M_{11} & M_{12} & \mathbf{b_1} \\ M_{21} & M_{22} & \mathbf{b_2} \\ {}^t\mathbf{a_1} & {}^t\mathbf{a_2} & c \end{pmatrix}.$$

Then the components of this matrix satisfy the following equations:

$$\left(\frac{d}{dt} + \Delta_0 - iZ\right) M_{11} - i\mathbf{Z}^t\mathbf{a_1} = 0, \quad M_{11}(0) = I, \tag{4.1}$$

$$\left(\frac{d}{dt} + \Delta_0 + iZ\right) M_{21} + i\bar{\mathbf{Z}}^t\mathbf{a_1} = 0, \quad M_{21}(0) = 0, \tag{4.2}$$

$$\left(\frac{d}{dt} + \Delta_0 + n\right) ({}^t\mathbf{a_1}) - i\left({}^t\bar{\mathbf{Z}}M_{11} -^t \mathbf{Z}M_{21}\right) = 0, \quad \mathbf{a_1}(0) = 0. \tag{4.3}$$

According to (4.1) we have:

$$\begin{aligned} M_{11} &= e^{-t(\Delta_0 - iZ)} + i\int_0^t e^{-(t-s)(\Delta_0 - iZ)}\mathbf{Z}^t\mathbf{a_1}(s)ds \\ &= e^{-t(\Delta_0 - iZ)} + i\mathbf{Z}\left(\int_0^t e^{-(t-s)(\Delta_0 + iZ)}({}^t\mathbf{a_1}(s))ds\right) \\ &= e^{-t(\Delta_0 - iZ)} + \mathbf{Z}^t\mathbf{m_1}(t) \end{aligned}$$

with

$$\mathbf{m_1}(t) = i\int_0^t e^{-(t-s)(\Delta_0 + iZ)}\mathbf{a_1}(s)ds. \tag{4.4}$$

Similarly, by (4.2):

$$\begin{aligned} M_{21} &= -i\int_0^t e^{-(t-s)(\Delta_0 + iZ)}\bar{\mathbf{Z}}^t\mathbf{a_1}(s)ds \\ &= -i\bar{\mathbf{Z}}\left(\int_0^t e^{-(t-s)(\Delta_0 - iZ)}({}^t\mathbf{a_1}(s))ds\right) \\ &= \bar{\mathbf{Z}}^t\mathbf{m_2}(t) \end{aligned}$$

with

$$\mathbf{m_2}(t) = -i\int_0^t e^{-(t-s)(\Delta_0 - iZ)}\mathbf{a_1}(s)ds. \tag{4.5}$$

According to (4.4) one obtains:

$$\left(\frac{d}{dt} + \Delta_0 + iZ\right)\mathbf{m_1}(t) = i\mathbf{a_1}(t), \quad \mathbf{m_1}(0) = 0.$$

Similarly, by (4.5)

$$\left(\frac{d}{dt} + \Delta_0 - iZ\right)\mathbf{m_2}(t) = -i\mathbf{a_1}(t), \quad \mathbf{m_2}(0) = 0.$$

Hence (4.1)∼(4.3) are reduced to

$$\left(\frac{d}{dt}+\Delta_0+iZ\right)\mathbf{m_1}-i\mathbf{a_1}=0,\quad \mathbf{m_1}(0)=0,\tag{4.6}$$

$$\left(\frac{d}{dt}+\Delta_0-iZ\right)\mathbf{m_2}+i\mathbf{a_1}=0,\quad \mathbf{m_2}(0)=0,\tag{4.7}$$

$$\left(\frac{d}{dt}+\Delta_0+n\right)(\mathbf{a_1})-ie^{-t(\Delta_0+iZ)}\bar{\mathbf{Z}}-i\left({}^t\bar{\mathbf{Z}}\mathbf{Z}\mathbf{m_1}-{}^t\,\mathbf{Z}\bar{\mathbf{Z}}\mathbf{m_2}\right)=0,\quad \mathbf{a_1}(0)=0,\tag{4.8}$$

where we use

$$\begin{aligned}{}^t\bar{\mathbf{Z}}M_{11}-{}^t\,\mathbf{Z}M_{21}&={}^t\,\bar{\mathbf{Z}}(e^{-t(\Delta_0-iZ)})+{}^t\,\bar{\mathbf{Z}}\mathbf{Z}{}^t\mathbf{m_1}(t)-{}^t\,\mathbf{Z}\bar{\mathbf{Z}}{}^t\mathbf{m_2}(t)\\&=(e^{-t(\Delta_0+iZ)})^t\bar{\mathbf{Z}}+{}^t\,\bar{\mathbf{Z}}\mathbf{Z}{}^t\mathbf{m_1}(t)-{}^t\,\mathbf{Z}\bar{\mathbf{Z}}{}^t\mathbf{m_2}(t).\end{aligned}$$

