Free hyperplane arrangements over arbitrary fields and their computation with CoCoA

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Abstract

We study the class of free hyperplane arrangements. Specifically, we investigate the relations between freeness over a field of finite characteristic and freeness over $\mathbb{Q}$. Moreover, we describe the package `arrangements` for the software CoCoA.

1 Introduction

An arrangement of hyperplanes is a finite collection of codimension one affine subspaces in a finite dimensional vector space. Freeness of an arrangement is a key notion which connects arrangement theory with algebraic geometry and combinatorics. We refer to [6] for a comprehensive treatment of the subject.

The study of free arrangements was started by Saito [9] and a remarkable factorization theorem was proved by Terao [11]. This theorem asserts that the characteristic polynomial of a free arrangement completely factors into linear polynomials over the integers. This imposes a necessary condition on the structure of the intersection lattice for an arrangement to be free. The Terao conjecture is the converse problem, i.e. to understand if the structure of the intersection lattice characterize freeness of arrangements.

We will study the connections between freeness over a field of characteristic zero and over a finite field, and to describe in which cases the two situations are related and how. Moreover, we will describe the new package `arrangements` for the free computer algebra software CoCoA.

These results are part of [7] and [8]. All the computations in the paper are done using the computer algebra system CoCoA, see [1], [2], [3] and [8].

2 Hyperplane arrangements

Let K be a field. A finite set of affine hyperplanes $\mathcal{A} = \{H_1, \dots, H_n\}$ in K^l is called a **hyperplane arrangement**. For each hyperplane H_i we fix a defining polynomial $\alpha_i \in S = S^*(K^l) = K[x_1, \dots, x_l]$ such that $H_i = \alpha_i^{-1}(0)$, and let $Q(\mathcal{A}) = \prod_{i=1}^n \alpha_i$. An arrangement $\mathcal{A}$ is called **central** if each H_i contains the origin of K^l.

Let $L(\mathcal{A}) = \{\bigcap_{H\in\mathcal{B}} H \mid \mathcal{B} \subseteq \mathcal{A}\}$ be the **lattice of intersection** of $\mathcal{A}$, ordered by reverse inclusion, i.e. $X \leq Y$ if and only if $Y \subseteq X$, for $X, Y \in L(\mathcal{A})$. Define a rank function on $L(\mathcal{A})$ by $\mathrm{rk}(X) = \mathrm{codim}(X)$. $L(\mathcal{A})$ plays a fundamental role in the study of hyperplane arrangements, in fact it determines the combinatorics of the arrangement.

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Let $\mu\colon L(\mathcal{A}) \to \mathbb{Z}$ be the **Möbius function** of $L(\mathcal{A})$ defined by

$$\mu(X) = \begin{cases} 1 & \text{for } X = K^l, \\ -\sum_{Y<X}\mu(Y) & \text{if } X > K^l. \end{cases}$$

Then the **characteristic polynomial** of $\mathcal{A}$ is defined by

$$\chi(\mathcal{A}, t) = \sum_{X\in L(\mathcal{A})} \mu(X)t^{\dim(X)}.$$

We denote by $\mathrm{Der}_{K^l} = \{\sum_{i=1}^{l} f_i\partial_{x_i} \mid f_i \in S\}$ the S-module of **polynomial vector fields** on K^l (or S-derivations). Let $\delta = \sum_{i=1}^{l} f_i\partial_{x_i} \in \mathrm{Der}_{K^l}$. Then δ is said to be **homogeneous of polynomial degree** d if $f_1, \dots, f_l$ are homogeneous polynomials of degree d in S. In this case, we write $\mathrm{pdeg}(\delta) = d$.

A central arrangement $\mathcal{A}$ is said to be **free with exponents** $(e_1, \dots, e_l)$ if and only if the module of vector fields logarithmic tangent to $\mathcal{A}$,

$$D(\mathcal{A}) = \{\delta \in \mathrm{Der}_{K^l} \mid \delta(\alpha_i) \in \langle\alpha_i\rangle S, \forall i\},$$

is a free S-module and there exists a basis $\delta_1, \dots, \delta_l \in D(\mathcal{A})$ such that $\mathrm{pdeg}(\delta_i) = e_i$, or equivalently $D(\mathcal{A}) \cong \bigoplus_{i=1}^{l} S(-e_i)$. If $\mathcal{A}$ is free with exponents $(e_1, \dots, e_l)$, we can suppose $e_1 \leq e_2 \leq \cdots \leq e_l$, and if $\mathcal{A}$ is essential then $e_1 = 1$.

