Adapted Sequence for Polyhedral Realization of Crystal Bases

Yuki Kanakubo (Sophia University, Faculty of Science and Technology)

1 Introduction

The crystals $B(\infty)$ $B(\lambda)$ are, roughly speaking, a basis of the subalgebra $U_q^-(\mathfrak{g})$ and irreducible integrable highest weight modules of the quantum group $U_q(\mathfrak{g})$ at $q=0$, where $\mathfrak{g}$ is a symmetrizable Kac-Moody Lie algebra with an index set $I=\{1,2,\cdots,n\}$, λ is an integral weight [6, 8]. Since then the theory of crystal bases has influenced many areas of mathematics and physics, e.g., algebraic combinatorics, statistical mechanics, cellular automaton, etc. In order to apply the theory of crystal bases to these areas, it is required to realize the crystal bases in suitable forms, like as tableaux realizations, path realizations, geometric realizations, etc. The *polyhedral realization* is one of descriptions of crystal bases.

In [12], the polyhedral realization for $B(\infty)$ has been introduced as an image of 'Kashiwara embedding' $\Psi_\iota : B(\infty) \hookrightarrow \mathbb{Z}^\infty$, where ι is an infinite sequence of entries in I and $\mathbb{Z}^\infty$ is an infinite $\mathbb{Z}$-lattice with certain crystal structure (see subsections 2.2, 2.3). In the same paper, some set of linear functions $\Xi_\iota \subset (\mathbb{Q}^\infty)^*$ associated with the sequence ι and the subset $\Sigma_\iota \subset \mathbb{Z}^\infty$ defined by

$$\Sigma_\iota = \{\mathbf{x} \in \mathbb{Z}^\infty | \varphi(\mathbf{x}) \geq 0, \quad \forall \varphi \in \Xi_\iota\}$$

are treated. Then under some condition on ι called 'positivity condition', we find that $\mathrm{Im}(\Psi_\iota) = \Sigma_\iota \cong B(\infty)$, which implies that the crystal $B(\infty)$ is realized as a polyhedral convex cone in $\mathbb{Z}^\infty$. In [11], the polyhedral realization for $B(\lambda)$ is introduced and an algebraic method to calculate it is found under the condition on the pair (ι, λ) called 'ample condition'.

To confirm the positivity condition, ample condition for a given ι, it is necessary to obtain the whole linear functions of Ξ_ι, which requires a lot of calculations. So far, in [2, 3, 11, 12] it has been shown that the specific sequence $\iota = (\cdots,2,1,n,\cdots,2,1,n\cdots,2,1)$ satisfies the positivity and ample condition for all simple Lie algebras and several affine Lie algebras.

In this article, we discuss the following two problems in the case $\mathfrak{g}$ is a classical Lie algebra:

(1) Find a sufficient condition of the positivity condition and ample condition.

(2) Find explicit forms of the polyhedral realizations.

We will give an answer for (1) in Theorem 3.7, 3.11. As for (2), we will give an explicit formula of the polyhedral realizations in terms of column tableaux in the case ι satisfies the sufficient condition of (1) (Theorem 3.6, 3.11 and Corollary 3.8, 3.12). This is a joint work with Toshiki Nakashima in Sophia university.

2 Crystal and its polyhedral realization

Let us recall the definition of *crystals* [7].

2.1 Notations

Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra over $\mathbb{Q}$ with a Cartan subalgebra $\mathfrak{t}$, a weight lattice $P \subset \mathfrak{t}^*$, the set of simple roots $\{\alpha_i : i \in I\} \subset \mathfrak{t}^*$, and the set of coroots $\{h_i : i \in I\} \subset \mathfrak{t}$, where $I = \{1, 2, \cdots, n\}$ is a finite index set. Let $\langle h, \lambda\rangle = \lambda(h)$ be the pairing between $\mathfrak{t}$ and $\mathfrak{t}^*$, and (α, β) be an inner product on $\mathfrak{t}^*$ such that $(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{\geq 0}$ and $\langle h_i, \lambda\rangle = \frac{2(\alpha_i,\lambda)}{(\alpha_i,\alpha_i)}$ for $\lambda \in \mathfrak{t}^*$ and $A := (\langle h_i, \alpha_j\rangle)_{i,j}$ be the associated generalized symmetrizable Cartan matrix. Let $P^* = \{h \in \mathfrak{t} : \langle h, P\rangle \subset \mathbb{Z}\}$ and $P_+ := \{\lambda \in P : \langle h_i, \lambda\rangle \in \mathbb{Z}_{\geq 0}\}$. The quantum algebra $U_q(\mathfrak{g})$ is the associative $\mathbb{Q}(q)$-algebra generated by e_i, f_i $(i \in I)$, and q^h $(h \in P^*)$ satisfying the usual relations. The algebra $U_q^-(\mathfrak{g})$ is the subalgebra of $U_q(\mathfrak{g})$ generated by the f_i $(i \in I)$.

For the irreducible highest weight module of $U_q(\mathfrak{g})$ with the highest weight $\lambda \in P_+$, we denote it by $V(\lambda)$ and its *crystal base* we denote $(L(\lambda), B(\lambda))$. Similarly, for the crystal base of the algebra $U_q^-(\mathfrak{g})$ we denote $(L(\infty), B(\infty))$ (see [5, 6]). Let $u_\infty \in B(\infty)$, $u_\lambda \in B(\lambda)$ be the highest weight vectors. For positive integers l and m with $l \leq m$, we set $[l, m] := \{l, l+1, \cdots, m-1, m\}$.

2.2 Crystals

Definition 2.1. [7] A **crystal** is a set $\mathcal{B}$ together with maps $\mathrm{wt} : \mathcal{B} \to P$, $\varepsilon_i, \varphi_i : \mathcal{B} \to \mathbb{Z} \cup \{-\infty\}$ and $\tilde{e}_i, \tilde{f}_i : \mathcal{B} \to \mathcal{B} \cup \{0\}$ $(i \in I)$ satisfying the following: For $b, b' \in \mathcal{B}$, $i, j \in I$,

(1) $\varphi_i(b) = \varepsilon_i(b) + \langle \mathrm{wt}(b), h_i\rangle$,

(2) $\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$ if $\tilde{e}_i(b) \in \mathcal{B}$, $\quad \mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i$ if $\tilde{f}_i(b) \in \mathcal{B}$,

(3) $\varepsilon_i(\tilde{e}_i(b)) = \varepsilon_i(b) - 1$, $\;\varphi_i(\tilde{e}_i(b)) = \varphi_i(b) + 1$ if $\tilde{e}_i(b) \in \mathcal{B}$,

(4) $\varepsilon_i(\tilde{f}_i(b)) = \varepsilon_i(b) + 1$, $\;\varphi_i(\tilde{f}_i(b)) = \varphi_i(b) - 1$ if $\tilde{f}_i(b) \in \mathcal{B}$,

(5) $\tilde{f}_i(b) = b'$ if and only if $b = \tilde{e}_i(b')$,

(6) if $\varphi_i(b) = -\infty$ then $\tilde{e}_i(b) = \tilde{f}_i(b) = 0$.