By (4.6)∼(4.8) we may put

$$\mathbf{m_1}(t)=e^{-t\Delta_0}\tilde{\alpha}(t)\bar{\mathbf{Z}},\quad \mathbf{m_2}(t)=e^{-t\Delta_0}\tilde{\beta}(t)\bar{\mathbf{Z}},\quad \mathbf{a_1}(t)=e^{-t\Delta_0}\tilde{a}(t)\bar{\mathbf{Z}}.$$

Then we obtain

$$\begin{aligned}&\left(\frac{d}{dt}+iZ\right)\tilde{\alpha}(t)-i\tilde{a}(t)=0,\quad \tilde{\alpha}(0)=0,\\&\left(\frac{d}{dt}-iZ\right)\tilde{\beta}(t)+i\tilde{a}(t)=0,\quad \tilde{\beta}(0)=0,\\&\left(\frac{d}{dt}+n\right)\tilde{a}(t)-ie^{-itZ}+\frac{i}{2}\Delta_{sub}\big(\tilde{\alpha}(t)-\tilde{\beta}(t)\big)-\frac{n}{2}Z\big(\tilde{\alpha}(t)+\tilde{\beta}(t)\big)=0,\quad \tilde{a}(0)=0.\end{aligned}$$

Here we have applied the relation:

$${}^t\bar{\mathbf{Z}}\mathbf{Z}=-\frac{1}{2}\left(\Delta_{sub}+inZ\right),\quad {}^t\mathbf{Z}\bar{\mathbf{Z}}=-\frac{1}{2}\left(\Delta_{sub}-inZ\right).$$

We obtain a part of the assertion of Theorem 4.1 via the next lemma.

Lemma 4.2. *The above system of equations is solved by:*

$$\begin{aligned}\tilde{\alpha}(t)&=\Phi(t)-2\,h(t)*e^{-itZ},\\\tilde{\beta}(t)&=\Phi(t),\\\tilde{a}(t)&=Z\Phi(t)+ih(t).\end{aligned}$$

Proof. Set

$$x_1(t)=\frac{1}{2}(\tilde{\alpha}-\tilde{\beta}),\quad y_1(t)=\frac{1}{2}(\tilde{\alpha}+\tilde{\beta}).$$

Then we have

$$\begin{aligned}&\frac{d}{dt}x_1-i(\tilde{a}-Zy_1)=0,\quad x_1(0)=0,\\&\frac{d}{dt}y_1+iZx_1=0,\quad y_1(0)=0,\\&\frac{d}{dt}\tilde{a}-ie^{-itZ}I+i\Delta_{sub}x_1+n(\tilde{a}-Zy_1)=0,\quad \tilde{a}(0)=0.\end{aligned}$$

These equations lead the following equation for $x_1(t)$;

$$\left(\frac{d^2}{dt^2} + n\frac{d}{dt} - \Delta_0\right) x_1(t) + e^{-itZ} I = 0, \quad x_1(0) = 0, \quad \frac{d}{dt}x_1(0) = 0. \tag{4.9}$$

The unique solution of (4.9) is given

$$x_1(t) = -h(t) * e^{-itZ}.$$

Then we have

$$y_1(t) = -iZ \int_0^t x_1(s)ds = \Phi(t) - h(t) * e^{-itZ}$$

and

$$\begin{aligned}
\tilde{\alpha}(t) &= x_1(t) + y_1(t) = \Phi(t) - 2\,h(t) * e^{-itZ}, \\
\tilde{\beta}(t) &= y_1(t) - x_1(t) = \Phi(t), \\
\tilde{a}(t) &= -i\frac{d}{dt}x_1(t) + Zy_1(t) = i\left(\frac{d}{dt} + iZ\right)\left(h(t) * e^{-itZ}\right) + Z\Phi(t) \\
&= ih_0(t) + Z\Phi(t).
\end{aligned}$$

