THE INTEGRAL TABLES OF THE ELLIPTIC FUNCTIONS BY LEGENDRE AND ITS APPLICATION BY POISSONN.

流体数理古典理論研究所
増田 茂 (SHIGERU MASUDA)

Contents

Abstract. We introduce Legendre's elliptic functions with his theory and construction of the Tables in origin of 1825-26 [1]. Legendre emphasizes his theory's superiority to Euler's integral theory, (as you know, this is entitled with 'Eulerlian integral' on the total title, of theses huge volumes 1-3 of books), above all, the concept of echelon and the complete functions on which we discuss specially.

Succedingly, we introduce Poisson's applications of Legendre's elliptic functions to the capillary model owing to the theory and table by Legendre, which Poisson gives up the self made theories based on the same elliptic functions including tables. Poisson discusses the heat problems, in which he also applies the elliptic functions and tables in the Earth's science, 1n 1835. Both application may be the first orthodox applications of Legendre's theory to the nonlinear differential equations. Hence, all these topics are the translations from the original by me.

Mathematics Subject Classification 2020 : 11-XX, 33E05, 35-XX, 35K05, 35QXX, 44-XX 44A11, 44A15, 91F10

Key words : Elliptic function, mathematical physics, Legendre's Tables of elliptic functions, Poisson's applications of elliptic functions by Legendre, mathematical history.

1. Legendre's Elliptic Functions.

1.1. **General consideration on the echelon of the modules and on the properties of the function F related to different terms of this echelon.** [1]

§ **79**. The complete function F^1c [2] can explain from two manners, the one method from the decreasing module c, c°, $c^{\circ\circ}$ etc. ; the another method from increasing modules c, c', c''. etc., The primary expression $F^1c = \frac{\pi}{2}K$, or $K = (1+c^\circ)\ (1+c^{\circ\circ})\ (1+c^{\circ\circ\circ})\cdots$, we can also put K under the form $K = \frac{2\sqrt{c^\circ}}{c}\cdot\frac{2\sqrt{c^\circ\circ}}{c^\circ}\cdot\frac{2\sqrt{c^\circ\circ\circ}}{c^{\circ\circ}}$, or more simply again, under the form $K = \sqrt{\frac{1}{b}\cdot b^\circ b^{\circ\circ}, \text{etc.}}$, where, we

Date: 2019/12/08.

[1] (⇓) Remark : hence, our footnotes are showed with (⇓), the original footnote by authors with 'sic'.

[2] (⇓) In the articles § **79** and § **80**, the complete function F^1c is spelled as $F'c$, we doubt it is the miss with the TeX editing, so that we correct these symbols. Because in the last line of the § **80**, it is correctly used.

TABLE 1. Legendre's definitions and discussions of three kinds of integral : F, E, Π.

no	chapter article	name of items	symbols	definitions, meaning and discussions
1	§5,a11	constant of complement	b, c	$b^2 + c^2 = 1$, namely, $b = \sqrt{1-c^2}$. cf. no. 4.
2	§5,a13	three kinds of integral	F, E, Π	F : first ellliptic function (e.f.), E : second e.f., Π : tertiary e.f.
		length of arc	H	$H = A' \int \frac{d\varphi}{\Delta} + B' \int \Delta d\varphi + C' \int \frac{d\varphi}{(1+n\sin^2\varphi)\Delta}$.
		$d(\Delta\tan\varphi)$		$d(\Delta\tan\varphi) = \frac{d\varphi}{\Delta\cos^2\varphi} - \frac{d\varphi}{\Delta} + \Delta d\varphi$.
3	§5,a15	three kinds of integral	F, E, Π n, Δ, c	$F = \int \frac{d\varphi}{\Delta}, \quad E = \int \Delta d\varphi, \quad \Pi = \int \frac{d\varphi}{(1+n\sin^2\varphi)\Delta}$, where, $n = \pm, \in R, C, \quad \Delta = \sqrt{1-c^2\sin^2\varphi}, \quad c < 1$.
4	§15,a18	complete function	F^1	$F(\varphi) + F(\psi) = F(\frac{1}{2}\pi) = F^1$
5	§15,a45	relation between F^1, E^1 (Legendre's relation.)	$F^1(c), E^1(c)$ $F^1(b), E^1(b)$	$\frac{\pi}{2} = F^1(c)E^1(b) + F^1(b)E^1(c) - F^1(b)F^1(c)$.
6	§15,a53	three cases of n	n, b, c	$I : n = \cot^2\theta, \quad II : n = -1 + b^2\sin^2\theta, \quad b^2 + c^2 = 1$. $III : n = -c^2\sin^2\theta. \quad I, II$, for circle, III for logarithm.
7	§20, a76-a87	echelon of the modules	$b,\ c,\ F$	general consideration of echelon of the modules
8	§22,a97	Π	Π	method of applied approximation to the elliptic function of the tertiary kind
9	§23,a105	n	Π, n	$n = \cot^2\theta$.
	§23,a111			$n = -1 + b^2\sin^2\theta$.
	§23,a115			$n = -c^2\sin^2\theta$.
10	§35,a232	Π^1 (complete function)	Π^1	Π^1 is expressed with F^1 and E^1.

will remember that the sequence b, b°, $b^{\circ\circ}$, $b^{\circ\circ\circ}$, converge rapidly toward an equal limit to the unit.

The second expression, owing to the formula of the (art. 72), is $F^1 c = \frac{K'}{2} \log \cdot \frac{4}{b'^\nu}$, where, we have

$$K' = \frac{2\sqrt{b'}}{b} \cdot \frac{2\sqrt{b''}}{b'} \cdot \frac{2\sqrt{b'''}}{b''} \cdots = \sqrt{\frac{1}{c} \cdot c'c''c'''} \ ;$$

we suppose in this formula, b'^ν considerably small for that $1 - c'^\nu$ were negligible.

Equalizing between them the two values, from $F^1 c$, we will get this general formula

$$(E) \qquad \frac{\pi}{2}\sqrt{\frac{b^{\circ\mu}\cdots b^{\circ\circ\circ}b^{\circ\circ}bb'b''\cdots b'^{\nu-1}}{b'^\nu}} = \log \cdot \frac{4}{b'\ \nu},$$

where, we see that the product under the root must be prolonged to right up to a term $b^{\circ\mu}$ which doesn't differ sensibly with the unit, and to right up to term $b'^{\nu-1}$, considerably small for that the following $b'\ ^\nu$ or at least its square, were of the order of the negligible quantities.

If we change b with c, we will have sensibly

$$(F) \qquad \frac{\pi}{2}\sqrt{\frac{c'\ ^\nu\cdots c'''c''c'c\ c^\circ c^{\circ\circ}\cdots c^{\circ\mu-1}}{c^{\circ\mu}}} = \log \cdot \frac{4}{c^\circ\ ^\nu}, \qquad (1)$$

formula which suppose $1 - c'^\nu$ negligible hence as $1 - b^\circ\ ^\mu$.

If we multiple these formula of precedent article, we will have this very remarkable result,

$$(G) \qquad \frac{\pi^2}{4} \cdot 2^{\mu+\nu} = \log \frac{4}{b'\ ^\nu} \cdot \log \frac{4}{c^\circ\ ^\mu} \qquad (2)$$

equation which suppose negligible the $\frac{1}{4}$ of $b'\ ^\nu$ and that of $c^{\circ\mu}$.

When $b = c$, we have generally $b^\mu = c^{\circ\mu}$. Hence, in supposing the $\frac{1}{4}$ of $c^{\circ\mu}$, that of $b'\ ^\nu$ will be also it, so that we can put $\nu = \mu$, and the preceding equation turns into

$$(H) \qquad \frac{\pi}{2} \cdot 2^\mu = \log \cdot \frac{4}{c^\circ\ ^\mu}. \qquad (3)$$

These equations aren't approached, however, in putting the suitable values to the number μ and ν, the approximation will be made to a certain degree proposed, more rapidly than we could make with all other method. Let's give some examples.

§ **80**. Let's remark at first,

1°. that we can arrive directly at the equation (2) by means of the double values following

$$F^1 c = \frac{\pi}{2} K = \frac{K'}{2^\nu} \log \cdot \frac{4}{b'^\nu}, \qquad F^1 b = \frac{\pi}{2} K' = \frac{K}{2^\mu} \log \cdot \frac{4}{c^{\circ\mu}} \ ;$$

because in multiplying these values, we obtain immediately $\frac{\pi^2}{4} = \frac{1}{2^{\mu+\nu}} \cdot \log \cdot \frac{4}{b'^\nu} \cdot \log \cdot \frac{4}{c^{\circ\mu}}$.

2°. That when $c = \sqrt{\frac{1}{2}}$, the comparison of the equation (1) to the equation (3) gives

$$4^\mu = \frac{c'^\mu \cdots c''' c'' c' c \ c^\circ c^{\circ\circ} \cdots c^{\circ\mu - 1}}{c^{\circ\mu}}, \tag{4}$$

[3] where, we suppose $1 - c'^\mu$ negligible. This one would obtain immediately from the equation (**??**) in putting $b = c$, this one which give $b^{\circ\nu} = c'^\nu$.

3°. That in the equation (1) we can suppose c already considerably small for that $1 - b$ were negligible, hence we will be able to put $c^{\circ\mu} = c$, this one which will give this formula more simple

$$(I) \qquad \log \frac{4}{c} = \frac{\pi}{2} \sqrt{\frac{1}{c} \cdot c' c'' c''' \cdots 1}.$$

This formula, for the rest, isn't another thing with the equation $\frac{\pi}{2} K' = F^b = \log \frac{4}{c}$.