We call $\tilde{e}_i, \tilde{f}_i$ *Kashiwara operators*.

The crystal bases $B(\infty)$, $B(\lambda)$ are important crystals. Let us see another example of crystals.

Example 2.2. Let $R_\lambda := \{r_\lambda\}$ $(\lambda \in P)$ and

$$\mathrm{wt}(r_\lambda) = \lambda, \quad \varepsilon_i(r_\lambda) = -\lambda(h_i), \quad \varphi_i(r_\lambda) = 0, \quad \tilde{e}_i(r_\lambda) = \tilde{f}_i(r_\lambda) = 0.$$

Then R_λ is a crystal.

Example 2.3. For $i \in I$, we set

$$B_i := \{(m)_i | m \in \mathbb{Z}\}$$

and define ε_j, φ_j, wt, $\tilde{e}_j$ and $\tilde{f}_j$ $(j \in I)$ as follows:

- $\varepsilon_j((m)_i) = \varphi_j((m)_i) = -\infty$ $(j \neq i)$,
- $\mathrm{wt}((m)_i) = m\alpha_i$, $\varepsilon_i((m)_i) = -m$, $\varphi_i((m)_i) = m$,
- $\tilde{e}_j((m)_i) = \tilde{f}_j((m)_i) = 0$ $(j \neq i)$,

$$\cdots (-2)_i \xrightarrow{\tilde{e}_i} (-1)_i \xrightarrow{\tilde{e}_i} (0)_i \xrightarrow{\tilde{e}_i} (1)_i \xrightarrow{\tilde{e}_i} (2)_i \xrightarrow{\tilde{e}_i} \cdots$$

$$\cdots (-2)_i \xleftarrow{\tilde{f}_i} (-1)_i \xleftarrow{\tilde{f}_i} (0)_i \xleftarrow{\tilde{f}_i} (1)_i \xleftarrow{\tilde{f}_i} (2)_i \xleftarrow{\tilde{f}_i} \cdots.$$

Then B_i is a crystal.

Definition 2.4. The **tensor product** $\mathcal{B}_1 \otimes \mathcal{B}_2$ of crystals $\mathcal{B}_1$, $\mathcal{B}_2$ is defined to be the set $\mathcal{B}_1 \times \mathcal{B}_2$ whose crystal structure is defined as follows:

(1) $\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2)$,

(2) $\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, \mathrm{wt}(b_1)\rangle)$,

(3) $\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_2), \varphi_i(b_1) + \langle h_i, \mathrm{wt}(b_2)\rangle)$,

(4) $\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$

(5) $\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2), \end{cases}$

for $i \in I$.

Definition 2.5. A **strict morphism** $\psi : \mathcal{B}_1 \to \mathcal{B}_2$ of crystals $\mathcal{B}_1$, $\mathcal{B}_2$ is a map $\mathcal{B}_1 \bigsqcup \{0\} \to \mathcal{B}_2 \bigsqcup \{0\}$ such that $\psi(0) = 0$ and for $i \in I$, $b \in \mathcal{B}_1$,

(1) $\mathrm{wt}(\psi(b)) = \mathrm{wt}(b)$, $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$, $\varphi_i(\psi(b)) = \varphi_i(b)$ if $\psi(b) \neq 0$,

(2) $\psi(\tilde{e}_i(b)) = \tilde{e}_i \psi(b)$, $\psi(\tilde{f}_i(b)) = \tilde{f}_i \psi(b)$.

An injective strict morphism is said to be **strict embedding**.

We put

$$\mathbb{Z}^\infty := \{\mathbf{x} = (\cdots, x_4, x_3, x_2, x_1) | x_k \in \mathbb{Z},\ x_l = 0 (l \gg 0)\}$$

and fix an infinite sequence $\iota := (\cdots, i_3, i_2, i_1)$ of I such that $i_k \neq i_{k+1}$ $(k \in \mathbb{Z}_{>0})$ and $\sharp\{k \in \mathbb{Z}_{>0} | i_k = j\} = \infty$ (for any $j \in I$).

We define a crystal structure on $\mathbb{Z}^\infty$ associated with ι as follows: One define the linear functions σ_k $(k \in \mathbb{Z}_{>0})$ as

$$\sigma_k(\mathbf{x}) := x_k + \sum_{j>k} \langle h_{i_k}, \alpha_{i_j}\rangle x_j \qquad (k \in \mathbb{Z}_{\geq 1},\ \mathbf{x} \in \mathbb{Z}^\infty).$$

By $x_j = 0$ $(j \gg 0)$, the linear function $\sigma_k(\mathbf{x})$ is well-defined on $\mathbb{Z}^\infty$. We also get

$$\sigma_k(\mathbf{x}) = 0 \quad (k \gg 0). \tag{2.1}$$

Next, $\sigma^{(i)}(\mathbf{x}) := \max_{k \in \mathbb{Z}_{\geq 1};\ i_k = i} \sigma_k(\mathbf{x})$ $(i \in I)$. It follows from (2.1) that $\sigma^{(i)}(\mathbf{x}) \geq 0$.

Setting

$$M^{(i)} = M^{(i)}(\mathbf{x}) := \{k \in \mathbb{Z}_{\geq 1} | i_k = i,\ \sigma_k(\mathbf{x}) = \sigma^{(i)}(\mathbf{x})\},$$

by (2.1),

$$\max M^{(i)} < \infty \Leftrightarrow \sigma^{(i)}(\mathbf{x}) > 0.$$

Now we define $\tilde{f}_i : \mathbb{Z}^\infty \to \mathbb{Z}^\infty$ and $\tilde{e}_i : \mathbb{Z}^\infty \to \mathbb{Z}^\infty \cup \{0\}$ as

$$(\tilde{f}_i(\mathbf{x}))_k := x_k + \delta_{k, \min M^{(i)}},$$

$$(\tilde{e}_i(\mathbf{x}))_k := x_k - \delta_{k, \max M^{(i)}} \ \text{ if } \sigma^{(i)}(\mathbf{x}) > 0, \quad (\tilde{e}_i(\mathbf{x}))_k := 0 \ \text{ if } \sigma^{(i)}(\mathbf{x}) = 0.$$

We also define

$$\mathrm{wt}(\mathbf{x}) = -\sum_{j \in \mathbb{Z}_{\geq 1}} x_j \alpha_{i_j},$$

$$\varepsilon_i(\mathbf{x}) = \sigma^{(i)}(\mathbf{x}), \quad \varphi_i(\mathbf{x}) = \langle h_i, \mathrm{wt}(\mathbf{x})\rangle + \varepsilon_i(\mathbf{x}).$$

Theorem 2.6. *[12]* $(\mathbb{Z}^\infty, \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i, \mathrm{wt})$ *is a crystal. We denote it by* $\mathbb{Z}^\infty_\iota$.