□

It is easy to see that:

$$M_{22} = \overline{M_{11}}, \quad M_{12} = \overline{M_{21}}, \quad \mathbf{a_2} = \overline{\mathbf{a_1}}.$$

Now for $\mathbf{b_1}, \mathbf{b_2}, c$ we have the following system:

$$\left(\frac{d}{dt} + \Delta_0 - iZ\right)\mathbf{b_1} - i\mathbf{Z}c = 0, \quad \mathbf{b_1}(0) = 0, \tag{4.10}$$

$$\left(\frac{d}{dt} + \Delta_0 + iZ\right)\mathbf{b_2} + i\bar{\mathbf{Z}}c = 0, \quad \mathbf{b_2}(0) = 0, \tag{4.11}$$

$$\left(\frac{d}{dt} + \Delta_0 + n\right)c - i({}^t\bar{\mathbf{Z}}\mathbf{b_1} - {}^t\,\mathbf{Z}\mathbf{b_2}) = 0, \quad c(0) = 1. \tag{4.12}$$

Using a similar argument for M_{11}, M_{21} and $\mathbf{a_1}$ we find:

$$\mathbf{b_1} = \mathbf{Z}e^{-t\Delta_0}\tilde{b}_1(t), \quad \mathbf{b_2} = \bar{\mathbf{Z}}e^{-t\Delta_0}\tilde{b}_2(t), \quad c(t) = e^{-t\Delta_0}\tilde{c}(t)$$

with

$$\tilde{b}_1(t) = \tilde{a}(t), \quad \tilde{b}_2(t) = \overline{\tilde{a}(t)}, \quad \tilde{c}(t) = \left(\frac{d^2}{dt^2} + Z^2\right)\Phi(t).$$

Then we get the assertion of theorem.

5. Properties of operators

We list some properties of the operators which were introduced in the previous sections:

Proposition 5.1. *The operators defined above fulfill the following relations:*

$$Ad_H + d_H A = 0, \quad B\delta_H + \delta_H B = 0, \tag{5.1}$$

$$(d_H)^2 = A^2 = -LZ, \quad (\delta_H)^2 = B^2 = \Lambda Z \tag{5.2}$$

$$[B, L] = -d_H, \quad [\Lambda, A] = -\delta_H \tag{5.3}$$

$$[d_H, L] = [\delta_H, \Lambda] = 0, \quad [A, L] = [B, \Lambda] = 0, \quad [\Delta_H, L] = [\Delta_H, \Lambda] = 0, \tag{5.4}$$

$$AB + BA = \Delta_H, \quad A\delta_H + \delta_H A = (n - Q)Z, \quad Bd_H + d_H B = -(n - Q)Z, \tag{5.5}$$

$$[P, d_H] = [\Delta_H, d_H] = -AZ, \quad [P, A] = [\Delta_H, A] = d_H Z, \tag{5.6}$$

$$[P, \delta_H] = [\Delta_H, \delta_H] = -BZ, \quad [P, B] = [\Delta_H, B] = \delta_H Z. \tag{5.7}$$

6. The heat kernel on p-forms

For the rest of the paper we study the Laplacian acting on p-forms on the Heisenberg group and we assume that $0 \le p \le n$. We make a suitable *Ansatz* and assume that the functions $u(t)$ and $v(t)$ in Section 3 are of the following form:

$$u(t) = e^{-tP} x_0(t) + \sum_{j=1}^{p} u_j(t),$$
$$v(t) = e^{-tP} y_1(t) + \sum_{j=2}^{p} v_j(t),$$

where

$$u_{2k}(t) = L^{k-1}\Big(Le^{-tP} x_{2k}(t) - \mathrm{A}\, \mathrm{d}_H e^{-tP} \tilde{x}_{2k}(t)\Big),$$
$$u_{2k+1}(t) = L^{k}\Big(Ae^{-tP} x_{2k+1}(t) + d_H e^{-tP} \tilde{x}_{2k+1}(t)\Big),$$
$$v_{2k}(t) = L^{k-1}\Big(Ae^{-tP} y_{2k}(t) + d_H e^{-tP} \tilde{y}_{2k}(t)\Big),$$
$$v_{2k+1}(t) = L^{k-1}\Big(Le^{-tP} y_{2k+1}(t) - \mathrm{A}\, \mathrm{d}_H e^{-tP} \tilde{y}_{2k+1}(t)\Big), \quad k \ge 1,$$

and $\{x_j, \tilde{x}_j, y_j, \tilde{y}_j\}$ for $0 \le j \le p$ are $(p-j)$-form valued functions on the real line. Moreover, we note that $\{x_j, \tilde{x}_j, y_j, \tilde{y}_j\}$ vanish for all $j \ge p + 1$. We may set

$$x_j \equiv 0 \quad (j \le -1), \quad \tilde{x}_j \equiv 0 \quad (j \le 0), \quad y_j \equiv 0 \quad (j \le 0), \quad \tilde{y}_j \equiv 0 \quad (j \le 1).$$