The module $D(\mathcal{A})$ is a graded S-module and we have that $D(\mathcal{A}) = \{\delta \in \mathrm{Der}_{K^l} \mid \delta(Q(\mathcal{A})) \in \langle Q(\mathcal{A})\rangle S\}$. In particular, since the arrangement $\mathcal{A}$ is central, then the Euler vector field $\delta_E = \sum_{i=1}^{l} x_i\partial_{x_i}$ belongs to $D(\mathcal{A})$. In this case, if the characteristic of K does not divide n, $D(\mathcal{A}) \cong S{\cdot}\delta_E \oplus D_0(\mathcal{A})$, where $D_0(\mathcal{A}) = \{\delta \in \mathrm{Der}_{K^l} \mid \delta(Q(\mathcal{A})) = 0\}$.

In general the exponents of an arrangement depend on the characteristic of K. In fact, we have the following example.

Example 2.1 ([6], Example 4.35)
Consider the arrangement $\mathcal{A}$ in K^3 with defining polynomial $Q(\mathcal{A}) = xyz(x-y)(x+z)(y+z)(x+y+z)$. Then $\mathcal{A}$ is free for any K, but its exponents depend on the characteristic of K.

If the characteristic of K is different from 2, then $\mathcal{A}$ is free with exponents $(1, 3, 3)$, in fact we can take as basis of $D(\mathcal{A})$ the following vector fields δ_E, $\delta_2 = x(x+z)(x+y+z)\partial_x + y(y+z)(x+y+z)\partial_y$ and $\delta_3 = x(x+z)(2y+z)\partial_x + y(y+z)(2x+z)\partial_y$.

If the characteristic of K is 2, then $\mathcal{A}$ is free with exponents $(1, 2, 4)$, in fact we can take as basis of $D(\mathcal{A})$ the following vector fields δ_E, $\delta_2 = x^2\partial_x + y^2\partial_y + z^2\partial_z$ and $\delta_3 = x^4\partial_x + y^4\partial_y + z^4\partial_z$.

Let $\delta_1, \dots, \delta_l \in D(\mathcal{A})$. Then $\det(\delta_i(x_j))_{i,j}$ is divisible by $Q(\mathcal{A})$. One of the most famous characterization of freeness is due to Saito [9] and it uses the determinant of the coefficient matrix of $\delta_1, \dots, \delta_l$ to check if the arrangement $\mathcal{A}$ is free or not. Notice that the original statement is for characteristic 0, but in [10] Terao showed that this statement holds true for any characteristic.

Theorem 2.2 (Saito's criterion)
Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a central arrangement in K^l and $\delta_1, \dots, \delta_l \in D(\mathcal{A})$. Then the following facts are equivalent

1. *$D(\mathcal{A})$ is free with basis $\delta_1, \dots, \delta_l$, i. e. $D(\mathcal{A}) = S \cdot \delta_1 \oplus \cdots \oplus S \cdot \delta_l$.*
2. *$\det(\delta_i(x_j))_{i,j} = cQ(\mathcal{A})$, where $c \in K \setminus \{0\}$.*
3. *$\delta_1, \dots, \delta_l$ are linearly independent over S and $\sum_{i=1}^{l} \mathrm{pdeg}(\delta_i) = n$.*

Freeness has several consequences. For example we recall the following.

Theorem 2.3 ([11])
Suppose that $\mathcal{A}$ is a free arrangement in K^l with exponents $(e_1, \ldots, e_l)$. Then

$$\chi(\mathcal{A}, t) = \prod_{i=1}^{l} (t - e_i).$$

The previous theorem imposes a necessary condition on the structure of $L(\mathcal{A})$ for the arrangement $\mathcal{A}$ to be free. The Terao conjecture is the problem to ask the converse, i.e if the structure of $L(\mathcal{A})$ characterize freeness of $\mathcal{A}$. This conjecture is still unsolved. In [12], Yoshinaga gave an affirmative result when $K = \mathbb{F}_p$ and $l = 3$. Specifically, he proved the following result.