2.3 Kashiwara embedding and Polyhedral Realizations of $B(\infty)$

Theorem 2.7. [7] *For any $i \in I$, there uniquely exists a strict embedding*

$$\Psi_i : B(\infty) \hookrightarrow B(\infty) \otimes B_i, \quad u_\infty \mapsto u_\infty \otimes (0)_i.$$

*The map Ψ_i is called a **Kashiwara embedding**.*

Using this theorem repeatedly, for a sequence $i_l, \cdots, i_1 \in I$, we obtain a strict embedding

$$\Psi_{i_l,\cdots,i_1} : B(\infty) \hookrightarrow B(\infty) \otimes B_{i_l} \otimes B_{i_{l-1}} \otimes \cdots \otimes B_{i_1}.$$

Let $\iota = (\cdots, i_3, i_2, i_1)$ be an infinite sequence of I as in the previous subsection. Using Theorem 2.7, we can construct a strict embedding $\Psi_\iota : B(\infty) \hookrightarrow \mathbb{Z}^\infty_\iota = \{(\cdots, a_l, \cdots, a_2, a_1) | a_l \in \mathbb{Z}, a_k = 0 (k \gg 0)\}$ of crystals as follows:

For $b \in B(\infty)$, taking $m \gg 0$, we get

$$\Psi_{i_m,\cdots,i_1}(b) \quad = \quad u_\infty \otimes (-a_m)_{i_m} \otimes (-a_{m-1})_{i_{m-1}} \otimes \cdots \otimes (-a_1)_{i_1}$$

with some $a_j \in \mathbb{Z}_{\geq 0}$. Then we set $\Psi_\iota(b) := (\cdots, 0, 0, a_m, \cdots, a_1)$.

Proposition 2.8. [12] *The map*

$$\Psi_\iota : B(\infty) \hookrightarrow \mathbb{Z}^\infty_{\geq 0} \subset \mathbb{Z}^\infty_\iota \tag{2.2}$$

is a strict embedding of crystals such that $\Psi_\iota(u_\infty) = (\cdots, 0, \cdots, 0, 0)$.

Definition 2.9. The image $\mathrm{Im}\Psi_\iota (\cong B(\infty))$ is called a **polyhedral realization** of $B(\infty)$.

2.4 Polyhedral Realizations of $B(\lambda)$

Theorem 2.10. *[11]*

(*i*) *There exists a strict embedding of crystals $\Omega_\lambda : B(\lambda) \hookrightarrow B(\infty) \otimes R_\lambda$ such that $\Omega_\lambda(u_\lambda) = u_\infty \otimes r_\lambda$.*

(*ii*) *The following map is the unique strict embedding of crystals s.t. $\Psi^{(\lambda)}_\iota(u_\lambda) = (\cdots, 0, 0, 0) \otimes r_\lambda$:*

$$\Psi^{(\lambda)}_\iota = \Psi_\iota \otimes id \circ \Omega_\lambda : B(\lambda) \hookrightarrow B(\infty) \otimes R_\lambda \hookrightarrow \mathbb{Z}^\infty_\iota \otimes R_\lambda =: \mathbb{Z}^\infty_\iota[\lambda].$$

Definition 2.11. $\mathrm{Im}\Psi^{(\lambda)}_\iota (\cong B(\lambda))$ is called a **polyhedral realization** of $B(\lambda)$.

We identify $\mathrm{Im}\Psi^{(\lambda)}_\iota$ as a subset of $\mathbb{Z}^\infty_\iota$.

2.5 Calculations of Polyhedral realizations for $B(\infty)$

Let us consider the infinite dimensional vector space

$$\mathbb{Q}^\infty := \{\mathbf{a} = (\cdots, a_k, \cdots, a_2, a_1) : a_k \in \mathbb{Q} \text{ and } a_k = 0 \text{ for } k \gg 0\},$$

and its dual space $(\mathbb{Q}^\infty)^* := \mathrm{Hom}(\mathbb{Q}^\infty, \mathbb{Q})$. Let $x_k \in (\mathbb{Q}^\infty)^*$ be the linear function defined as $x_k((\cdots, a_k, \cdots, a_2, a_1)) := a_k$ for $k \in \mathbb{Z}_{\geq 1}$. We can write each linear form $\varphi \in (\mathbb{Q}^\infty)^*$ as $\varphi = \sum_{k \geq 1} \varphi_k x_k$ $(\varphi_k \in \mathbb{Q})$.

For the fixed infinite sequence $\iota = (i_k)$ and $k \geq 1$ we set $k^{(+)} := \min\{l : l > k \text{ and } i_k = i_l\}$ and $k^{(-)} := \max\{l : l < k \text{ and } i_k = i_l\}$ if it exists, or $k^{(-)} = 0$ otherwise. We set $\beta_0 = 0$ and

$$\beta_k := x_k + \sum_{k<j<k^{(+)}} \langle h_{i_k}, \alpha_{i_j} \rangle x_j + x_{k^{(+)}} \in (\mathbb{Q}^\infty)^* \qquad (k \geq 1). \tag{2.3}$$

We define the piecewise-linear operator $S_k = S_{k,\iota}$ on $(\mathbb{Q}^\infty)^*$ by

$$S_k(\varphi) := \begin{cases} \varphi - \varphi_k\beta_k & \text{if } \varphi_k > 0, \\ \varphi - \varphi_k\beta_{k^{(-)}} & \text{if } \varphi_k \le 0. \end{cases} \tag{2.4}$$

Here we set

$$\Xi_\iota := \{S_{j_l}\cdots S_{j_2}S_{j_1}x_{j_0} \mid l \ge 0, j_0, j_1, \cdots, j_l \ge 1\}, \tag{2.5}$$

$$\Sigma_\iota := \{\mathbf{x} \in \mathbb{Z}^\infty \subset \mathbb{Q}^\infty \mid \varphi(\mathbf{x}) \ge 0 \text{ for any } \varphi \in \Xi_\iota\}. \tag{2.6}$$

We impose on ι the following **positivity condition**:

$$\text{if } k^{(-)} = 0 \text{ then } \varphi_k \ge 0 \text{ for any } \varphi = \textstyle\sum_k \varphi_k x_k \in \Xi_\iota. \tag{2.7}$$

Theorem 2.12. [12] *Let ι be a sequence of indices in the subsection 2.2 satisfying* (2.7). *Then we have* $\mathrm{Im}(\Psi_\iota)(\cong B(\infty)) = \Sigma_\iota$.