§ **83**. Let's suppose now that the two complete functions $F'b$ and $F'c$, as we have additionally $F'c = \frac{\pi}{2} K$ and $F'b = \frac{K}{2^\mu} \log \frac{4}{c^{\circ\mu}}$, it will result from here $\frac{n\pi}{2} = \frac{1}{2^\mu} \log \frac{4}{c^{\circ\mu}}$. Hence $\frac{n\pi}{2}$ will be equal to the limit toward which tend rapidly the terms of the sequence $\frac{1}{2} \log \frac{4}{c^\circ}$, $\frac{1}{4} \log \frac{4}{c^{\circ\circ}}$, $\frac{1}{8} \log \frac{4}{c^{\circ\circ\circ}}$, etc., the case of $b = c$ is contained in this formula in putting $n = 1$; however, there is another case where we can make application from it.

Hence, we have found that in putting $c = \sin 15^\circ = \frac{1}{2}\sqrt{1 - \sqrt{3}}$, we have $F'b = \sqrt{3} \cdot F'c$; hence, in calculating the sequence c°, $c^{\circ\circ}$, etc., owing to the module $c = \sin 15^\circ$, $\frac{\pi\sqrt{3}}{2}$ will be equal to the limit of the sequence $\frac{1}{2} \log \frac{4}{c^\circ}$, $\frac{1}{4} \log \frac{4}{c^{\circ\circ}}$, etc. The approximation is such that from the primary term we have $\pi = \frac{2}{\sqrt{3}} \log \frac{2}{(\tan 7^\circ)^{\frac{1}{2}}} = 3.141636$, value which doesn't differ from the veritable which in the fifth decimal.

We will see below, that in putting $\sin 2\alpha = \tan^2 15^\circ$ and $c = \sin \alpha$, we have $F'b = 3F'c$; hence, in calculating the sequence of the module $c^\circ, c^{\circ\circ}$, etc., owing to the value $c = \sin \alpha$, we will have $\frac{3\pi}{2}$ equal to the limit of the sequence $\frac{1}{2} \ \log \frac{4}{c^\circ}$, $\frac{1}{4} \ \log \frac{4}{c^{\circ\circ}}$, etc. With the primary term, we obtain already $\pi = 3.1415926627$, the error being only one decimal of the eighth order, from here, we can conclude that at fifth term, the approximate equivalate to 128 decimals.

§ **84**. Let's suppose that we would want to find the module c such that the relation of the two complementary functions $F'b$ and $F'c$ were equal to a given number n, rational or irrational, it will need to satisfy with the equation $\frac{n\pi}{2} = \frac{1}{2^\mu} \log \frac{4}{c^{\circ\mu}}$, where the index μ will turn to be taken owing to the degree of approximation which we wish to obtain, in order that the quantities of the order $1 - b^{\circ\mu}$ or $(c^{\circ\mu})^2$, were negligible.

Let, for example, $n = \sqrt{2}$, if we take $\mu = 3$, then we will have $\log \frac{4}{c^{\circ\circ\circ}} = \pi \cdot 2^{\frac{5}{2}}$, this one gives, in usual logarithms, $\log c^{\circ\circ\circ} = 2.83398\ 18161\ 9082$. Knowing $c^{\circ\circ\circ}$, we deduce from it successively $c^{\circ\circ}$, c°, etc., and finally c, of which the logarithmic values are

$c^{\circ\circ}$ 6.74302 08075 1135, $\quad c^\circ$ 8.67250 01689 4325, $\quad c$ 9.61722 43146 6214.

With this logarithm, we find $c = \sqrt{2} - 1$, and in effect, in the case of this module, we have exactly $F^1 b = \sqrt{2} \cdot F^1 c$.

---

[3](⇓)In doubly multiplying in both hand-sides of (3), we get $\left(2^\mu\right)^2 = 4^\mu$ of the left hand-side of (4).

§ **85**. If we consider the ratio of the complementary functions in two consecutive degrees of the echelon of the modules, namely : $\frac{F^1b}{F^1c}$, $\frac{F^1b^\circ}{F^1c^\circ}$, we will find the second ratio is double from the primary. In effect, we have the equation $F^1c = (1+c^\circ)F^1c^\circ$, from here, results also $F^1b^\circ = (1+b)F^1b$; and because $c^\circ = \frac{1-b}{1+c}$, we have $1+c^\circ = \frac{2}{1+b}$; hence $\frac{F^1b}{F^1c} = \frac{1}{2}\cdot\frac{F^1b^\circ}{F^1c^\circ}$; similarly $\frac{F^1b^\circ}{F^1c^\circ} = \frac{1}{2}\frac{F^1b^{\circ\circ}}{F^1c^{\circ\circ}}$, etc., so that in general

$$\frac{F^1b}{F^1c} = \frac{1}{2^\mu}\cdot\frac{F^1b^{\circ\mu}}{F^1c^{\circ\mu}}. \tag{5}$$

We would have similarly, in continuing the echelon in another sense, $\frac{F^1b}{F^1c} = 2\frac{F^1b'}{F^1c'} = 4\frac{F^1b''}{F^1c''}$, etc.

This properties is general, whatever the echelon of the modules. In the particular case where $b = c = \sin 45^\circ$, we have hence $F^1b = F^1c$, $F^1b^\circ = 2F^1c^\circ$, $F^1b^{\circ\circ} = 4F^1c^{\circ\circ}$, etc., $F^1b' = \frac{1}{2}F^1b^\circ$, $F^1b'' = \frac{1}{4}F^1b^{\circ\circ}$, etc.

If the module $c = \sqrt{2}-1$, we will have $b' = c$ and $c' = b$; hence $\frac{F^1b}{F^1c} = 2\frac{F^1c}{F^1b}$. Hence we have in this case $F^1b = \sqrt{2}F^1c$, and successively, $F^1b^\circ = 2\sqrt{2}.F^1c^\circ$, $F^1b^{\circ\circ} = 4\sqrt{2}\cdot F^1c^{\circ\circ}$, etc. ; this is also this one which we would obtain immediately with the equation $F^1c' = (1+c)F^1c$, where $c' = b$.

The two modules which we have given just now for example, are the only ones in which the echelon of the modules is the same, to the near order, than that of the complementary theirs.

2. Construction of the Table.

2.1. The calculation of the Complete Functions.

§ **643**. (On the calculus of the complete functions.) Let's suppose in general that we wish to calculate the logarithms of the functions of which it is the problem up to 14 decimals, because this number is that which comport (contain) the tables the most extended which would have been published up to present, namely : the *Arithmetica logarithmica* of Briggs and the *Trigonometriac Britannica* of the same author. The examples which supply in this hypothesis make judge easily the simplifications of which the calculations are susceptible, when we will wish to obtain that ten or less number of exact decimals. We will see soon that the same given which serve to calculate the functions F^1c, E^1c, serve also to calculate the complementary functions F^1b, E^1b ; this is because we consider only the values of c less than $\sin 45^\circ$. When the module proposes will be greater than $\sin 45^\circ$, we will exchange between them the letters c and b, in order that c designates always the smallest of the two.

It needs at first to form the echelon of the modules c, c°, $c^{\circ\circ}$, etc. and that of their complements b, b°, $b^{\circ\circ}$, etc. ; however, the number of terms to calculate vary following the largeness of the primitive module, and it implies to establish the general divisions which fix, with a precise manner, the number of these terms.

§ **644**. The object which we propose being to obtain as far as it is possible 14 exact decimals, if we are arrived to a term b^μ such as $-\log b^\mu$ were less than a half-unit decimal of the 14° order, then we will be able to regard $\log b^\mu$ as null, and to more strong reason, the terms following $\log b^{\mu+1}$, $\log b^{\mu+2}$, etc. ; therefore $b^{\mu-1}$ will be the last of the terms b, b°, $b^{\circ\circ}$, of which it needs to regard.

The series of the modules c, c°, $c^{\circ\circ}$, etc. always compose a term of more ; it will turn in consequence, be terminated at the module c^μ. This reason is that we have then $c^\mu = \left(\frac{1}{2}c^{\mu-1}\right)^2\cdot\frac{1}{b^{\mu-1}}$, and which therefore the logarithm of $b^{\mu-1}$ is necessary to compose the value of $\log c^\mu$.

Passed the term c^μ, there isn't location to consider the following $c^{\mu+1}$, because we will have without sensible error $c^{\mu+1} = \left(\frac{1}{2}c^\mu\right)^2$, and because, therefore, the quantity $\frac{1}{2^\mu}\log\frac{4}{c^\mu}$ doesn't change in putting $\mu+1$ instead of μ.

Posed thus, it is easy to see that we will know the limits of the different cases, in beginning with determining the value of the module c which gives for its complement $\log b = \frac{1}{2}10^{-14}$.

The module supposed c being extremely small, we have from a manner sufficiently exact

$b = 1 - \frac{1}{2}c^2$ and $\log b = -\frac{1}{2}mc^2$; [4] therefore $c^2 = M10^{-14}$, then $c = 10^{-7}\sqrt{M}$, namely [5] $\log c = 3.1811078$.

If we assimilate c at the sin of an arc, we will find that this arc isn't only fraction of secondary and that we have $c = \sin 0'' 03130$.