The initial conditions are

$$x_0(0) = U_0, \quad y_1(0) = V_0,$$
$$x_j(0) = 0, \quad \tilde{x}_j(0) = 0 \quad (1 \le j \le p), \quad y_j(0) = 0, \quad \tilde{y}_j(0) = 0 \quad (2 \le j \le p).$$

If $p = 1$, then

$$u(t) = e^{-tP} x_0(t) + A e^{-tP} x_1(t) + d_H e^{-tP} \tilde{x}_1(t),$$
$$v(t) = e^{-tP} y_1(t).$$

If $p = 2$, then

$$u(t) = e^{-tP} x_0(t) + A e^{-tP} x_1(t) + d_H e^{-tP} \tilde{x}_1(t) + L e^{-tP} x_2(t) - A d_H e^{-tP} \tilde{x}_2(t),$$
$$v(t) = e^{-tP} y_1(t) + A e^{-tP} y_2(t) + d_H e^{-tP} \tilde{y}_2(t).$$

Definition 6.1. *For $j = 0, 1, 2, \cdots$ we put*

$$a_j = j(n - p + j), \quad b_j = j(n - p + j + 1).$$

Definition 6.2. *(1)*

$$\alpha = \frac{1}{2}(A + i d_H), \quad \bar{\alpha} = \frac{1}{2}(A - i d_H),$$
$$\beta = \frac{1}{2}(B + i \delta_H), \quad \bar{\beta} = \frac{1}{2}(B - i \delta_H).$$

(2)

$$T_\beta(t) = \beta e^{-itZ}, \quad T_{\bar{\beta}}(t) = \bar{\beta} e^{itZ},$$
$$T_B(t) = T_\beta(t) + T_{\bar{\beta}}(t), \quad T_{\delta_H}(t) = -i \left(T_\beta(t) - T_{\bar{\beta}}(t) \right).$$

Using Proposition 5.1 we obtain the following theorem on a relation between the coefficients of $u(t)$ and $v(t)$:

Theorem 6.1. *(I)*

$$\left(\frac{d}{dt} + b_k \right) x_{2k+1}(t) - Z \tilde{x}_{2k+1}(t) + y_{2k+1}(t) - T_B(t) \tilde{x}_{2k}(t) + \Lambda x_{2k-1}(t) = 0,$$
$$\text{and} \quad x_{2k+1}(0) = 0, \quad 0 \le k \le \left[\frac{p-1}{2} \right],$$
$$\left(\frac{d}{dt} + b_k \right) \tilde{x}_{2k+1}(t) + Z \left(x_{2k+1}(t) + \tilde{y}_{2k+1}(t) \right) - T_{\delta_H}(t) \tilde{x}_{2k} + \Lambda \tilde{x}_{2k-1}(t) = 0,$$
$$\text{and} \quad \tilde{x}_{2k+1}(0) = 0, \quad 0 \le k \le \left[\frac{p-1}{2} \right],$$
$$\left(\frac{d}{dt} + a_{k+1} \right) y_{2k+1}(t) + \Delta_H \left(x_{2k+1}(t) + \tilde{y}_{2k+1}(t) \right) - (n - p + k + 1) Z \tilde{x}_{2k+1}(t)$$
$$- T_{\delta_H}(t) y_{2k}(t) + T_B(t) \left(x_{2k}(t) + \tilde{y}_{2k}(t) \right) + \Lambda y_{2k-1}(t) = 0,$$
$$\text{and} \quad y_{2k+1}(0) = \delta_{k,0} V_0, \quad 0 \le k \le \left[\frac{p-1}{2} \right],$$
$$\left(\frac{d}{dt} + a_k \right) \tilde{y}_{2k+1}(t) - k x_{2k+1}(t) + T_B(t) \tilde{x}_{2k}(t) + \Lambda \tilde{y}_{2k-1}(t) = 0,$$
$$\text{and} \quad \tilde{y}_{2k+1}(0) = 0, \quad 1 \le k \le \left[\frac{p-1}{2} \right].$$