Example 2.13. Let $\mathfrak{g}$ be of type A_2 and $\iota = (\cdots, 2, 1, 2, 1, 2, 1)$. It follows $1^- = 2^- = 0$, $k^- > 0$ $(k > 2)$. We rewrite a vector $(\cdots, x_6, x_5, x_4, x_3, x_2, x_1)$ as

$$(\cdots, x_{3,2}, x_{3,1}, x_{2,2}, x_{2,1}, x_{1,2}, x_{1,1}),$$

that is, $x_{2l-1} = x_{l,1}$, $x_{2l} = x_{l,2}$ for $l \in \mathbb{Z}_{\ge 1}$. Recall that positivity condition means that the coefficients of $x_1 = x_{1,1}$ and $x_2 = x_{1,2}$ in each $\varphi \in \Xi_\iota$ are non-negative. Similarly, we rewrite $S_{2l-1} = S_{l,1}$, $S_{2l} = S_{l,2}$. For $k \in \mathbb{Z}_{\ge 1}$, the action of the operators are the following:

$$x_{k,1} \underset{S_{k+1,1}}{\overset{S_{k,1}}{\rightleftarrows}} x_{k,2} - x_{k+1,1} \underset{S_{k+1,2}}{\overset{S_{k,2}}{\rightleftarrows}} -x_{k+1,2},$$

$$x_{k,2} \underset{S_{k+1,2}}{\overset{S_{k,2}}{\rightleftarrows}} x_{k+1,1} - x_{k+1,2} \underset{S_{k+2,1}}{\overset{S_{k+1,1}}{\rightleftarrows}} -x_{k+2,1},$$

and other actions are trivial. Thus we obtain

$$\Xi_\iota = \{x_{k,1},\ x_{k,2} - x_{k+1,1},\ -x_{k+1,2},\ x_{k,2},\ x_{k+1,1} - x_{k+1,2}, -x_{k+2,1} | k \ge 1\}.$$

Note that the coefficients of $x_{1,1}$ and $x_{1,2}$ in each $\varphi \in \Xi_\iota$ are non-negative. Therefore ι satisfies the positivity condition (2.7) and it follows from Theorem 2.12 that

$$\mathrm{Im}(\Psi_\iota) = \Sigma_\iota = \{\mathbf{x} \in \mathbb{Z}^\infty_\iota | \varphi(\mathbf{x}) \ge 0, \quad \forall \varphi \in \Xi_\iota\}.$$

For $\mathbf{x} = (\cdots, x_{3,2}, x_{3,1}, x_{2,2}, x_{2,1}, x_{1,2}, x_{1,1}) \in \mathrm{Im}(\Psi_\iota)$, combining the inequalities $x_{k,1} \ge 0$, $-x_{k+2,1} \ge 0$ $(k \ge 1)$, we obtain $x_{k,1} = 0$ $(k \ge 3)$. Similarly, by $x_{k,2} \ge 0$, $-x_{k+1,2} \ge 0$ $(k \ge 1)$, we get $x_{k,2} = 0$ $(k \ge 2)$. Hence, we obtain

$$\mathrm{Im}(\Psi_\iota) = \Sigma_\iota = \{\mathbf{x} \in \mathbb{Z}^\infty_\iota | x_{k+1,1} = x_{k,2} = 0 \text{ for } k \in \mathbb{Z}_{\ge 2}, \quad x_{1,2} \ge x_{2,1} \ge 0,\ x_{1,1} \ge 0\}.$$

Example 2.14. [11] Let $\mathfrak{g}$ be of type A_3 and $\iota = (\cdots, 2, 1, 2, 3, 2, 1)$. We obtain

$$x_1 \overset{S_1}{\to} -x_5 + x_4 + x_2 \overset{S_2}{\to} -x_5 + x_3 \overset{S_5}{\to} -x_4 + x_3 - x_2 + x_1.$$

Thus, $-x_4 + x_3 - x_2 + x_1 \in \Xi_\iota$ and $2^- = 0$. Therefore ι does **not** satisfy the positivity condition.

2.6 Calculations of Polyhedral realizations for $B(\lambda)$

- Let $\beta_k,\ \beta_k^{(-)}$ ($k\in\mathbb{Z}_{\geq 1}$) be

$$\beta_k = x_k + \sum_{k<j<k^+} \langle h_{i_k}, \alpha_{i_j}\rangle x_j + x_{k^+},$$

$$\beta_k^{(-)} := \begin{cases} x_{k^-} + \sum_{k^-<j<k} \langle h_{i_k}, \alpha_{i_j}\rangle x_j + x_k = \beta_{k^-} & \text{if } k^- > 0, \\ -\langle h_{i_k}, \lambda\rangle + \sum_{1\leq j<k} \langle h_{i_k}, \alpha_{i_j}\rangle x_j + x_k & \text{if } k^- = 0. \end{cases}$$

- For $\varphi = \sum c_k x_k + c$ ($\sum c_k x_k \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^\infty, \mathbb{Z})$, c: constant term) and $k\in\mathbb{Z}_{\geq 1}$, we define $\hat{S}_k(\varphi)$ as

$$\hat{S}_k(\varphi) := \begin{cases} \varphi - c_k\beta_k & \text{if } c_k \geq 0, \\ \varphi - c_k\beta_k^{(-)} & \text{if } c_k < 0. \end{cases}$$

For $\iota = (\cdots, i_3, i_2, i_1)$ and $i\in I$, let $\iota^{(i)} := \min\{k\in\mathbb{Z}_{\geq 1} | i_k = i\}$,

$$\lambda^{(i)} := \langle h_i, \lambda\rangle - \sum_{1\leq j<\iota^{(i)}} \langle h_i, \alpha_{i_j}\rangle x_j - x_{\iota^{(i)}}.$$

$$\Xi_\iota[\lambda] := \{\hat{S}_{j_l}\cdots\hat{S}_{j_1} x_{j_0} | l\geq 0, j_0, j_1, \cdots, j_l \geq 1\} \cup \{\hat{S}_{j_l}\cdots\hat{S}_{j_1}\lambda^{(i)} | l\geq 0, j_1,\cdots, j_l\geq 1,\ i\in I\}.$$

Definition 2.15. [11] If

$$(\cdots, 0, 0, 0) \in \{\mathbf{x}\in\mathbb{Z}^\infty[\lambda] | \varphi(\mathbf{x}) \geq 0, \quad \forall\varphi\in\Xi_\iota[\lambda]\}$$

then we say the pair (ι, λ) is *ample*.

Theorem 2.16. *[11] If the pair (ι,λ) is ample then*

$$\mathrm{Im}(\Psi_\iota^{(\lambda)})(\cong B(\lambda)) = \{\boldsymbol{x}\in\mathbb{Z}^\infty[\lambda] | \varphi(\boldsymbol{x}) \geq 0, \quad \forall\varphi\in\Xi_\iota[\lambda]\}.$$

2.7 Infinite sequences adapted to A

Definition 2.17. Let $A = (a_{i,j})$ be the generalized symmetrizable Cartan matrix of $\mathfrak{g}$ and ι a sequence of indices in the subsection 2.2. If ι satisfies the following condition, we say ι is **adapted** to A : For $i, j\in I$ with $i\neq j$ and $a_{i,j}\neq 0$, the subsequence of ι consisting of all i, j is

$$(\cdots, i, j, i, j, i, j, i, j) \quad \text{or} \quad (\cdots, j, i, j, i, j, i, j, i).$$

If the Cartan matrix is fixed then the sequence ι is shortly said to be **adapted**.