It needs now to start from the module very small to form the sequence of the module increasing $c,\ c',\ c'',\ C'''$, etc. ; this is a calculus which we will be able to make from a sufficiently exact manner for our object, with a Table to seven decimals only. We will have,

- at first, $c' = \frac{2\sqrt{c}}{1+c}$, or, simply $c' = 2\sqrt{c}$, this one, which gives $\log c' = 6.8915839$ and $c' = \sin 0°2'\ 40''\ 70$.
- To have c'' I put $c' = \tan^2\frac{1}{2}\theta$, I have $l\tan\frac{1}{2}\theta = 8.4457919$, $\frac{1}{2}\theta = 1°\ 35'\ 55''\ 78$, $\theta = 3°\ 11'\ 51''\ 56$; therefore $c'' = \sin 3°\ 11'\ 51''\ 56$ and $\log c'' = 8.7464836$.
- If we put again $c'' = \tan^2\frac{1}{2}\theta'$, we will have $l\tan\frac{1}{2}\theta' = 9.3732418$, $\frac{1}{2}\theta' = 13°\ 17'\ 18''\ 84$, $\theta' = 26°\ 34'\ 37''\ 68$; therefore $c''' = \sin 26°\ 34'\ 37''\ 68$ and $\log c''' = 9.6506981$.
- Let finally, $c''' = \tan^2\frac{1}{2}\theta''$, then we will have $l\tan\frac{1}{2}\theta'' = 9.8253490$, $\frac{1}{2}\theta'' = 33°\ 46'\ 40''\ 15$, $\theta' = 67°\ 33'\ 20''\ 30$; therefore $c^{IV} = \sin 67°\ 33'\ 20''\ 30$ and $\log c^{IV} = 9.9657898$.

§ **645**. It results from the preceding calculus.

1°. that from $c = \sin 67°33'$ until $c = \sin 26°34'$, we will turn to restrict to calculate the four terms $b,\ b°,\ b^{°°},\ b^{°°°}$, and the fives $c,\ c°,\ c^{°°},\ c^{°°°},\ c^{°°°°}$;

2°. That from $c = \sin 26°34'$ until $c = \sin 3°\ 11'$, then we will have to calculate the three terms $b,\ b°,\ b^{°°}$, and the fours $c,\ c°,\ c^{°°},\ c^{°°°}$;

3°. That from $c = \sin 3°11'$ until $c = \sin 0°\ 2'\ 40''$, then it will suffice the two terms $b,\ b°$, and the threes $c,\ c°,\ c^{°°}$;

4°. That from $c = \sin 0°2'\ 40''$ until $c = \sin 0''\ 0313$, it will suffice to calculate the term b and the twos $c,\ c°$;

5°. Finally, that below from $c = \sin 0''\ 0313$, we haven't necessity the only one term c.

Such is the number of the terms of the series of the modules and of that (module) of their complements, which it will be necessary to calculate in the different cases, to obtain 14 exact decimals in the logarithms of the functions $F^1c,\ E^1c,\ F^1b,\ E^1b$. We are going to see now how the calculus of these modules can be effectuated in the easiest manner.

2.1.1. *Formation of the echelon of the modules.* § **646**. (Formation of the echelon of the modules.) Knowing the logarithms of c and b, it is important to find these of the terms following $c°$ and $b°$. For this, let $c° = x$ be, the equation $b°c = 2\sqrt{bc°}$ will give $x = \frac{\left(\frac{1}{2}c\right)^2}{b}(1-x^2)$, and in putting $p = \frac{\left(\frac{1}{2}c\right)^2}{b}$, the value of x developed in regular series will be

$$x = p - \frac{1}{4}\cdot 4p^3 + \frac{1\cdot 3}{4\cdot 6}\cdot 16p^5 - \frac{1\cdot 3\cdot 5}{4\cdot 6\cdot 8}\cdot 64p^7 + \text{etc.}$$

But, it is important to calculate directly $\log x$; namely, the value $x = \frac{\sqrt{1+4p^2}-1}{2p}$ gives

$$\frac{dx}{x} = \frac{dp}{p\sqrt{1+4p^2}} = \frac{dp}{p}\Big(1^{\frac{1}{2}}\cdot 4p^2 + \frac{1\cdot 3}{2\cdot 4}\cdot 16p^5 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot 64p^6 + \text{etc.}\Big),$$

from here, we get in integrating

$$\log x = \log p - p^2 + \frac{3}{2}\cdot p^4 - \frac{3\cdot 5}{2\cdot 3}\cdot\frac{4p^6}{3} + \frac{3\cdot 5\cdot 7}{2\cdot 3\cdot 4}\cdot\frac{8p^8}{4} - \text{etc.}$$

[4]In all the logarithmic calculus which follow, let's designate constantly with the letter m, the known number 0.43429, etc., of which the logarithm is 9.63758 43113 00537 and with the letter M its inverse 2.30258, etc., of which the logarithm is 0.36221 56866 99463.

[5]($\Downarrow$) m means $\log_{10} e = 0.43429\ 44819\ 03252, \cdots$, and M means $\log_e 10 = 2.30258\ 50929\ 94046\ \cdots$.

These logarithms are hyperbolics ; to convert them into vulgar logarithms, it needs to multiply the algebraic parties with m ; this is because putting $P = mp^2 - \frac{3}{2}mp^4 + \frac{10}{3}mp^6 -$ etc., then we will have $\log x$, namely, $\log c^\circ = \log p - P$ and $\log b^\circ = -\frac{1}{2}P$; therefore we will know at once $\log c^\circ$ and $\log b^\circ$.

§ **650**. Let's always start from the hypothesis which we wish to have the logarithm of these four functions, approached until the fourteenth decimal ; additionally, we can always suppose $c < \sin 45\circ$. Posed thus, let's start with the case which requires the longest calculus, that one where the module c is composed between $\sin 45^\circ$ and $\sin 26^\circ 34'$; then the echelon of the modules needs to be prolonged up to the terms $b^{\circ\circ\circ}$, $c^{\circ\circ\circ\circ}$, inclusively. The other cases will be susceptible of diverse simplification to order that the module c will turn smaller.

The values of F^1c, E^1c are found at first immediately with the formulae

$$F^1c = \frac{\pi}{2}\cdot K, \qquad K = \sqrt{\frac{1}{b}b^\circ b^{\circ\circ} b^{\circ\circ\circ}}, \quad E^1c = LF^1c, \qquad L = \frac{b}{b^{\circ 2}}\Big(1 - \frac{1}{2}c^{\circ 2}c^{\circ\circ} - \frac{1}{4}c^{\circ 2}c^{\circ\circ}c^{\circ\circ\circ}\Big).$$

To simplify the calculus of the coefficient L, I observe that the two terms $\frac{1}{2}c^{\circ 2}c^{\circ\circ\circ}(1+\frac{1}{2})$ can be reduce to only one ; because we have of a sufficientlt exact manner, $1 + \frac{1}{2}c^{\circ\circ\circ} = \sqrt{1 + c^{\circ\circ\circ}} = \sqrt{\frac{2\sqrt{c^{\circ\circ}}}{c^{\circ\circ}}}$; on the other side, $\frac{2\sqrt{c^{\circ\circ\circ}}}{c^{\circ\circ}} = \frac{b^{\circ\circ\circ}}{\sqrt{b^{\circ\circ}}}$. Therefore

$$L = \frac{b}{b^{\circ 2}}\Big(1 - \frac{1}{2}c^{\circ 2}c^{\circ\circ}\cdot\sqrt{\frac{b^{\circ\circ\circ}}{\sqrt[4]{b^{\circ\circ}}}}\Big).$$

Therefore, putting $r = \frac{1}{2}c^{\circ 2}c^{\circ\circ}\cdot\sqrt{\frac{b^{\circ\circ\circ}}{\sqrt[4]{b^{\circ\circ}}}}$, then we will have $E^1c = \frac{b}{b^{\circ 2}}F^1c(1-r)$. When c is given under the form $\sin\theta$, and that the angle θ as well $\frac{1}{2}\theta$, is found immediately in the Table, we have more simply $\frac{b}{b^{\circ 2}} = \cos^4\frac{1}{2}\theta$. All is reduced therefore to find $\log(1-r)$, this one, which will make with the formula $\log(1-r) = -mr - \frac{1}{2}mr^2 - \frac{1}{3}$, of which it will suffice to calculate three terms at best.

The primary term mr of this value can be calculated with the precision sufficient with the Tables to 10 decimals ; because it con't have at most than ten effective number : and when similarly there would be an error of one or two units on the tenth effective number, which will be at the rank of the fourteenth decimal, this error will be mixed with those of which the other logarithms are susceptible ; because in pushing the approximation up to the fourteenth decimal, we can't pretend only the fourteenth decimal will always be exact.

2.1.2. *An example.* *($c = \sin\theta$, $\sin 2\theta = \tan^2 15^\circ$.)* § **666**. In this example which is related to the tertiary case of the (art. 645), we don't give directly neither the value of c, nor that of b ; it needs to deduce them from the equation $\sin 2\theta = \tan^2 15^\circ$ or $2bc = \tan^2 15^\circ$. Here, the process which we will use for this object.