(II)

$$\left(\frac{d}{dt}+b_k\right)x_{2k}(t)+\Delta_H\tilde{x}_{2k}(t)-Zy_{2k}(t)+T_B(t)\tilde{x}_{2k-1}(t)$$
$$-T_{\delta_H}(t)x_{2k-1}(t)+\Lambda x_{2k-2}(t)=0,$$
$$\text{and}\quad x_{2k}(0)=\delta_{k,0}U_0\quad 0\le k\le\left[\frac{p}{2}\right],$$
$$\left(\frac{d}{dt}+b_{k-1}\right)\tilde{x}_{2k}(t)-\tilde{y}_{2k}(t)+\Lambda\tilde{x}_{2k-2}(t)=0,$$
$$\text{and}\quad \tilde{x}_{2k}(0)=0\quad 1\le k\le\left[\frac{p}{2}\right],$$
$$\left(\frac{d}{dt}+a_k\right)y_{2k}(t)-Z\tilde{y}_{2k}(t)-(n-p+k)Z\tilde{x}_{2k}(t)$$
$$-T_B(t)\big(\tilde{y}_{2k-1}(t)+x_{2k-1}(t)\big)+\Lambda y_{2k-2}(t)=0,$$
$$\text{and}\quad y_{2k}(0)=0,\quad 1\le k\le\left[\frac{p}{2}\right],$$
$$\left(\frac{d}{dt}+a_k\right)\tilde{y}_{2k}(t)+Zy_{2k}(t)-kx_{2k}(t)-\Delta_H\tilde{x}_{2k}(t)-T_{\delta_H}(t)\tilde{y}_{2k-1}(t)$$
$$-T_B(t)\tilde{x}_{2k-1}(t)+\Lambda\tilde{y}_{2k-2}(t)=0,$$
$$\text{and}\quad \tilde{y}_{2k}(0)=0,\quad 1\le k\le\left[\frac{p}{2}\right].$$

7. Reduction to a system of the equations

We note that

$$x_0(t)\equiv U_0.$$

Indeed x_0 is characterized by the equation:

$$\frac{d}{dt}x_0(t)=0,\quad x_0(0)=U_0$$

Definition 7.1. *Define $w_j(t)$ and $z_j(t)$ for $(0\le j\le p)$ as:*

$$w_j(t)=x_j(t)+\tilde{y}_j(t),\qquad z_j(t)=y_j(t)-Z\tilde{x}_j(t)$$

We note that

$$x_0(t)=w_0(t)\equiv U_0,\quad w_1(t)=x_1(t),\quad z_0(t)\equiv 0,\quad z_1(0)=V_0$$

Definition 7.2. *(1) Define vectors $X_j(t)$ for $0\le j\le p$ as follows:*

$$X_{2k+1}(t)=\begin{pmatrix} w_{2k+1}(t)\\ z_{2k+1}(t)\\ \tilde{x}_{2k+1}(t)\\ x_{2k+1}(t)\end{pmatrix}\quad (k\ge 0),\quad X_{2k}(t)=\begin{pmatrix}\tilde{x}_{2k}(t)\\ x_{2k}(t)\\ w_{2k}(t)\\ z_{2k}(t)\end{pmatrix}\quad (k\ge 0).$$

(2) Define 2×2 *matrices* N_j $(1\le j\le p)$ *and* 4×4 *matrices* $\tilde N_j$ $(1\le j\le p)$ *by:*

$$N_{2k}=\begin{pmatrix} b_{k-1} & 1\\ P & b_k\end{pmatrix},\quad N_{2k+1}=\begin{pmatrix} a_k & 1\\ P & a_{k+1}\end{pmatrix},$$

$$\tilde N_{2k}=\begin{pmatrix} b_{k-1} & 1 & -1 & 0\\ P & b_k & 0 & -Z\\ 0 & 0 & a_k & 0\\ 0 & 0 & 0 & a_k\end{pmatrix},\quad \tilde N_{2k+1}=\begin{pmatrix} a_k & 1 & 0 & 0\\ P & a_{k+1} & 0 & 0\\ Z & 0 & b_k & 0\\ 0 & 1 & 0 & b_k\end{pmatrix},(k\ge 0).$$