The notion of 'adapted to Cartan matrix' is similar to the one of 'adapted to a quiver' in [1].

Example 2.18. Let us consider the case $\mathfrak{g}$ is of type A_3, $\iota = (\cdots, 2, 1, 3, 2, 1, 3, 2, 1, 3)$.

- The subsequence consisting of 1, 2 : $(\cdots, 2, 1, 2, 1, 2, 1)$.
- The subsequence consisting of 2, 3 : $(\cdots, 2, 3, 2, 3, 2, 3)$.
- Since $a_{1,3} = 0$ we do not need consider the pair 1, 3.

Thus ι is an adapted sequence.

Example 2.19. Let us consider the case $\mathfrak{g}$ is of type A_3, $\iota = (\cdots, 2, 1, 2, 3, 2, 1)$, which is the same setting as in Example 2.14. The subsequence consisting of 1, 2 is $(\cdots, 2, 1, 2, 2, 1)$. Thus ι is not an adapted sequence.

3 Tableaux descriptions of Polyhedral realizations

In this section, we take $\mathfrak{g}$ as a finite dimensional simple Lie algebra of type A_n, B_n, C_n or D_n. In the rest of article, we follow Kac's notation [4].

3.1 Tableaux descriptions of Polyhedral realizations for $B(\infty)$

In what follows, we suppose $\iota = (\cdots, i_3, i_2, i_1)$ is adapted to the Cartan matrix $A = (a_{i,j})$ of $\mathfrak{g}$. Let $(p_{i,j})_{i\neq j,\ a_{i,j}\neq 0}$ be the set of integers such that

$$p_{i,j} = \begin{cases} 1 & \text{if the subsequence of } \iota \text{ consisting of } i,j \text{ is } (\cdots, j,i,j,i,j,i), \\ 0 & \text{if the subsequence of } \iota \text{ consisting of } i,j \text{ is } (\cdots, i,j,i,j,i,j). \end{cases} \tag{3.1}$$

For k $(2 \leq k \leq n)$, we set

$$P(k) := \begin{cases} p_{2,1} + p_{3,2} + \cdots + p_{n-2,n-3} + p_{n,n-2} & \text{if } k = n \text{ and } \mathfrak{g} \text{ is of type } \mathrm{D}_n, \\ p_{2,1} + p_{3,2} + p_{4,3} + \cdots + p_{k,k-1} & \text{if otherwise,} \end{cases}$$

and $P(0) = P(1) = P(n+1) = 0$. For $k \in \mathbb{Z}_{\geq 1}$, we rewrite x_k, β_k and S_k in 2.5 as

$$x_k = x_{s,j}, \quad S_k = S_{s,j}, \quad \beta_k = \beta_{s,j} \tag{3.2}$$

if $i_k = j$ and j is appearing s times in i_k, i_{k-1}, $\cdots, i_1$. For example, if $\iota = (\cdots, 2, 1, 3, 2, 1, 3, 2, 1, 3)$ then we rewrite $(\cdots, x_6, x_5, x_4, x_3, x_2, x_1) = (\cdots, x_{2,2}, x_{2,1}, x_{2,3}, x_{1,2}, x_{1,1}, x_{1,3})$.

Remark 3.1. *Note that the positivity condition (2.7) implies that for $T \in \Xi_\iota$ the coefficients of $x_{1,j}$ $(j \in I)$ in T are non-negative.*

We will use the both notation x_k and $x_{s,j}$.

Definition 3.2. Let us define the following (partial) ordered sets J_{A}, J_{B}, J_{C} and J_{D}:

- $J_{\mathrm{A}} := \{1, 2, \cdots, n, n+1\}$ with the order $1 < 2 < \cdots < n < n+1$.
- $J_{\mathrm{B}} = J_{\mathrm{C}} := \{1, 2, \cdots, n, \overline{n}, \cdots, \overline{2}, \overline{1}\}$ with the order

$$1 < 2 < \cdots < n < \overline{n} < \cdots < \overline{2} < \overline{1}.$$

- $J_{\mathrm{D}} := \{1, 2, \cdots, n, \overline{n}, \cdots, \overline{2}, \overline{1}\}$ with the partial order

$$1 < 2 < \cdots < n-1 < \begin{smallmatrix} n \\ \overline{n} \end{smallmatrix} < \overline{n-1} < \cdots < \overline{2} < \overline{1}.$$

For $j \in \{1, 2, \cdots, n\}$, we set $|j| = |\overline{j}| = j$.

Definition 3.3. (i) For $1 \leq j \leq n+1$ and $s \in \mathbb{Z}$, we set

$$\boxed{j}^{\mathrm{A}}_s := x_{s+P(j),j} - x_{s+P(j-1)+1,j-1} \in (\mathbb{Q}^\infty)^*,$$

where $x_{m,0} = x_{m,n+1} = 0$ for $m \in \mathbb{Z}$, and $x_{m,i} = 0$ for $m \in \mathbb{Z}_{\leq 0}$ and $i \in I$.

(ii) For $1 \leq j \leq n$ and $s \in \mathbb{Z}$, we set

$$\boxed{j}^{\mathrm{B}}_s := x_{s+P(j),j} - x_{s+P(j-1)+1,j-1} \in (\mathbb{Q}^\infty)^*,$$

$$\boxed{\overline{j}}^{\mathrm{B}}_s := x_{s+P(j-1)+n-j+1,j-1} - x_{s+P(j)+n-j+1,j} \in (\mathbb{Q}^\infty)^*,$$

where $x_{m,0} = 0$ for $m \in \mathbb{Z}$, and $x_{m,i} = 0$ for $m \in \mathbb{Z}_{\leq 0}$ and $i \in I$.

(iii) For $1 \le j \le n-1$ and $s \in \mathbb{Z}$, we set

$$\boxed{j}^{\mathrm{C}}_{s} := x_{s+P(j),j} - x_{s+P(j-1)+1,j-1}, \quad \boxed{n}^{\mathrm{C}}_{s} := 2x_{s+P(n),n} - x_{s+P(n-1)+1,n-1} \in (\mathbb{Q}^{\infty})^{*},$$

$$\boxed{\overline{n}}^{\mathrm{C}}_{s} := x_{s+P(n-1)+1,n-1} - 2x_{s+P(n)+1,n}, \quad \boxed{\overline{j}}^{\mathrm{C}}_{s} := x_{s+P(j-1)+n-j+1,j-1} - x_{s+P(j)+n-j+1,j} \in (\mathbb{Q}^{\infty})^{*},$$

$$\boxed{\overline{n+1}}^{\mathrm{C}}_{s} := x_{s+P(n),n} \in (\mathbb{Q}^{\infty})^{*},$$

where $x_{m,0} = 0$ for $m \in \mathbb{Z}$, and $x_{m,i} = 0$ for $m \in \mathbb{Z}_{\le 0}$ and $i \in I$.