From the equation $\sin 2\theta = \tan^2\lambda$, we get $\cos 2\theta = \frac{\sqrt{\cos 2\lambda}}{\cos^2\lambda}$ [6] Let therefore, $A = \frac{\sqrt{\cos 30^\circ}}{\cos^2 15^\circ}$, then we will have successively c with the equation $c = \frac{\tan^2 15^\circ}{2b}$. Knowing the logarithms of c and b, we will find with the ordinary method, that of c°, b°, successively that of $c^{\circ\circ}$, this one, which suffices in the present case to complish the series of the methods.

§ **667**. The echelon of the modules being terminated, we will calculate as if follows the quantities F^1c, E^1c Let's now start calculating F^1b, it will make with the equation $F^1b = KMh$, where we have $h = \frac{1}{4}\log\frac{4}{c^{\circ\circ}}$. We see that between the logarithms calculated from F^1b and F^1c, the difference responds exactly to the logarithm of 3, this one which accords with the property of these functions.

We can again make see that the value found for F^1c satisfies exactly with the equation $F^1c =$

[6]($\Downarrow$) By using $\sin^2 2\theta + \cos^2 2\theta = 1$, we get $\cos 2\theta = \frac{\sqrt{cos^4 - \sin^4}}{cos^2\lambda}$, then $\underbrace{(\cos^2\lambda + \sin^2\lambda)}_{1}(cos^2\lambda - \sin^2\lambda) = (\cos^2\lambda - \sin^2\lambda) = 1 - 2\sin^2\lambda = 1 - 2(1-\cos^2\lambda) = 1 - 2 + 2\cos^2\lambda = -1 + 2\cos^2\lambda = \cos 2\lambda$. Finally we get $\cos 2\theta = \frac{\sqrt{\cos 2\lambda}}{\cos^2\lambda}$.

$\frac{2\cos 15^\circ}{\sqrt[4]{27}} F^1(\sin 45^\circ)$, given the (art. 155). value which accords perfectly with the result of the preceding calculation. There isn't more than to calculate the log of $E^1 b$; for this, let's follow the formula of the (art. 655). The values which we are going to find for $E^1 c,\ E^1 b$ can be verified with the formulae of the (art. 158) ; or at once, with the formula $E^1 b = 2E^1 c - 2F^1(\sin 45^\circ)$.

§ **669**. To find the function $E^1 c$, we have he formula reduced $E^1 c = \frac{b}{b^{\circ 2}} F^1 c(1-r)$, namely, simply $E^1 c = \frac{b}{b^{\circ 2}} F^1 c$,

$b \cdots$	9.99999 97172 733 14
$1 : b^{\circ 2} \cdots$	46
$F^1 c \cdots$	0.19612 00183 934 92
+)................................	
$E^1 c \cdots$	0.19611 97356 668 52.

[7] The function $F^1 b$ will be calculated with the formulae $h = \frac{1}{4} \log \frac{4}{c^{\circ\circ}}, \quad F^1 b = hM\sqrt{\frac{b^\circ}{b}}$.

$4 \cdots$	0.60205 99913 27962	$h \cdots$	0.54958 60704 10184
$c^{\circ\circ} \cdots$	6.42304 49983 30089	$M \cdots$	0.36221 56886 99463
$-)$..............................			
$4h =$	14.17901 49929 97873	$\sqrt{\frac{b^\circ}{b}} \cdots$	1413 63331
$h =$	3.54475 37482 49468	+).............................	
		$F^1 b \cdots$	0.91180 19004 72978
		$F^1 c \cdots$	0.19612 00183 93492
		$diff \cdots$	0.71568 18820 79486.

The difference of these two logarithms responds to very near to $\sqrt{27}$, and in effect we need to have exactly $F^1 b = 3\sqrt{3} F^1 c$. [8]

The value of $E^1 c,\ E^1 b$ will be able to be verified with the formulae of the (art. 169)

$$E^1 c = \left(\frac{1}{2} + \frac{2n-1}{2\sqrt{3}}\right) F^1 c + \frac{\pi}{4F^1 b}, \qquad E^1 b = \left(\frac{1}{2} + \frac{1-2n}{2\sqrt{3}}\right) F^1 b + \frac{\pi}{4F^1 c}.$$

and the results will accord also exactly with that we can desire it.

§ **685**. It is important in general to find the logarithms of the functions $F^1 b,\ E^1 b$, when b differs few from the unit or when its complement c is the sine of an angle of a small number of degrees. In this case, we will find easily, with the interpolations, the complementary functions $F^1 c,\ E^1 c$, and this is with the mean of $F^1 c$ which must determine $F^1 b,\ E^1 b$.

For this I observe at first that in the case of which let's occupy, we would be able to suppose $b^{\circ\circ} = 1$; however, let's contain to suppose $b^{\circ\circ\circ} = 1$, in order that the solution is applied to a greater number of cases ; then the general formulae give (art. 654).

$$K = \sqrt{\frac{b^\circ\ b^{\circ\circ}}{b}}, \quad F^1 c = \frac{1}{2}\pi K, \qquad F^1 B = \frac{KM}{8} \log \frac{4}{c^{\circ\circ\circ}}.$$

It needs hence to seek if we can explain $F^1 b$ with the only data $b,\ c,\ F^1 c$, with having help at the auxiliaries $b^\circ,\ b^{\circ\circ}, c^{\circ\circ\circ}$.

At first, K is known with the value $K = \frac{F^1 c}{\frac{1}{2}\pi}$. Let successively $c^\circ = x, \quad c^{\circ\circ} = y$; from the

[7](⇓) This verifies $\log E^1 c = \log b + \log \frac{1}{b^{\circ 2}} + \log F^1 c$.

[8](⇓) $\frac{F^1 b}{F^1 c} = 3\sqrt{3} = \sqrt{27}$, namely, $\log \sqrt{27} = 0.71568\ 1882 \cdots$. $\log \frac{F^1 b}{F^1 c} = \log F^1 b - \log F^1 c$, which is expressed as $diff$ in the bottom of the above tableau.

equations $bK^2 = b^\circ b^{\circ\circ}$, $\quad cb^\circ = 2\sqrt{bc^\circ}$, $\quad c^\circ b^{\circ\circ} = 2\sqrt{b^\circ\ c^{\circ\circ}}$, $\quad c^{\circ\circ} = 2\sqrt{b^{\circ\circ}\ c^{\circ\circ\circ}}$, we will deduce

$$b^\circ = \frac{2\sqrt{bx}}{c}, \quad b^{\circ\circ} = \frac{bK^2}{b^\circ} = \frac{1}{2}cK^2\sqrt{\frac{b}{x}}, \quad c^\circ b^{\circ\circ} = \frac{1}{2}K^2c\sqrt{bx} = 2\sqrt{b^\circ y}.$$

This last being squared gives $K^4c^2bx = 16b^\circ y$;[9] squaring again and substituting the value of b°, then we will have $K^8c^4b^2x^2 = y^2 \cdot \frac{4bx}{c^2}$; hence $y^2 = \frac{K^8c^6}{4^3}bx$. This equation doesn't suffice to determine x and y ; however, we have additionally, $b^{\circ\circ} = (1-y^2)^{\frac{1}{2}} = \frac{K^2c}{2}\sqrt{\frac{b}{x}}$; from here, we get

$$x = \frac{\frac{1}{4}bK^4c^2}{1-y^2} = \frac{1}{4}bK^4c^2(b^{\circ\circ})^{-2}, \qquad y = \frac{K^6c^4b}{2^6}(b^{\circ\circ})^{-1}.$$

Let $K^2b = \alpha^4$, then this last equation will give $\frac{4}{y} = \left(\frac{4}{cK\alpha}\right)^4 b^{\circ\circ}$; however, $\frac{4}{c^{\circ\circ\circ}} = \left(\frac{4}{c^{\circ\circ}}\right)^2 b^{\circ\circ} = \left(\frac{4}{y}\right)^2 b^{\circ\circ} = \left(\frac{4}{cK\alpha}\right)^4 \left(b^{\circ\circ}\right)^3$; therefore $F^1b = MK\log\left[\frac{4}{cK\alpha}\left(b^{\circ\circ}\right)^{\frac{3}{8}}\right]$. Let $\beta = \left(\frac{1}{b^{\circ\circ}}\right)^{\frac{3}{8}}$, then we will have finally

$$F^1b = MK\log\left(\frac{4}{cK\alpha\beta}\right)\cdots\begin{cases}\alpha = \sqrt[4]{K^2b}, \\ \log\beta = \frac{3}{8}\log\frac{1}{b^{\circ\circ}} = \frac{3}{4}M\Big(\log\frac{1}{\alpha}\Big)^2\end{cases}.$$

Therefore we see that in the calculation of $\log F^1b$, it enters only the quantities $b,\ c,\ K$, of which we have the logarithms, in order that we avoid therefore the direct interpolation for F^1b, which is reduced to the interpolation of F^1c which hasn't difficulty.