Remark 7.1. We have:

$$X_0(t)=\begin{pmatrix}0\\ U_0\\ U_0\\ 0\end{pmatrix},\quad X_1(t)=\begin{pmatrix}x_1(t)\\ z_1(t)\\ \tilde x_1(t)\\ x_1(t)\end{pmatrix}.$$

The following statement follows from Theorem 6.1:

Theorem 7.1. *(1)* $X_1(t)$ *solves the initial value problem:*

$$\left(\frac{d}{dt}+\tilde N_1\right)X_1(t)+\begin{pmatrix}0\\ T_B(t)U_0\\ 0\\ 0\end{pmatrix}=0\qquad with\qquad X_1(0)=\begin{pmatrix}0\\ V_0\\ 0\\ 0\end{pmatrix}.$$

(2) Using the above notations we obtain the following equations for $2\le j\le p$*:*

$$\left(\frac{d}{dt}+\tilde N_j\right)X_j(t)+S(t)X_{j-1}(t)+\Lambda X_{j-2}(t)=0\qquad with\qquad X_j(0)=\begin{pmatrix}0\\ 0\\ 0\\ 0\end{pmatrix}$$

and

$$S(t)=\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & T_B(t) & -T_{\delta_H}(t)\\ -T_{\delta_H}(t) & 0 & 0 & 0\\ -T_B(t) & 0 & 0 & 0\end{pmatrix}.$$

We present an exact form of $X_1(t)$. Then $X_j(t)$ for $2\le j\le p$ is given in the form:

$$X_j(t)=-e^{-t\tilde N_j}*\Big(S(t)X_{j-1}(t)+\Lambda X_{j-2}(t)\Big).$$

Moreover, we obtain $y_j(t),\tilde y_j(t)$ $(1\le j\le p)$ as follows:

$$y_j(t)=z_j(t)+Z\tilde x_j(t),\quad \tilde y_j(t)=w_j(t)-x_j(t).$$

8. The formula for $e^{-t\tilde{N}_j}$

In this section we give the exact form of $e^{-t\tilde{N}_j}$ $(1 \leq j \leq p)$.

Definition 8.1. *(1) Define a system of p second order ordinary differential operators L_j where $1 \leq j \leq p$:*

$$L_{2k} = \left(\frac{d}{dt} + b_{k-1}\right)\left(\frac{d}{dt} + b_k\right) - P, \quad 1 \leq k \leq \left[\frac{p}{2}\right],$$
$$L_{2k+1} = \left(\frac{d}{dt} + a_k\right)\left(\frac{d}{dt} + a_{k+1}\right) - P, \quad 0 \leq k \leq \left[\frac{p-1}{2}\right].$$

(2) Let $h_j(t)$ for $1 \leq j \leq p$ be operator-valued functions such that

$$L_j h_j(t) = 0, \quad h_j(0) = 0, \quad \left(\frac{d}{dt}h_j\right)(0) = 1.$$

(3) Define $\Phi_j(t)$ for $1 \leq j \leq p$ by:

$$\Phi_{2k} = h_{2k} * e^{-a_k t},$$

$$\Phi_{2k+1} = h_{2k+1} * e^{-b_k t}.$$

Proposition 8.1. *(1) The functions $\Phi_j(t)$ satisfy the equations:*

$$\left(\frac{d}{dt} + a_k\right) L_{2k}\Phi_{2k}(t) = 0,\ \Phi_{2k}(0) = 0,\ \left(\frac{d}{dt}\Phi_{2k}\right)(0) = 0,\ \left(\frac{d^2}{dt^2}\Phi_{2k}\right)(0) = 1,$$

$$\left(\frac{d}{dt} + b_k\right) L_{2k+1}\Phi_{2k+1}(t) = 0,\ \Phi_{2k+1}(0) = 0,\ \left(\frac{d}{dt}\Phi_{2k+1}\right)(0) = 0,\ \left(\frac{d^2}{dt^2}\Phi_{2k+1}\right)(0) = 1.$$

(2) The fundamental solutions e^{-tN_j} $(1 \leq j \leq p)$ have the form:

$$e^{-tN_{2k+1}} = \begin{pmatrix} \frac{d}{dt} + a_{k+1} & -1 \\ -P & \frac{d}{dt} + a_k \end{pmatrix} h_{2k+1}(t), \quad 0 \leq k \leq \left[\frac{p-1}{2}\right],$$

$$e^{-tN_{2k}} = \begin{pmatrix} \frac{d}{dt} + b_k & -1 \\ -P & \frac{d}{dt} + b_{k-1} \end{pmatrix} h_{2k}(t), \quad 1 \leq k \leq \left[\frac{p}{2}\right].$$