(iv) For $s \in \mathbb{Z}$, we set

$$\boxed{j}^{\mathrm{D}}_{s} := x_{s+P(j),j} - x_{s+P(j-1)+1,j-1} \in (\mathbb{Q}^{\infty})^{*}, \quad (1 \le j \le n-2,\ j = n),$$

$$\boxed{n-1}^{\mathrm{D}}_{s} := x_{s+P(n-1),n-1} + x_{s+P(n),n} - x_{s+P(n-2)+1,n-2} \in (\mathbb{Q}^{\infty})^{*},$$

$$\boxed{\overline{n}}^{\mathrm{D}}_{s} := x_{s+P(n-1),n-1} - x_{s+P(n)+1,n} \in (\mathbb{Q}^{\infty})^{*},$$

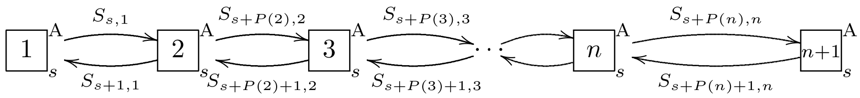
$$\boxed{\overline{n-1}}^{\mathrm{D}}_{s} := x_{s+P(n-2)+1,n-2} - x_{s+P(n-1)+1,n-1} - x_{s+P(n)+1,n} \in (\mathbb{Q}^{\infty})^{*},$$

$$\boxed{\overline{j}}^{\mathrm{D}}_{s} := x_{s+P(j-1)+n-j,j-1} - x_{s+P(j)+n-j,j} \in (\mathbb{Q}^{\infty})^{*}, \quad (1 \le j \le n-2),$$

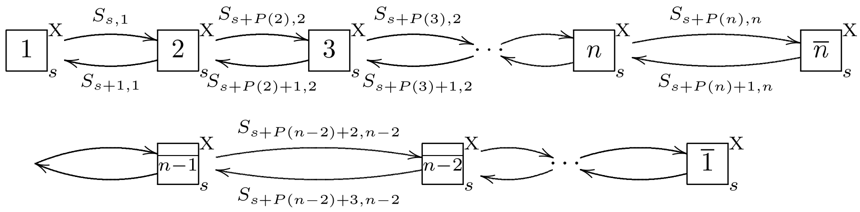
$$\boxed{\overline{n+1}}^{\mathrm{D}}_{s} := x_{s+P(n),n} \in (\mathbb{Q}^{\infty})^{*},$$

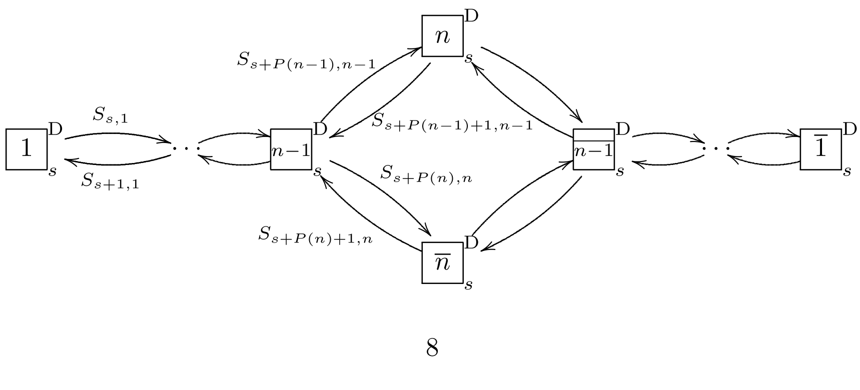
where $x_{m,0} = 0$ for $m \in \mathbb{Z}$, and $x_{m,i} = 0$ for $m \in \mathbb{Z}_{\le 0}$ and $i \in I$.

These boxes are related each other via operators S_k $(k \in \mathbb{Z}_{\ge 1})$:



For X = B or C,





Definition 3.4. (i) For X = A, B, C or D,

$$\boxed{\begin{array}{c} j_1 \\ \hline j_2 \\ \hline \vdots \\ \hline j_{k-1} \\ \hline j_k \end{array}}^{\mathrm{X}}_{s} := \boxed{j_k}^{\mathrm{X}}_{s} + \boxed{j_{k-1}}^{\mathrm{X}}_{s+1} + \cdots + \boxed{j_2}^{\mathrm{X}}_{s+k-2} + \boxed{j_1}^{\mathrm{X}}_{s+k-1} \in (\mathbb{Q}^{\infty})^*.$$

(ii) For X = A, B,

$$\mathrm{Tab}_{\mathrm{X},\iota} := \{ \boxed{\begin{array}{c} j_1 \\ \hline j_2 \\ \hline \vdots \\ \hline j_k \end{array}}^{\mathrm{X}}_{s} | k \in I,\ j_i \in J_X,\ s \geq 1 - P(k),\ (*)^{\mathrm{X}}_{k} \},$$

$(*)^{\mathrm{A}}_{k} : 1 \leq j_1 < j_2 < \cdots < j_k \leq n+1,$

$$(*)^{\mathrm{B}}_{k} : \begin{cases} 1 \leq j_1 < j_2 < \cdots < j_k \leq \overline{1} & \text{for } k < n, \\ 1 \leq j_1 < j_2 < \cdots < j_n \leq \overline{1},\ \ |j_l| \neq |j_m|\ (l \neq m) & \text{for } k = n. \end{cases}$$

$$\mathrm{Tab}_{\mathrm{C},\iota} := \{ \boxed{\begin{array}{c} j_1 \\ \hline j_2 \\ \hline \vdots \\ \hline j_k \end{array}}^{\mathrm{C}}_{s} \ \Bigg| \begin{array}{l} j_1 \in J_{\mathrm{C}} \cup \{\overline{n+1}\}, j_2, \cdots, j_k \in J_{\mathrm{C}}, \\ \text{if } j_1 \neq \overline{n+1} \text{ then } k \in [1, n-1]\ 1 \leq j_1 < j_2 < \cdots < j_k \leq \overline{1},\ s \geq 1 - P(k), \\ \text{if } j_1 = \overline{n+1} \text{ then } k \in [1, n+1],\ \overline{n} \leq j_2 < \cdots < j_k \leq \overline{1},\ s \geq 1 - P(n). \end{array} \}$$