§ **686**. To judge the exactitude of this formula, let's take $c = \sin 15^\circ$, and let's give to $\log K$ the exact value up to fourteen decimals, which we find with the direct calculation, and additionally, for the Table to give immediately. We will have therefore the givens

$c\cdots 9.41299\ 62305\ 6934, \qquad b\cdots 9.98494\ 37781\ 0270, \qquad K\cdots 0.00749\ 54886\ 8247.$

By means of these data, the calculation of $h = \frac{1}{8}\log\frac{4}{c^{\circ\circ\circ}}$ will be made as it follows :

$4\cdots$	0.60205 9991 3 2796	$\sqrt{b}\cdots$	9.99247 18890 5135
$c\cdots$	9.51299 6230 5 6934	$K\cdots$	749 54886 8247
	−)............................	+)	....
$\frac{4}{c}\cdots$	1.18906 3760 7 5862	$\alpha^2\cdots$	9.99996 73777 3382
$K\cdots$	0.00749 5488 6 8247	$\alpha\cdots$	9.99998 36888 6691
	−)............................	$\log\frac{1}{\alpha} =$	0.00001 63111 3309= p
$\frac{4}{cK}\cdots$	1.18156 8272 0 7615	$\log\beta = \frac{3}{4}Mp^2$	
$\alpha\cdots$	9.99998 3588 8 6691	$p\cdots$	5.21248 413
	−)............................	$p^2\cdots$	0.42496 826
$\frac{4}{cK\alpha}\cdots$	1.18158 4583 2 0924	$\frac{3}{4}M\cdots$	0.23727 695
		+)	....
$\beta\cdots$	4 5946	$l\beta\cdots$	0.66224 52
	−)............................		
$h\cdots$	1.18158 45827 4978.		

[9](⇓) By squaring both sides of the expression : $\frac{1}{2}K^2c\sqrt{bx} = 2\sqrt{b^\circ y}$, we get this equation.

[10] This value of h accords exactly with that which would give $\frac{1}{8}\log\frac{4}{c^{\circ\circ\circ}}$, calculated with the direct method [11], up to the fifteenth decimal. Therefore in substituting in the formula $F^1b = KMh$, then we will have similarly an exact value of $\log F^1b$, up to the fifteenth decimal, and which will satisfy with the equation $F^1b = \sqrt{3}.F^1c$, explaining a particular property of this functions. [12]

3. Poisson's application of Legendre's elliptic function and his Table.

3.1. **The capillarity action. Preface**. We will find, in the following chapters, the applications of these general equations to the equilibrium of the liquid in the tubes of a very small diameter and to the other analogous question, and we will be capable to remark the usage which I have made the elliptic tables by Mr. Legendre, for the rigorous solution of problems which we couldn't have solved in any other way, without this aid, except for approximation.

§ **87**. I designate with h the value of z which responds to the point C ; by reason of $\frac{dz}{dx} = 0$ in this point, we will have $a^2 = b - h^2$; and in eliminating b of the equation (6) :

$$(1)_6 \qquad \frac{a^2}{\left(1+\frac{d^2z}{dx^2}\right)^{\frac{1}{2}}} = b - z^2, \tag{6}$$

the right hand-side of the equation (6) will turn into $a^2 + h^2 - z^2$. The radical being a positive quantity, it needs that z^2 weren't greater than $a^2 + h^2$; and, by reason of that the left hand-side of the equation is less than a^2, it needs that z^2 were not smaller than h^2. Then, we see already that without consideration of the sign, the variable z is composed between the limits h and $\sqrt{a^2 + h^2}$; it will be positive or negative, according as the curve will turn its concavity or its convexity with upward.

We get from this equation

$$dx = \frac{(a^2 + h^2 - z^2)dz}{\sqrt{(z^2 - h^2)\,(h^2 + 2a^2 - z^2)}}.$$

I will consider separately the two parts of the curve which arrives at the point C ; in each of them, the variable x will be regarded as positive and regarded from this point ; and for that it cross from this point up to each edge of the curve, I will suppose the radical of the same sign with dz.

Posed thus, to explain x in elliptic function, I put $z^2 = \frac{(h^2+2a^2)h^2}{h^2+2a^2\cos\varphi}$; from here we get

$$\tan^2\varphi = \frac{(h^2 + 2a^2)(z^2 - h^2)}{h^2(h^2 + 2a^2 - z^2)};$$

and the variable z^2 is neither less than h^2, nor greater than $h^2 + a^2$, this value of $\tan^2\varphi$ will be positive ; this one suffices for that φ were a real angle. The expression of dx will turn

$$dx = \frac{(a^2 + h^2)d\varphi}{\sqrt{h^2 + 2a^2\cos^2\varphi}} - \frac{\left(h^2 + 2a^2\right)h^2 d\varphi}{\left(h^2 + a^2\cos^2\varphi\right)^{\frac{3}{2}}},$$

consequently, this one is the same thing,

$$dx = \frac{(2 - c^2)a}{c\sqrt{2}}\,\frac{d\varphi}{\sqrt{1 - c^2\sin^2\varphi}} - \frac{2(1 - c^2)a}{c\sqrt{2}}\,\frac{d\varphi}{(1 - c^2\sin^2\varphi)^{\frac{3}{2}}},$$

[10](⇓) $l\beta = \log\beta = \frac{3}{4}Mp^2$, where, $Mp^2 = M\log\frac{1}{\alpha}$. $\log h = \log\frac{4}{cK\alpha\beta} = \log\frac{4}{cK\alpha} - \log\beta$, $l\beta$ means $\log_{10}\beta = p^2 + \frac{3}{4}M = \log_{10} 4.5946 = 0.6622452\cdots$.

[11](⇓) cf. the (art. 664).

[12](⇓) cf. the (art. 83)

in designating with c a quantity positive, less than the unit, and given with the equation $c^2 = \frac{2a^2}{2a^2+h^2}$. Additionally, we have identically

$$d\Big(\frac{\sin\varphi\cos\varphi}{\sqrt{1-c^2\sin^2\varphi}}\Big) = \frac{1}{c^2}\sqrt{1-c^2\sin^2\varphi}d\varphi - \frac{(1-c^2)}{c^2}\ \frac{d\varphi}{(1-c^2\sin^2\varphi)^{\frac{3}{2}}};$$

from here, it results

$$dx = \frac{(2-c^2)a}{c\sqrt{2}}\ \frac{d\varphi}{\sqrt{1-c^2\sin^2\varphi}} - \frac{2a}{c\sqrt{2}}\ \sqrt{1-c^2\sin^2\varphi}d\varphi + ac\sqrt{2}d\Big(\frac{\sin\varphi\cos\varphi}{\sqrt{1-c^2\sin^2\varphi}}\Big)$$

Owing to the notation known of Mr. Legendre, we have also

$$\int\frac{d\varphi}{\sqrt{1-c^2\sin^2\varphi}} = F(c,\varphi), \qquad \int\sqrt{1-c^2\sin^2\varphi}d\varphi = E(c,\varphi)\ ;$$

the integrals starting with the variable φ. In integrating, we will have then

$$(3)_6 \qquad \frac{x\sqrt{2}}{a} = \frac{2-c^2}{c}F(c,\varphi) - \frac{2}{c}E(c,\varphi) + \frac{c\sin 2\varphi}{\sqrt{1-c^2\sin^2\varphi}} \tag{7}$$

We don't add the constant arbitrary, because that x is null at the point C, for which we have $z = h$, this one, which gives $\varphi = 0$ and makes evaporate the right hand-side of this equation. We will have at the same time

$$(4)_6 \qquad z^2 = \frac{2a^2(1-c^2)}{\sqrt{1-c^2\sin^2\varphi}} \tag{8}$$

and these equations (7) and (8) make known the x and z of each of the points of the curve, the functions of a third variable φ, when we will have determined the module c.

Consequently, if we put $\frac{\frac{dz}{dx}}{\left(1+\left(\frac{dz}{dx}\right)^2\right)^{\frac{1}{2}}} = -\cos\omega$, where ω will be the angle which is given at the two extremities of the layer, and which depend, at each of these points, on the material of corps terminated with the vertical plane, and on that of liquid. In designating with k the value of z which responds to the one of these two points, and eliminating $\frac{dz}{dx}$ in the equation (**??**) and the precedent, it turns into $k = h^2 + a^2(1-\sin\ \omega)$; in regarding to the value of c^2, we will have then

$$(5)_6 \qquad h^2 = \frac{2a^2(1-c^2)}{c^2}, \qquad k^2 = \frac{a^2}{c^2}(2-c^2-c^2 sin\ \omega)\ ; \tag{9}$$

and if we call θ the value of φ which responds to $z = k$, it will result

$$(6)_6 \qquad \tan^2\theta = \frac{1-\sin\omega}{(1+\sin\omega)(1-c^2)}. \tag{10}$$

Let α be the value corresponding to x, namely, the distance from the point C to the one of two vertical planes ; we will have

$$(7)_6 \qquad \frac{x\sqrt{2}}{a} = \frac{2-c^2}{c}F(c,\theta) - \frac{2}{c}E(c,\theta) + \frac{c\sin 2\theta}{\sqrt{1-c^2\sin^2\theta}} \tag{11}$$