Theorem 2.4
Let $\mathcal{A}$ be a central arrangement in $\mathbb{F}_p^3$. Then the following facts hold true.

1. *When $|\mathcal{A}| \geq 2p$, $\mathcal{A}$ is free if and only if $\chi(\mathcal{A}, p) = 0$.*
2. *When $|\mathcal{A}| = 2p - 1$, $\mathcal{A}$ is free if and only if either $\chi(\mathcal{A}, p) = 0$ or $\chi(\mathcal{A}, t) = (t-1)(t-p+1)^2$.*
3. *When $|\mathcal{A}| = 2p - 2$, $\mathcal{A}$ is free if and only if either $\chi(\mathcal{A}, p) = 0$ or $\chi(\mathcal{A}, t) = (t-1)(t-p+1)(t-p+2)$.*

More in general, in [12], Yoshinaga proved the following.

Theorem 2.5
Let $\mathcal{A}$ be a central arrangement in $\mathbb{F}_p^l$. If $\chi(\mathcal{A}, p^{l-2}) = 0$, then $\mathcal{A}$ is free with exponents $(1, p, \ldots, p^{l-2}, |\mathcal{A}| - 1 - p - \cdots - p^{l-2})$.

3 Change of characteristic

From now on we will assume that $\mathcal{A} = \{H_1, \ldots, H_n\}$ is a central arrangement in $\mathbb{Q}^l$. After getting rid of the denominators, we can suppose that $\alpha_i \in \mathbb{Z}[x_1, \ldots, x_l]$ for all $i = 1, \ldots, n$, and hence that $Q(\mathcal{A}) = \prod_{i=1}^{n} \alpha_i \in \mathbb{Z}[x_1, \ldots, x_l]$. Moreover, we can also assume that there exists no prime number p that divides any α_i.

Let p be a prime number. Consider the image of $Q(\mathcal{A})$ under the canonical homomorphism

$$\pi_p \colon \mathbb{Z}[x_1, \ldots, x_l] \to \mathbb{F}_p[x_1, \ldots, x_l].$$

If $\pi_p(Q(\mathcal{A}))$ is reduced, we will say that the prime number p is **good for** $\mathcal{A}$. Notice that there is a finite number of primes p that are not good for $\mathcal{A}$.

Let now p be a good prime for $\mathcal{A}$, and consider $\mathcal{A}_p$ the arrangement in $\mathbb{F}_p^l$ defined by $\pi_p(Q(\mathcal{A}))$. Hence, by construction, $Q(\mathcal{A}_p) = \pi_p(Q(\mathcal{A})) \neq 0$ and it is reduced.

Theorem 3.6
If $\mathcal{A}$ is free in $\mathbb{Q}^l$ with exponents $(e_1, \ldots, e_l)$, then $\mathcal{A}_p$ is free in $\mathbb{F}_p^l$ with exponents $(e_1, \ldots, e_l)$, for all good primes except possibly a finite number of them.

Since the number of not good primes for $\mathcal{A}$ is finite, we have the following.

Corollary 3.7
Let $\mathcal{A}$ be a central arrangement in $\mathbb{Q}^l$ and p a large prime number. If $\mathcal{A}$ is free in $\mathbb{Q}^l$ with exponents $(e_1, \ldots, e_l)$, then $\mathcal{A}_p$ is free in $\mathbb{F}_p^l$ with exponents $(e_1, \ldots, e_l)$.

Example 3.8
Consider $\mathcal{A}$ the arrangement in $\mathbb{Q}^4$ as the cone of $\mathcal{A}^{[-2,2]}$ the Shi-Catalan arrangement of type B. As described in [4], $\mathcal{A}$ is free with exponents $(1, 13, 15, 17)$. Now, $5, 7$ and 11 are all good prime numbers for $\mathcal{A}$. A direct computation shows that the arrangement $\mathcal{A}_5$ over $\mathbb{F}_5$ is free with exponents $(1, 5, 15, 25)$. However, both $\mathcal{A}_7$ over $\mathbb{F}_7$ and $\mathcal{A}_{11}$ over $\mathbb{F}_{11}$ are not free. Moreover, for any other good prime number p, $\mathcal{A}_p$ over $\mathbb{F}_p$ is free with exponents $(1, 13, 15, 17)$.