(3) The fundamental solutions $e^{-t\tilde{N}_j}$ $(1 \leq j \leq p)$ have the form:

$$e^{-t\tilde{N}_{2k+1}} = \begin{pmatrix} e^{-tN_{2k+1}} & 0 \\ G_{2k+1}(t) & e^{-b_k t} I_2 \end{pmatrix}, \quad 0 \leq k \leq \left[\frac{p-1}{2}\right],$$

$$e^{-t\tilde{N}_{2k}} = \begin{pmatrix} e^{-tN_{2k}} & G_{2k}(t) \\ 0 & e^{-a_k t} I_2 \end{pmatrix}, \quad 1 \leq k \leq \left[\frac{p}{2}\right],$$

where $G_j(t)$ are 2×2 matrices which are defined by

$$\begin{aligned} G_{2k+1}(t) &= \begin{pmatrix} -Z(\frac{d}{dt}+a_{k+1}) & Z \\ P & -(\frac{d}{dt}+a_k) \end{pmatrix} \Phi_{2k+1}(t) \\ &= -\begin{pmatrix} Z & 0 \\ 0 & 1 \end{pmatrix} \left(e^{-tN_{2k+1}} * e^{-b_k t}\right), \\ G_{2k}(t) &= \begin{pmatrix} (\frac{d}{dt}+b_k) & -Z \\ -P & Z(\frac{d}{dt}+b_{k-1}) \end{pmatrix} \Phi_{2k}(t) = \left(e^{-tN_{2k}} * e^{-a_k t}\right) \begin{pmatrix} 1 & 0 \\ 0 & Z \end{pmatrix}. \end{aligned}$$

Remark 8.1. By using the operator$P$ the functions $h_j(t)$ for $1\le j\le p$ can be written as follows:

$$\begin{aligned} h_{2k}(t) =& e^{-\frac{1}{2}(b_{k-1}+b_k)t} \frac{\sinh\left\{t\sqrt{\frac{1}{4}(b_k-b_{k-1})^2+P}\right\}}{\sqrt{\frac{1}{4}(b_k-b_{k-1})^2+P}}, \\ h_{2k+1}(t) =& e^{-\frac{1}{2}(a_{k+1}+a_k)t} \frac{\sinh\left\{t\sqrt{\frac{1}{4}(a_{k+1}-a_k)^2+P}\right\}}{\sqrt{\frac{1}{4}(a_{k+1}-a_k)^2+P}}. \end{aligned}$$

9. An exact form of $X_1(t)$

In the present section we give an exact form of the function $X_1(t)$. It satisfies the following equation:

$$\left(\frac{d}{dt}+\tilde{N}_1\right) X_1(t) + \begin{pmatrix} 0 \\ T_B(t)U_0 \\ 0 \\ 0 \end{pmatrix} = 0, \quad X_1(0) = \begin{pmatrix} 0 \\ V_0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, we have

$$X_1(t) = e^{-t\tilde{N}_1} X_1(0) - e^{-t\tilde{N}_1} * \begin{pmatrix} 0 \\ T_B(t)U_0 \\ 0 \\ 0 \end{pmatrix}.$$

Theorem 9.1. *The function $X_1(t)$ has the form:*

$$X_1(t) = \begin{pmatrix} -\frac{d}{dt}k_1(t) \\ \frac{d^2}{dt^2}k_1(t) \\ Zk_1(t) \\ -\frac{d}{dt}k_1(t) \end{pmatrix},$$

where

$$k_1(t) = \Phi_1(t)V_0 - \Big(\Phi_1(t) * T_B(t)\Big)U_0.$$

Proof. We have

$$e^{-t\tilde{N}_1}\begin{pmatrix}0\\W\\0\\0\end{pmatrix}=\begin{pmatrix}-h_1(t)W\\\frac{d}{dt}h_1(t)W\\Z\Phi_1(t)W\\-\frac{d}{dt}\Phi_1(t)W\end{pmatrix}=\begin{pmatrix}-\frac{d}{dt}\Phi_1(t)W\\\frac{d^2}{dt^2}\Phi_1(t)W\\Z\Phi_1(t)W\\-\frac{d}{dt}\Phi_1(t)W\end{pmatrix}$$