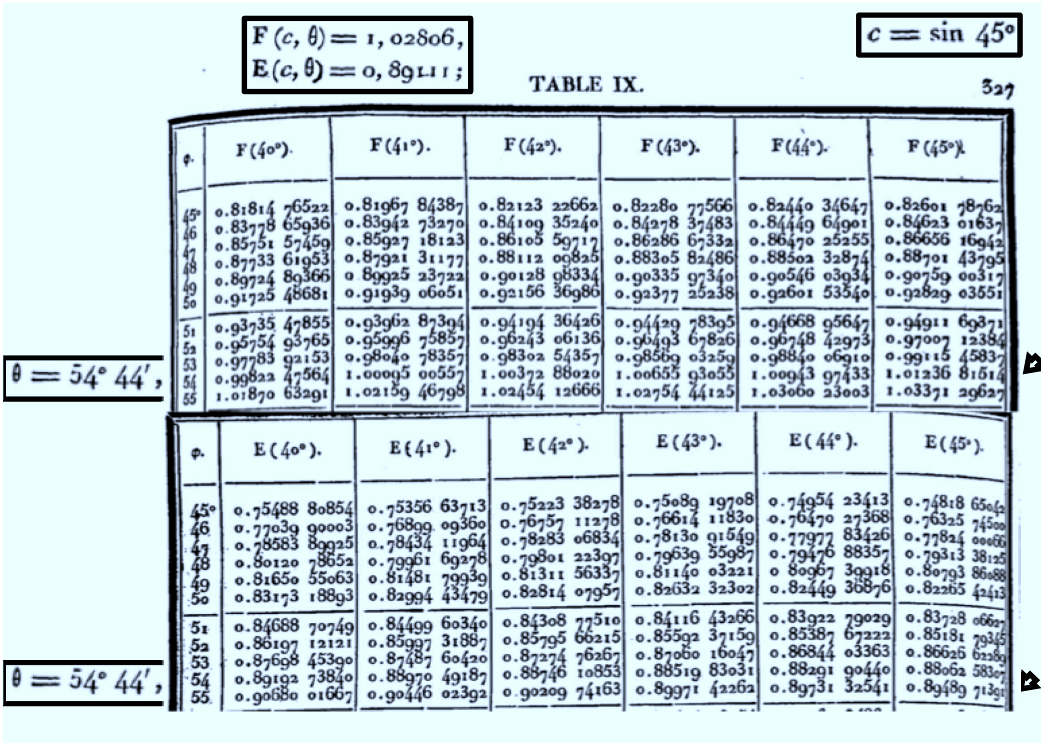
If we designate with α' and ω' the distance and the angle relative to the other vertical plane, and with θ' this one turns into θ, when we put ω' instead of ω, we will have a second equation which will be deduced from the precedent, in changing α and θ, with α' and θ'. I add these two equations, and I put $\alpha + \alpha' = \delta$, so that δ were the distance composed in the two vertical planes

; it turns

$$(8)_6 \qquad \frac{x\sqrt{2}}{a} = \frac{2-c^2}{c}\Big[F(c,\theta)+F(c,\theta')\Big]-\frac{2}{c}\Big[E(c,\theta)+E(c,\theta')\Big] + \frac{c\sin 2\theta}{\sqrt{1-c^2\sin^2\theta}}+\frac{c\sin 2\theta'}{\sqrt{1-c^2\sin^2\theta'}} \tag{12}$$

for the equation which will serve to determine c.

§ **88**. When we will have $\omega=\omega'$, the distances α and α' will be equal between them and to $\frac{1}{2}\delta$. If these angles are, in additionally, zero or π, we will have simply $\cos\theta=\sqrt{1-c^2}$. To consider with relation to c, the equation (11) which is transcendental, it would need to give to c a series of values ascending with the very small differences, from $c=0$ to $c=1$; let calculate by means of elliptic tables by Mr. Legendre, the value corresponding to the right hand-side of this equation[13] ; and let form then a table of the values of $\frac{\alpha\sqrt{2}}{a}$, relative to all these value of c : these being, when the distance δ or $s\alpha$, and constant a, and in consequence, the quantity $\frac{\alpha\sqrt{2}}{a}$ would be given, we would seek in this table, the value corresponding to c. But, the problem will be moreover simple if we give the elevation h of this point C and the constant a, and if we demand how long it must exist that the distance 2α composed between two planes. Let suppose, for example, which we musty have $h^2=2a^2$; it will result $c=\frac{1}{\sqrt{2}}$, $\cot\theta=\frac{1}{\sqrt{2}}$, $\theta=54°44'$, [14] and the equation (11) will turn $\frac{\delta}{h}=\frac{3}{\sqrt{2}}F(c,\theta)-2\sqrt{2}E(c,\theta)+\sqrt{\frac{2}{3}}$. For these values of c and θ, the tables by Mr. Legendre give $F(c,\theta)=1.02806$, $E(c,\theta)=0.89111$; from here, we conclude $\frac{\delta}{h}=0.4776$, for the ratio of the distance from the two planes to the smallest ordinate of the curve.



$\boxed{F(c,\theta) = 1,02806, \; E(c,\theta) = 0,89111;}$ $\boxed{c = \sin 45°}$

TABLE IX. 327

φ.	F(40°).	F(41°).	F(42°).	F(43°).	F(44°).	F(45°).
45°	0.81814 76522	0.81967 84387	0.82123 22662	0.82280 77566	0.82440 34647	0.82601 78762
46	0.83778 65936	0.83942 73270	0.84109 35240	0.84278 37483	0.84449 64901	0.84623 01637
47	0.85751 57459	0.85927 18123	0.86105 59717	0.86286 67332	0.86470 25255	0.86656 16942
48	0.87733 61953	0.87921 31177	0.88112 09825	0.88305 82486	0.88502 32874	0.88701 43795
49	0.89724 89366	0.89925 23722	0.90128 98334	0.90335 97340	0.90546 03934	0.90759 00317
50	0.91725 48681	0.91939 06051	0.92156 36986	0.92377 25238	0.92601 53540	0.92829 03551
51	0.93735 47855	0.93962 87394	0.94194 36426	0.94429 78395	0.94668 95647	0.94911 69371
52	0.95754 93765	0.95996 75857	0.96243 06136	0.96493 67826	0.96748 42973	0.97007 12384
53	0.97783 92153	0.98040 78357	0.98302 54357	0.98569 03259	0.98840 06910	0.99115 45837
54	0.99822 47564	1.00095 00557	1.00372 88020	1.00655 93055	1.00943 97433	1.01236 81514
55	1.01870 63291	1.02159 46798	1.02454 12666	1.02754 44125	1.03060 23003	1.03371 29627

φ.	E(40°).	E(41°).	E(42°).	E(43°).	E(44°).	E(45°).
45°	0.75488 80854	0.75356 63713	0.75223 38278	0.75089 19708	0.74954 23413	0.74818 65042
46	0.77039 90003	0.76899 09360	0.76757 11278	0.76614 11830	0.76470 27368	0.76325 74500
47	0.78583 89925	0.78434 11964	0.78283 06834	0.78130 91549	0.77977 83426	0.77824 00066
48	0.80120 78652	0.79961 69278	0.79801 22397	0.79639 55987	0.79476 88357	0.79313 38125
49	0.81650 55063	0.81481 79939	0.81311 56337	0.81140 03221	0.80967 39918	0.80793 86088
50	0.83173 18893	0.82994 43479	0.82814 07957	0.82632 32302	0.82449 36876	0.82265 42413
51	0.84688 70749	0.84499 60340	0.84308 77510	0.84116 43266	0.83922 79029	0.83728 06627
52	0.86197 12121	0.85997 31887	0.85795 66215	0.85592 37159	0.85387 67222	0.85181 79345
53	0.87698 45390	0.87487 60420	0.87274 76267	0.87060 16047	0.86844 03363	0.86626 62289
54	0.89192 73840	0.88970 49187	0.88746 10853	0.88519 83031	0.88291 90440	0.88062 58307
55	0.90680 01667	0.90446 02392	0.90209 74163	0.89971 42262	0.89731 32541	0.89489 71391

$\boxed{\theta = 54°\,44',}$ $\boxed{\theta = 54°\,44',}$

fig.1 Legendre's table of the elliptic functions.

13(⇓) cf. This means the equation (11).

14(⇓) We show the fig.1 of $F(c,\theta)$ and $E(c,\theta)$ of the Legendre's table (1825) in a few part of the relating page. ($c=\frac{1}{\sqrt{2}}$, $\theta=54°44'$) cf. [1, vol.2, p.327].

Owing to the equation (9), the greatest ordinate is $k = h\sqrt{\frac{3}{2}}$; the average ordinate is then $z = \frac{1}{2}h\left(1 + h\sqrt{\frac{3}{2}}\right)$; and the value corresponding to φ will be $\varphi = 38°16'30''$. For this value of φ and $c = \sin 45°$, we see, in the tables of Mr. Legendre, $F(c,\theta) = 0.69500$, $E(c,\theta) = 0.64437$; and the equation (7) gives successively $x = h(0.2061)$. In comparing this value of x with that of α, consequently of $\frac{1}{2}\delta$, $x = \alpha(0.8628)$; so that, in this example, the average ordinates are very more approached from the planes vertical than the point C ; this one give at the few of curvature of the liquid near this point.

3.2. **Motion of the heat of the interior and on the surface of the Earth.**

We define $\varphi(t)$ as follows :

$$(13)_{PS12} \qquad \varphi(t) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(t')\ dt' + \frac{1}{\pi}\sum_{i=0}^{\infty}\Big[\int_0^{2\pi} \cos\ i\ (t-t')\ \varphi(t')\ dt'\Big]. \qquad (13)$$

§ **215**. (To apply the formula (13) to the function V.) Let ψ_i' be, which is the one ψ_i turns out when we change v in another variable v' ; in applying the formula (13) to the function V, we will have, according to the preceding value of this quantity,

$$\begin{aligned}(15)_H\ V \quad &= \quad \frac{\sin\ \mu\ \sin\ \gamma}{2\pi}\int_0^{2\pi} \psi_i'\ \sin\ v'\ dv' + \frac{\cos\ \mu}{2\pi}\int_0^{2\pi}\sqrt{1-\sin^2\ \gamma\ \sin^2\ v'}\ \sin\ \psi_i'\ dv' \\ &+ \quad \frac{\sin\ \mu\ \sin\ \gamma}{2\pi}\sum\Big[\int_0^{2\pi} \psi_i'\ \cos\ i\ (v-v')\ \sin\ v'\ dv'\Big] \\ &+ \quad \frac{\cos\ \mu}{2\pi}\sum\Big[\int_0^{2\pi}\sqrt{1-\sin^2\ \gamma\ \sin^2\ v}\ \cos\ i\ (v-v')\ \sin\ \psi_i'\ dv'\Big]. \qquad (14)\end{aligned}$$

Supposing that the latitude μ of the point O were northern ; and consider first of all, the case where we have $\mu < 90° - \gamma$, so that ψ_i' were a continuous function of v'.