Denote by $J(\mathcal{A})_{\mathbb{Z}}$ the ideal of $\mathbb{Z}[x_1, \ldots, x_l]$ generated by $Q(\mathcal{A})$ and its partial derivatives.

Lemma 3.9
The number of distinct prime numbers that are zero divisors in $\mathbb{Z}[x_1, \ldots, x_l]/J(\mathcal{A})_{\mathbb{Z}}$ is finite. Moreover, these zero divisors can be computed via the computation of a minimal strong Gröbner basis of $J(\mathcal{A})_{\mathbb{Z}}$.

Theorem 3.10
Let $\mathcal{A}$ be a central arrangement in $\mathbb{Q}^l$. Let p be a good prime number for $\mathcal{A}$ that does not divide n and that is a non-zero divisor in $\mathbb{Z}[x_1, \ldots, x_l]/J(\mathcal{A})_{\mathbb{Z}}$. If $\mathcal{A}_p$ is free in $\mathbb{F}_p^l$ with exponents $(e_1, \ldots, e_l)$, then $\mathcal{A}$ is free in $\mathbb{Q}^l$ with exponents $(e_1, \ldots, e_l)$.

In Theorem 3.10, the assumption that the prime p is a non-zero divisor in $\mathbb{Z}[x_1, \ldots, x_l]/J(\mathcal{A})_{\mathbb{Z}}$ is fundamental. In fact we have the following.

Example 3.11
Consider the arrangement $\mathcal{A} \subseteq \mathbb{Q}^3$ with defining polynomial $Q(\mathcal{A}) = z(x+2y-4z)(y+4z)(x+3y-6z)$. $\mathcal{A}$ is non free and both 2 and 3 are zero divisors in $\mathbb{Z}[x_1, \ldots, x_l]/J(\mathcal{A})_{\mathbb{Z}}$. In fact, we have that $3(y^2z^2 + 2yz^3 - 8z^4) \in J(\mathcal{A})_{\mathbb{Z}}$ but $y^2z^2 + 2yz^3 - 8z^4 \notin J(\mathcal{A})_{\mathbb{Z}}$, and similarly $2(xyz^2 + 4y^2z^2 + 4xz^3 + 8yz^3 - 32z^4) \in J(\mathcal{A})_{\mathbb{Z}}$ but $xyz^2 + 4y^2z^2 + 4xz^3 + 8yz^3 - 32z^4 \notin J(\mathcal{A})_{\mathbb{Z}}$. However, both A_2 and A_3 are free with exponents $(1, 1, 2)$.

By Lemma 3.9, the number of prime numbers that are zero divisors in $\mathbb{Z}[x_1, \ldots, x_l]/J(\mathcal{A})_{\mathbb{Z}}$ is finite. Hence, putting together Corollary 3.7 and Theorem 3.10, we have the following.

Corollary 3.12
Let $\mathcal{A}$ be a central arrangement in $\mathbb{Q}^l$ and p a large prime number. $\mathcal{A}_p$ is free in $\mathbb{F}_p^l$ with exponents $(e_1, \ldots, e_l)$ if and only if $\mathcal{A}$ is free in $\mathbb{Q}^l$ with exponents $(e_1, \ldots, e_l)$.

From Theorems 2.4 and 3.10, we have the following.

Theorem 3.13
Let $\mathcal{A}$ be a central arrangement in $\mathbb{Q}^3$. Let p be a good prime number for $\mathcal{A}$ that does not divide $|\mathcal{A}|$ and that is a non-zero divisor in $\mathbb{Z}[x_1, x_2, x_3]/J(\mathcal{A})_{\mathbb{Z}}$. Then the following facts hold true.