$\mathrm{Tab}_{\mathrm{D},\iota} :=$

$$\{ \boxed{\begin{array}{c} j_1 \\ \hline j_2 \\ \hline \vdots \\ \hline j_k \end{array}}^{\mathrm{D}}_{s} \ \Bigg| \begin{array}{l} j_1 \in J_{\mathrm{D}} \cup \{\overline{n+1}\}, j_2, \cdots, j_k \in J_{\mathrm{D}}, \\ \text{if } j_1 \neq \overline{n+1} \text{ then } k \in [1, n-2] \text{ and } j_1 \ngeq j_2 \ngeq \cdots \ngeq j_k,\ s \geq 1 - P(k), \\ \text{if } j_1 = \overline{n+1} \text{ and } k \text{ is even then } k \in [1, n+1],\ \overline{n} \leq j_2 < \cdots < j_k \leq \overline{1},\ s \geq 1 - P(n-1), \\ \text{if } j_1 = \overline{n+1} \text{ and } k \text{ is odd then } k \in [1, n+1],\ \overline{n} \leq j_2 < \cdots < j_k \leq \overline{1},\ s \geq 1 - P(n). \end{array} \}$$

Remark 3.5. *Similar notations to Definition 3.3 and 3.4 (i) can be found in [9, 10].*

Theorem 3.6. *For* X = A, B, C *or* D, *we suppose that* ι *is adapted to the Cartan matrix of type* X. *Then*

$$\Xi_{\iota} = \mathrm{Tab}_{\mathrm{X},\iota}.$$

The following theorem implies that the condition of **adapted** is a sufficient condition of the positivity condition.

Theorem 3.7. *In the setting of Theorem 3.6,* ι *satisfies the positivity condition.*

We get an explicit form of the polyhedral realization in terms of column tableaux.

Corollary 3.8. *In the setting of Theorem 3.6, we have*

$$\mathrm{Im}(\Psi_\iota)=\{\boldsymbol{a}\in\mathbb{Z}_\iota^\infty|\varphi(\boldsymbol{a})\geq 0,\ \text{for all }\varphi\in\mathrm{Tab}^n_{\mathrm{X},\iota},\ a_{m,i}=0\text{ for }m>n,\ i\in I\},$$

where $\mathrm{Tab}^n_{\mathrm{X},\iota}:=\{\begin{array}{|c|}\hline j_1\\\hline \vdots\\\hline j_k\\\hline\end{array}{}^{\mathrm{X}}_{s}\in\mathrm{Tab}_{\mathrm{X},\iota}|s\leq n\}.$

Example 3.9. Let $\mathfrak{g}$ be the Lie algebra of type A_3 and $\iota=(\cdots,3,1,2,3,1,2)$. The sequence ι is adapted to the Cartan matrix of type A_3. We get $p_{2,1}=1$, $p_{3,2}=0$, $P(2)=P(3)=1$ and

$$\begin{aligned}
\mathrm{Tab}_{\mathrm{A},\iota}&=\{\begin{array}{|c|}\hline j\\\hline\end{array}{}^{\mathrm{A}}_{s}|s\geq 1,\ j\in[1,4]\}\cup\{\begin{array}{|c|}\hline i\\\hline j\\\hline\end{array}{}^{\mathrm{A}}_{s}\mid\begin{array}{l}s\geq 0,\\ 1\leq i<j\leq 4.\end{array}\}\cup\{\begin{array}{|c|}\hline i\\\hline j\\\hline k\\\hline\end{array}{}^{\mathrm{A}}_{s}\mid\begin{array}{l}s\geq 0,\\ 1\leq i<j<k\leq 4.\end{array}\}\\
&=\{x_{s,1},\ x_{s+1,2}-x_{s+1,1},\ x_{s+1,3}-x_{s+2,2},\ -x_{s+2,3}|s\geq 1\}\\
&\cup\ \{x_{s+1,2},\ x_{s+1,3}-x_{s+2,2}+x_{s+1,1},\ x_{s+1,1}-x_{s+2,3},\ x_{s+1,3}-x_{s+2,1},\\
&\quad x_{s+2,2}-x_{s+2,1}-x_{s+2,3},\ -x_{s+3,2}|s\geq 0\}\\
&\cup\ \{x_{s+1,3},\ x_{s+2,2}-x_{s+2,3},\ x_{s+2,1}-x_{s+3,2},\ -x_{s+3,1}|s\geq 0\}.
\end{aligned}\tag{3.3}$$

By Theorem 3.6, we have $\Xi_\iota=\mathrm{Tab}_{\mathrm{A},\iota}$. The explicit form (3.3) means ι satisfies the positivity condition. Hence,

$$\mathrm{Im}(\Psi_\iota)=\Sigma_\iota=\{\mathbf{x}\in\mathbb{Z}_\iota^\infty|\varphi(\mathbf{x})\geq 0,\quad\forall\varphi\in\Xi_\iota\}.$$

For $\mathbf{x}=(\cdots,x_{2,3},x_{2,1},x_{2,2},x_{1,3},x_{1,1},x_{1,2})\in\mathrm{Im}(\Psi_\iota)$, combining inequalities $x_{s,1}\geq 0$ $(s\geq 1)$, $-x_{s+3,1}\geq 0$ $(s\geq 0)$, we obtain $x_{s+3,1}=0$ $(s\geq 0)$. Similarly, by $x_{s+1,2}\geq 0$, $-x_{s+3,2}\geq 0$ $(s\geq 0)$, we get $x_{s+3,2}=0$ $(s\geq 0)$. We also get $x_{s+2,3}=0$ $(s\geq 1)$. Hence, simplifying the inequalities, we obtain

$$\begin{aligned}
\mathrm{Im}(\Psi_\iota)&=\{\mathbf{x}\in\mathbb{Z}_\iota^\infty|x_{s,1}=x_{s,2}=x_{s,3}=0\text{ for }s\in\mathbb{Z}_{\geq 3},\quad x_{2,2}-x_{2,1}\geq x_{2,3}\geq 0,\\
&\qquad x_{1,3}-x_{2,2}+x_{1,1}\geq 0,\ x_{1,1}\geq x_{2,3}\geq 0,\ x_{1,3}\geq x_{2,1}\geq 0,\ x_{1,2}\geq 0\}.
\end{aligned}$$