In integration by part, and observing that the values of ψ_i' which responds to two limits $v' = 0$ and $v' = 2\pi$, were equal, we will have $\int_0^{2\pi} \psi_i'\ \sin\ v'\ dv' = \int_0^{2\pi} \frac{d\ \psi_i'}{d\ v'}\ \cos\ v'\ dv'$; by suitable means, we will have

$$\begin{aligned}&\int_0^{2\pi} \psi_i'\ \cos\ (v-v')\ \sin\ v'\ dv' \\ &= \quad \frac{1}{2}\pi^2\ \sin\ v + \frac{1}{2}\sin\ \mu\ \sin\ \gamma\ \int_0^{2\pi}\left[\frac{1}{2}\ \cos\ (v-2v') - v'\ \sin\ v\right]\frac{\cos\ v'\ dv'}{\Delta},\end{aligned}$$

$$\begin{aligned}&\int_0^{2\pi} \psi_i'\ \cos\ i\ (v-v')\ \sin\ v'\ dv' \\ &= \quad \frac{1}{2}\sin\ \mu\ \sin\ \gamma\ \int_0^{2\pi}\left[\frac{\cos\ (iv-iv'-v')}{i+1} - \frac{\cos\ (iv-iv'+v')}{i-1}\right]\frac{\cos\ v'\ dv'}{\Delta},\end{aligned}$$

where, we put, for abridgement,

$$\Delta \equiv \left(1-\sin^2\ \gamma\ \sin^2\ v'\right)\sqrt{\cos^2\ \mu - \sin^2\ \gamma\ sin^2\ v'}.$$

I substitute these values and this $\sin\psi_i'$ in the formula (14). The integrals relative to v' of quantities which are for factor, the sines of a multiple even or odd, or the cosines of a multiple odd of this angle, is deleted as being composed, between the limits $v' = 0$ and $v' = 2\pi$, of the elements equal to two by two and the contrary signs. Relatively to integral which includes v' under the sign $\int$, on the outside of the sines and cosines, we have

$$\int_0^{2\pi}\frac{v'\ \cos\ v'\ dv'}{\Delta} = \int_0^{\pi}\frac{v'\ \cos\ v'\ dv'}{\Delta} - \int_0^{\pi}\frac{(v'+\pi)\ \cos\ v'\ dv'}{\Delta} = -\pi\int_0^{\pi}\frac{\cos\ v'\ dv'}{\Delta},$$

by reason of cos $(v+\pi) = -\cos\ v'$; and this last integral is zero, as being also composed of the elements which is deduced two by two. in respect to the integrals of the quantities which are for factor the cosines of an even multiple of v', we will be able to reduce their limits to $v' = 0$ and $v' = \frac{1}{2}\ \pi$, quadruplicate their values. Posed thus, we find

$$(16)_H \quad V = \frac{1}{2}\ \pi\ \sin\ \mu\ \sin\ \gamma\ \sin\ v + Q + Q_\imath\ \cos\ 2\ v + Q_{\imath\imath}\ \cos\ 4\ v + Q_{\imath\imath\imath}\ \cos\ 6\ v + \text{etc.}\ ; (15)$$

where, Q, $Q_\imath$, $Q_{\imath\imath}$, $Q_{\imath\imath\imath}$, etc., being the quantities of independent of v. We have in particular,

$$Q = \frac{2}{\pi} \int_0^{\frac{1}{2}\ \pi} \Big(\sqrt{\cos^2\ \mu - \sin^2\ \gamma\ sin^2\ v'} + \frac{\sin^2\ \mu\ \sin^2\ \gamma\ \cos^2\ v}{\Delta} \Big)\ dv',$$

and for a certain index i, different from zero,

$$\begin{aligned} Q_\imath &= \frac{4}{\pi} \int_0^{\frac{1}{2}\ \pi} \Big[\sqrt{\cos^2\ \mu - \sin^2\ \gamma\ sin^2\ v'}\ \cos\ 2\ i\ v' \\ &- \Big(\cos\ 2\ i\ v'\ \cos\ v' + 2\ i\ \sin\ 2\ i\ v'\ \sin\ v' \Big) \frac{\sin^2\ \mu\ \sin^2\ \gamma\ \cos\ v'}{(4\ i^2 - 1)\ \Delta} \Big]\ dv'. \end{aligned}$$

§ **216**. (Explanation with elliptic functions) All these integrals Q, $Q_\imath$, $Q_{\imath\imath}$, $Q_{\imath\imath\imath}$, etc., are explained with elliptic functions ; this one will permit additionally to calculate easily the numerical values.

For the primary, we have

$$\begin{aligned} Q &= \frac{2}{\pi} \Big(\int_0^{\frac{1}{2}\ \pi} \sqrt{\cos^2\ \mu - \sin^2\ \gamma\ \sin^2\ v'}\ dv' + \int_0^{\frac{1}{2}\ \pi} \frac{\sin^2\ \mu\ dv'}{\sqrt{\cos^2\ \mu - \sin^2\ \gamma\ \sin^2\ v'}} \\ &- \int_0^{\frac{1}{2}\ \pi} \frac{\sin^2\ \mu\ \cos^2\ \gamma\ dv'}{\big(1 - \sin^2\ \gamma\ \sin^2\ v'\big) \sqrt{\cos^2\ \mu - \sin^2\ \gamma\ \sin^2\ v'}} \Big)\ ; \end{aligned}$$

this one shows that Q will depend on a complete elliptic function, of each of three kinds, having a same module $\frac{\sin\ \gamma}{\cos\ \mu}$; quantity is little one than the unit with hypothesis. In putting, $\frac{\sin\ \gamma}{\cos\ \mu} = c$, $\ -\sin^2\ \gamma = n$, and using the notation of Legendre [15], we will have

$$Q = \frac{2}{\pi}\ E^1(c)\ \cos\ \mu + \frac{2}{\pi} \Big[F^1(c) - \Pi^1(c,\ n)\ \cos^2\ \gamma \Big]\ \sin\ \mu\ \tan\ \mu.$$

However, we see that the complete functions of third kind is explained by means of the functions complete and incomplete of primary kind and of the same module ; in putting $n = -c^2\ \sin^2\ \varphi$, from the above, we get $\varphi = \frac{1}{2}\ \pi - \mu$, the angle φ will be the amplitude of the incomplete function, and we will have [a]

$$(17)_H \quad \Pi^1(c,\ n) = F^1(c) + \frac{\tan\ \varphi}{\sqrt{1 - c^2\ \sin^2\ \varphi}} \Big[F^1\ (c)\ E(c,\ \varphi) - E^1\ (c)\ F(c, \varphi) \Big]\ ; \quad (16)$$

[a] sic. *Traité des Fonctions elliptiques*, tome I, page 141.

in consequence, the value of Q will turn out finally

$$Q = \frac{2}{\pi} \Big[E^1(c)\ \cos\ \mu + F^1\ (c)\ \sin^2\ \gamma\ \sin\ \mu\ \tan\ \mu - \Big\{ F^1(c)\ E(c,\ \varphi) - E^1\ (c)\ F(c,\ \varphi) \Big\}\ \cos\ \gamma\ \ \sin\ \mu\ \Big].$$

If we put $i = 1$ in the value of $Q_\imath$, it comes into

$$Q_\imath = \frac{4}{\pi} \int_0^{\frac{1}{2}\ \pi} \sqrt{\cos^2\ \mu - \sin^2\ \gamma\ \sin^2\ v'}\ \cos\ 2v'\ dv' - \frac{4\ \sin^2\ \mu\ \sin^2\ \gamma}{3\ \pi} \int_0^{\frac{1}{2}\ \pi} (1 + 2\ \sin^2\ v')\ \frac{\cos^2\ v'}{\Delta}\ dv'.$$

[15] (⇓) Legendre [1].