1. *When $|\mathcal{A}| \geq 2p$, $\mathcal{A}$ is free if $\chi(\mathcal{A}_p, p) = 0$.*
2. *When $|\mathcal{A}| = 2p - 1$, $\mathcal{A}$ is free if either $\chi(\mathcal{A}_p, p) = 0$ or $\chi(\mathcal{A}_p, t) = (t-1)(t-p+1)^2$.*
3. *When $|\mathcal{A}| = 2p - 2$, $\mathcal{A}$ is free if either $\chi(\mathcal{A}_p, p) = 0$ or $\chi(\mathcal{A}_p, t) = (t-1)(t-p+1)(t-p+2)$.*

More in general, from Theorems 2.5 and 3.10, we have the following.

Theorem 3.14
Let $\mathcal{A}$ be a central arrangement in $\mathbb{Q}^l$. Let p be a good prime number for $\mathcal{A}$ that does not divide $|\mathcal{A}|$ and that is a non-zero divisor in $\mathbb{Z}[x_1, \ldots, x_l]/J(\mathcal{A})_{\mathbb{Z}}$. If $\chi(\mathcal{A}_p, p^{l-2}) = 0$, then $\mathcal{A}$ is free with exponents $(1, p, \ldots, p^{l-2}, |\mathcal{A}| - 1 - p - \cdots - p^{l-2})$.

Example 3.15
Consider the arrangement $\mathcal{A}$ in $\mathbb{Q}^3$ with defining polynomial $Q(\mathcal{A}) = xyz(x-y)(x+y)(x-z)(x+z)(y-z)(y+z)$. Now, $p = 5$ is a good prime number for $\mathcal{A}$ that does not divide $|\mathcal{A}| = 9$ and that is a non-zero divisor in $\mathbb{Z}[x, y, z]/J(\mathcal{A})_{\mathbb{Z}}$. A direct computations shows that, $\chi(\mathcal{A}_5, t) = (t-1)(t-3)(t-5)$ and hence, by Theorem 3.13 or 3.14, $\mathcal{A}$ is free with exponents $(1, 3, 5)$. Notice that in this case, $\mathcal{A}$ and $\mathcal{A}_5$ have isomorphic intersection lattice, hence $p = 5$ is a "large prime number". However, in general, it is difficult to detect when a prime number is "large" enough.

4 Arrangement package for CoCoA

In order to test our theorems and play with examples, we wrote the package `arrangements` for the software CoCoA, see [8]. This package is part of the official release CoCoA-5.2.4.

This package allows the user to easily define any hyperplane arrangement as a list of linear polynomials. Moreover, several known families of arrangements are already implemented (see CoCoA's help function `?ArrFamily`). For example, we construct the Braid arrangement (i.e. the arrangement in K^l with equation $\prod_{1\leq i<j\leq l} x_i - x_j = 0$) in CoCoA as follows:

```
/**/ use S ::= QQ[x, y, z];
/**/ A := ArrBraid(S, 3); A;
[x-y,  x-z,  y-z]
```

With this package, we can compute several combinatorial invariants of hyperplane arrangements. For example, we construct the flats of the intersection lattice, the characteristic polynomial, the Tutte polynomials, and the Betti numbers of the Braid arrangement in CoCoA as follows:

```
/**/ ArrFlats(A);
[[ideal(0)],  [ideal(x-y), ideal(x-z), ideal(y-z)], [ideal(x-z, y-z)]]
/**/ ArrCharPoly(A);
t^3-3*t^2+2*t
/**/ ArrTuttePoly(A);
t[1]^2+t[1]+t[2]
/**/ ArrBettiNumbers(A);
[1,  3,  2]
```

We can also compute various algebraic invariants. For example, we construct the Orlik-Terao ideal and the Solomon-Terao ideal of the Braid arrangement in CoCoA as follows:

```
/**/ OrlikTeraoIdeal(A);
ideal(y[1]*y[2]-y[1]*y[3]+y[2]*y[3])
/**/ f := x^2+y^2+z^2;
/**/ SolomonTeraoIdeal(A, f);
ideal(2*x+2*y+2*z,  2*x*y-2*y^2+2*x*z-2*z^2,  2*x*y*z-2*x*z^2-2*y*z^2+2*z^3)
```