and

$$e^{-t\tilde{N}_1}*\begin{pmatrix}0\\T_B(t)U_0\\0\\0\end{pmatrix}=\begin{pmatrix}-\left(\frac{d}{dt}\Phi_1(t)*T_B(t)\right)U_0\\\left(\frac{d^2}{dt^2}\Phi_1(t)*T_B(t)\right)U_0\\Z\left(\Phi_1(t)*T_B(t)\right)U_0\\-\left(\frac{d}{dt}\Phi_1(t)*T_B(t)\right)U_0\end{pmatrix}=\begin{pmatrix}-\frac{d}{dt}\left(\Phi_1(t)*T_B(t)\right)U_0\\\frac{d^2}{dt^2}\left(\Phi_1(t)*T_B(t)\right)U_0\\Z\left(\Phi_1(t)*T_B(t)\right)U_0\\-\frac{d}{dt}\left(\Phi_1(t)*T_B(t)\right)U_0\end{pmatrix},$$

where we used

$$\Phi_1(0)=\frac{d}{dt}\Phi_1(0)=0.$$

□

Corollary 9.2. *(1)*

$$x_1(t)=-\frac{d}{dt}k_1(t),\quad \tilde{x}_1(t)=Zk_1(t),\quad y_1(t)=\left(\frac{d^2}{dt^2}+Z^2\right)k_1(t).$$

(2)

$$\begin{aligned}u_1(t)=&-\alpha e^{-tP}\left(\frac{d}{dt}+iZ\right)k_1(t)-\bar{\alpha}e^{-tP}\left(\frac{d}{dt}-iZ\right)k_1(t),\\v_1(t)=&e^{-tP}\left(\frac{d^2}{dt^2}+Z^2\right)k_1(t).\end{aligned}$$

According to Corollary 9.2 we obtain the exact form of the heat kernel for one-forms which coincides with the expression in Theorem 4.1 since $a_1=n$ for one-forms.

Remark 9.1.

$$h_1(t)=e^{-\frac{1}{2}a_1t}\,\frac{\sinh\left\{t\sqrt{\frac{1}{4}a_1^2+P}\right\}}{\sqrt{\frac{1}{4}a_1^2+P}},\quad (a_1=n-p+1),$$

$$\begin{aligned}\left(\frac{d^2}{dt^2}+Z^2\right)k_1(t)&=\left(\frac{d^2}{dt^2}+Z^2\right)\Phi_1(t)V_0-\left(\frac{d}{dt}-iZ\right)\Phi_1(t)\beta U_0-\left(\frac{d}{dt}+iZ\right)\Phi_1(t)\bar{\beta}U_0,\\\left(\frac{d}{dt}+iZ\right)k_1(t)&=\left(\frac{d}{dt}+iZ\right)\Phi_1(t)V_0-\Phi_1(t)\beta U_0+\left\{\Phi_1(t)-2\left(h_1(t)*e^{itZ}\right)(t)\right\}\bar{\beta}U_0,\\\left(\frac{d}{dt}-iZ\right)k_1(t)&=\left(\frac{d}{dt}-iZ\right)\Phi_1(t)V_0-\Phi_1(t)\bar{\beta}U_0+\left\{\Phi_1(t)-2\left(h_1(t)*e^{-itZ}\right)(t)\right\}\beta U_0,\end{aligned}$$

where we use

$$iZ\left(\Phi_1(t)*e^{-itZ}\right)(t)=\Phi_1(t)-\left(h_1(t)*e^{-itZ}\right)(t).$$

References

[1] D. Müller, M.M. Peloso, F. Ricci, *Analysis of the Hodge Laplacian on the Heisenberg group*, Mem. Amer. Math. Soc. 233 (2015), no. 1095.

[2] D. Müller, M.M. Peloso, F. Ricci, *L^p-spectral multipliers for the Hodge Laplacian acting on 1-forms on the Heisenberg group*, Geom. Funct. Anal. 17 (2007), no. 3, 852-886.

[3] M. Rumin, *Formes différentielles sur les variétés de contact* (French), J. Diff. Geom. 39 (1994), 182-330.

[4] M. Rumin, *Sub-Riemannian limit of the differential form spectrum of contact manifolds*, Geom. Funct. Anal. 10 (2000), no. 2, 407-452.

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