Moreover, we have identically, according to this one, which Δ represents,

$$\frac{(1+2\ \sin^2\ v')\ \cos^2\ v'}{\Delta} = -\frac{(2+\sin^2\ \gamma)\ \cos^2\ \gamma}{\Delta\ \sin^2\ \gamma} - \frac{2\sqrt{\cos^2\ \mu - \sin^2\ \gamma\ \sin^2\ v'}}{\sin^2\ \gamma} + \frac{1+\cos^2\ \gamma + 2\ \cos^2\ \mu}{\sin^4\ \gamma\ \sqrt{\cos^2\ \mu - \sin^2\ \gamma\ \sin^2\ v'}}.$$

Being thus, we will have in eliminating $\Pi^1\ (c,\ n)$ by means of the equation (16), it results from here

$$Q_\imath = \frac{4}{3\ \pi\ \sin^2\ \gamma}\Big[\Big\{F^1(c)\ E(c,\ \varphi) - E^1\ (c)\ F(c,\ \varphi)\Big\}(2+\sin^2\ \gamma)\ \cos\ \gamma\ \sin\ \mu + (2-\sin^2\ \gamma)\ \cos\ \mu\ E^1\ (c) - (2\ \cos^2\ \gamma\ \cos^2\ \mu + \sin^4\ \gamma\ \sin\ \mu\ \tan\ \mu\ F^1\ (c)\Big].$$

The two primary integrals are obtained with the ordinary rules, and have for values

$$\frac{1}{2\ i}\int_0^{\frac{1}{2}\ \pi} \frac{\sin^2\ \gamma\ \sin\ 2\ i\ v'\ \sin\ v'}{\sqrt{\cos^2\ \mu - \sin^2\ \gamma\ sin^2\ v'}} = \cos\ \mu - \sqrt{\cos^2\ \mu - \sin^3\ \gamma}, \tag{17}$$

$$\int_0^{\frac{1}{2}\ \pi} \frac{\sin^2\ \gamma\ \sin\ v'\ \cos\ v'\ dv'}{\Delta} = \frac{1}{\sin\ \mu}\left[\frac{1}{2}\ \pi - \mu - \arccos\frac{\sin\ \mu}{\cos\ \gamma}\right]. \tag{18}$$

in elliptic function, the value of the third turns in virtue of the formula (16), into

$$\int_0^{\frac{1}{2}\ \pi} \frac{\sin^2\ \gamma\ \cos^2\ v'\ dv'}{\Delta} = \frac{\sin^2\ \gamma}{\cos\ \mu}\ F^1\ (c) - \frac{\cos\ \gamma}{\sin\ \mu}\ \Big[F^1\ (c)\ E(c,\ \varphi) - E^1\ F(c,\ \varphi)\Big]. \tag{19}$$

For all the indices i different from zero, we will have hence,

$$\begin{aligned} \frac{\pi}{4}Q_\imath &< \frac{1}{2\ i}\Big(\cos\ \mu - \sqrt{\cos^2\ \mu - \sin^2\ \gamma}\Big) \\ &+ \frac{2\ i\ \sin^2\ \mu}{4\ i^2-1}\ \left[\frac{1}{2}\ \pi - \mu - \arccos\frac{\sin\ \mu}{\cos\ \gamma}\right] \\ &+ \frac{\sin\ \mu}{4\ i^2-1}\ \Big[\tan\ \mu\ \sin^2\ \gamma\ F^1\ (c) - \cos\gamma\ \Big\{F^1\ (c)\ E\ (c,\ \varphi) - E^1\ (c)\ F\ (c,\ \varphi)\Big\}\Big]. \end{aligned} \tag{20}$$

At the equator, where it has $\mu = 0$, we will have

$$Q = \frac{2}{\pi}\ E^1\ (c), \qquad Q_\imath = \frac{4}{3\ \pi\ \sin^2\ \gamma}\ \Big[(1+\cos^2\ \gamma)\ E^1\ (c) - 2\ \cos^2\ \gamma\ F^1\ (c)\Big],$$

and generally, $Q_\imath < \frac{2\ (1-\cos\ \gamma)}{i\ \pi}$. If it should hold $\gamma = 0$, this limit of $Q_\imath$ would be zero ; it should need hence that $Q_\imath$ should have it also ; this one result, in effect, from the value of $Q_\imath$ of the preceding number, when we put $\mu = 0$ and $\gamma = 0$. The module c is $\sin\ \gamma$ in case of $\mu = 0$; in developing the elliptic functions contained in the preceding value of $Q_\imath$, in accordance with the powers of c^2 or of $\sin^2\ \gamma$, we have

$$E^1\ (c) = \frac{\pi}{2} - \frac{\pi}{8}\ \sin^2\ \gamma + \text{etc.}, \qquad F^1\ (c) = \frac{\pi}{2} + \frac{\pi}{8}\ \sin^2\ \gamma + \text{etc.}\ ;$$

this one reduces also to zero this value of $Q_\imath$ in the case $\gamma = 0$, and that of Q at the unit. However, we have really $c = \sin\ \gamma = \sin\ 23^\circ\ 28'$; the tables by Legendre [16] gives, in ordinary logarithm, $\log_{10}\ E^1\ (c) = 0.1779800$, $\log_{10}\ F^1\ (c) = 0.2146639$, and we deduce from here [17] $Q = 0.95910$, $Q_\imath = 0.04132$, $Q_\imath < \frac{1}{i}\ (0.05265)$.

[16](⇓) Legendre [1].

[17](⇓) According to our calculation, Q gives $\frac{2}{\pi}E^1(c)$, and $E^1(c) = \frac{\pi}{2} - \frac{\pi}{8}\sin^2\gamma$ +etc, then $Q = \frac{2}{\pi}\cdot\frac{\pi}{8}(4-\sin^2\gamma) = 0.9602596$.

If we take for μ the latitude of Paris, [a] we will have

$$\mu = 48^\circ\ 50', \qquad \gamma = 23^\circ\ 28', \qquad \varphi = 41^\circ\ 10', \qquad c = \frac{\sin\ \gamma}{\cos\ \mu} = \sin\ 37^\circ\ 14'\ ;$$

and according to the same tables,

$E^1\ (c) = 1.41513, \qquad F^1\ (c) = 1.75490, \qquad E\ (c,\ \varphi) = 0.69511, \qquad F\ (c) = 0.73514$;

from the above, we conclude $Q = 0.66662, \qquad Q_\imath = 0.00253.$

[a]($\Downarrow$) The latitude of the location of the today's Paris Observatory is $48^\circ\ 50'11.18''N$, showed with μ.

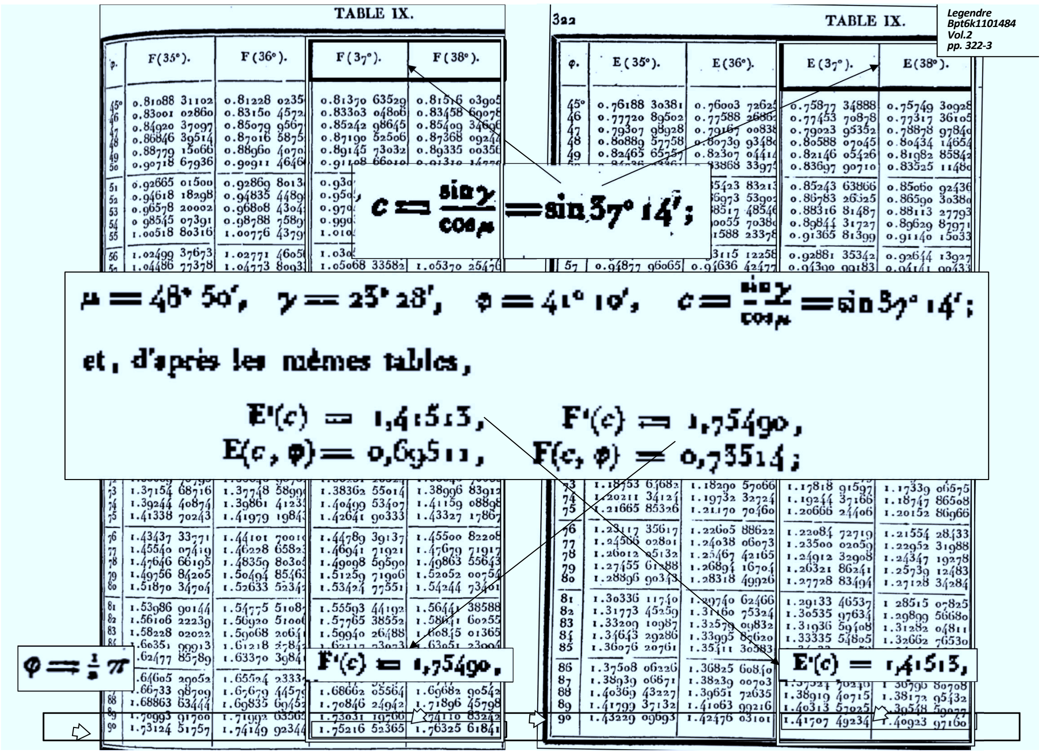


fig.2 Complete elliptic functions of Legendre's table

4. **Conclusions**

Legendre may be, we think, the only person in Poisson's all life, whom Poisson defeated in such academic arena in high esteem for the tremendous works by Legendre. Without his works, as Poisson says, his applications to the elliptic functions haven't put into practice.

References

[1] A.M. Legendre, *Traité des fonctions elliptiques et des intégrales eulériennes, avec des Tables pour faciliter le calcul numérique*, Paris. vol.1 1825, vol.2 1826, vol.3 1828. (vol.1) → http://gallica.bnf.fr/ark:/12148/bpt6k110147r (vol.2) → http://gallica.bnf.fr/ark:/12148/bpt6k1101484 (vol.3) → http://gallica.bnf.fr/ark:/12148/bpt6k110149h

[2] S.D.Poisson, *Nouvelle théorie de l'action capillaire*, Bachelier Pére et Fils, Paris, 1831. → http://gallica.bnf.fr/ark:/12148/bpt6k1103201

[3] S.D. Poisson, *Théorie mathématique de la chaleur*, Bachelier Pére et Fils, Paris, 1835. → http://www.e-rara.ch/doi/10.3931/e-rara-16666

Acknowledgments : This work was suppoted by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.