In addition, several functions for the class of free hyperplane arrangements are implemented. We check freeness, compute a Saito's matrix and the exponents of the arrangement in Example 2.1 in CoCoA as follows:

```
/**/ use S ::= QQ[x, y, z];
/**/ A := [x, y, z, x-y, x+z, y+z, x+y+z];
/**/ IsArrFree(A);
true
/**/ ArrDerModule(A);
matrix( /*RingWithID(4, "QQ[x, y, z]")*/
 [[x, 0, 0],
  [y, x*y*z-y^2*z, x^2*y-y^3],
  [z, -2*x*y*z-x*z^2-2*y*z^2-z^3, x^2*z+3*x*y*z+3*x*z^2+3*y*z^2+2*z^3]])
/**/ ArrExponents(A);
[1,  3,  3]
/**/ use F2 ::= ZZ/(2)[x, y, z];
/**/ A2 := [x, y, z, x-y, x+z, y+z, x+y+z];
/**/ IsArrFree(A2);
true
/**/ ArrDerModule(A2);
matrix( /*RingWithID(25, "RingWithID(24)[x, y, z]")*/
 [[x, 0, 0],
  [y, x*y+y^2, 0],
  [z, x*z+z^2, x^2*y*z+x*y^2*z+x^2*z^2+x*y*z^2+y^2*z^2+z^4]])
/**/ ArrExponents(A2);
[1,  2,  4]
```

This package allows also to do computations with multiarrangements. We construct the Ziegler's multirestriction of a given arrangement, compute its Saito's matrix and its multi-exponents in CoCoA as follows:

```
/**/ use S ::= QQ[x, y, z];
/**/ A := [x, y, z, x-y, x-y-z, x-y+2*z];
/**/ B := MultiArrRestrictionZiegler(A, z); B;
[[y[1], 1], [y[2], 1], [y[1]-y[2], 3]]
/**/ MultiArrDerModule(B);
matrix( /*RingWithID(18, "QQ[y[1], y[2]]")*/
 [[y[1]*y[2], y[1]^3],
  [y[1]*y[2], 3*y[1]^2*y[2]-3*y[1]*y[2]^2+y[2]^3]])
/**/ MultiArrExponents(B);
[2, 3]
/**/ ArrExponents(A);
[1, 2, 3]
```

References

[1] J. Abbott and A.M. Bigatti. CoCoALib: a C++ library for doing Computations in Commutative Algebra. Available at `http://cocoa.dima.unige.it/cocoalib`, 2016.

[2] J. Abbott and A.M. Bigatti. Gröbner bases for everyone with CoCoA-5 and CoCoALib. *Advanced Studies in Pure Mathematics*, 77:1–24, 2018.

[3] J. Abbott, A.M. Bigatti, and L. Robbiano. CoCoA: a system for doing Computations in Commutative Algebra. Available at `http://cocoa.dima.unige.it`.

[4] T. Abe and H. Terao. The freeness of Shi–Catalan arrangements. *European Journal of Combinatorics*, 32(8):1191–1198, 2011.

[5] A. Bigatti, E. Palezzato, and M. Torielli. New characterizations of freeness for hyperplane arrangements. *Journal of Algebraic Combinatorics*, https://doi.org/10.1007/s10801-019-00876- 9, 2019.

[6] P. Orlik and H. Terao. *Arrangements of hyperplanes*, volume 300 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.

[7] E. Palezzato and M. Torielli. Free hyperplane arrangements over arbitrary fields. *arXiv:1803.09908*, 2018.

[8] E. Palezzato and M. Torielli. Hyperplane arrangements in CoCoA. To appear in *Journal of Software for Algebra and Geometry. arXiv:1805.02366*, 2018.

[9] K. Saito. Theory of logarithmic differential forms and logarithmic vector fields. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 27(2):265–291, 1980.

[10] H. Terao. Free arrangements of hyperplanes over an arbitrary field. *Proceedings of the Japan Academy, Series A, Mathematical Sciences*, 59(7):301–303, 1983.

[11] H. Terao. Generalized exponents of a free arrangement of hyperplanes and Shepherd-Todd-Brieskorn formula. *Inventiones mathematicae*, 63(1):159–179, 1981.

[12] M. Yoshinaga. Free arrangements over finite field. *Proceedings of the Japan Academy, Series A, Mathematical Sciences*, 82(10):179–182, 2007.