Example 3.10. Let $\mathfrak{g}$ be the Lie algebra of type C_3 and $\iota=(\cdots,3,1,2,3,1,2)$. The sequence ι is adapted to the Cartan matrix of type C_3. We get $p_{2,1}=1$, $p_{3,2}=0$, $P(2)=P(3)=1$ and

$$\begin{aligned}
\mathrm{Tab}_{\mathrm{C},\iota}&=\{\begin{array}{|c|}\hline j\\\hline\end{array}{}^{\mathrm{C}}_{s}|s\geq 1,\ 1\leq j\leq\bar{1}\}\cup\{\begin{array}{|c|}\hline i\\\hline j\\\hline\end{array}{}^{\mathrm{C}}_{s}\mid\begin{array}{l}s\geq 0,\\ 1\leq i<j\leq\bar{1}.\end{array}\}\cup\{\begin{array}{|c|}\hline \bar{4}\\\hline j_2\\\hline\vdots\\\hline j_k\\\hline\end{array}{}^{\mathrm{C}}_{s}\mid\begin{array}{l}s\geq 0,\ k\in[1,4],\\ 3\leq j_2<\cdots<j_k\leq\bar{1}.\end{array}\}\\
&=\{x_{s,1},\ x_{s+1,2}-x_{s+1,1},\ 2x_{s+1,3}-x_{s+2,2},\ x_{s+2,2}-2x_{s+2,3},\ x_{s+2,1}-x_{s+3,2},\ -x_{s+3,1}|s\geq 1\}\\
&\cup\ \{x_{s+1,2},\ 2x_{s+1,3}-x_{s+2,2}+x_{s+1,1},\ x_{s+1,1}+x_{s+2,2}-2x_{s+2,3},\ x_{s+1,1}+x_{s+2,1}-x_{s+3,2},\\
&\quad x_{s+1,1}-x_{s+3,1},\ 2x_{s+1,3}-x_{s+2,1},\ 2x_{s+2,2}-x_{s+2,1}-2x_{s+2,3},\ x_{s+2,2}-x_{s+3,2},\\
&\quad x_{s+2,2}-x_{s+2,1}-x_{s+3,1},\\
&\quad 2x_{s+2,3}-2x_{s+3,2}+x_{s+2,1},\ 2x_{s+2,3}-x_{s+3,2}-x_{s+3,1},\ x_{s+2,1}-2x_{s+3,3},\\
&\quad x_{s+3,2}-2x_{s+3,3}-x_{s+3,1},\ -x_{s+4,2}|s\geq 0\}\\
&\cup\ \{x_{s+1,3},\ x_{s+2,2}-x_{s+2,3},\ x_{s+2,3}+x_{s+2,1}-x_{s+3,2},\ x_{s+2,3}-x_{s+3,1},\ x_{s+2,1}-x_{s+3,3},\\
&\quad x_{s+3,2}-x_{s+3,1}-x_{s+3,3},\ x_{s+3,3}-x_{s+4,2},\ -x_{s+4,3}|s\geq 0\}.
\end{aligned}\tag{3.4}$$

By Theorem 3.6, we have $\Xi_\iota = \mathrm{Tab}_{\mathrm{C},\iota}$. The explicit form (3.4) means ι satisfies the positivity condition. Hence,

$$\mathrm{Im}(\Psi_\iota) = \Sigma_\iota = \{\mathbf{x} \in \mathbb{Z}_\iota^\infty | \varphi(\mathbf{x}) \geq 0, \quad \forall \varphi \in \Xi_\iota\}.$$

Simplifying the inequalities, we obtain

$$\begin{aligned}
& \mathrm{Im}(\Psi_\iota) \\
= \ & \{\mathbf{x} \in \mathbb{Z}_\iota^\infty | x_{s,1} = x_{s,2} = x_{s,3} = 0 \text{ for } s \in \mathbb{Z}_{\geq 4}, \quad x_{2,2} \geq x_{2,1} \geq 0,\ 2x_{2,3} \geq x_{3,2} \geq x_{3,1} \geq 0, \\
& x_{3,2} - 2x_{3,3} \geq 0,\ 2x_{1,3} - x_{2,2} + x_{1,1} \geq 0, \\
& x_{1,1} + x_{2,2} - 2x_{2,3} \geq 0,\ x_{1,1} + x_{2,1} - x_{3,2} \geq 0,\ x_{1,1} \geq x_{3,1} \geq 0,\ 2x_{1,3} - x_{2,1} \geq 0, \\
& 2x_{2,2} - x_{2,1} - 2x_{2,3} \geq 0,\ x_{2,2} - x_{3,2} \geq 0,\ x_{2,2} - x_{2,1} - x_{3,1} \geq 0,\ 2x_{2,3} - 2x_{3,2} + x_{2,1} \geq 0, \\
& 2x_{2,3} - x_{3,2} - x_{3,1} \geq 0,\ x_{2,1} - 2x_{3,3} \geq 0,\ x_{3,2} - 2x_{3,3} - x_{3,1} \geq 0, \\
& x_{2,2} - x_{2,3} \geq 0,\ x_{2,3} + x_{2,1} - x_{3,2} \geq 0,\ x_{2,3} \geq x_{3,1} \geq 0,\ x_{2,1} \geq x_{3,3} \geq 0,\ x_{3,2} - x_{3,1} - x_{3,3} \geq 0, \\
& x_{1,2} \geq 0,\ x_{1,3} \geq 0\}.
\end{aligned}$$

3.2 Tableaux descriptions of Polyhedral realizations for $B(\lambda)$

For $k \in I$, we consider the following two conditions:

(1) $k < n$ and $\iota^{(k)} > \iota^{(k+1)}$, (2) $k > 1$ and $\iota^{(k)} > \iota^{(k-1)}$.

For $k \in I$, we set

$$\mathrm{Tab}_{\mathrm{A},\iota,\mathrm{k}}[\lambda] :=$$

$$\begin{cases} \{-x_{1,k} + \langle \lambda, h_k \rangle\} & \text{if } (1), (2) \text{ do not hold,} \\ \{\boxed{t}_{-P(k)} + \langle \lambda, h_k \rangle | k+1 \leq t \leq n+1\} & \text{if only (1) holds,} \\ \left\{ \begin{matrix} [j_1, \cdots, j_{k-1}, k+1, \cdots, n, n+1]_{-P(k-1)-n+k} \\ +\langle \lambda, h_k \rangle \end{matrix} \middle| \begin{matrix} 1 \leq j_1 < \cdots \\ < j_{k-1} \leq k \end{matrix} \right\} & \text{if only (2) holds,} \\ \{[j_1, \cdots, j_k]_{-P(k-1)} + \langle \lambda, h_k \rangle | 1 \leq j_1 < \cdots < j_k \leq n+1,\ j_k > k\} & \text{if both (1) and (2) hold.} \end{cases}$$

For $k \in \{1, \cdots, n-1\}$,

$$\mathrm{Tab}_{\mathrm{B},\iota,\mathrm{k}}[\lambda] := \begin{cases} \{-x_{1,k} + \langle \lambda, h_k \rangle\} & \text{if } (1), (2) \text{ do not hold,} \\ \{\boxed{t}_{1-P(k+1)} + \langle \lambda, h_k \rangle | k+1 \leq t \leq \overline{1}\} & \text{if only (1) holds,} \\ \{\boxed{t}_{-P(k-1)-n+k} + \langle \lambda, h_k \rangle | \overline{k} \leq t \leq \overline{1}\} & \text{if only (2) holds,} \\ \left\{ \begin{matrix} [j_1, \cdots, j_k]_{-P(k-1)} \\ +\langle \lambda, h_k \rangle \end{matrix} \middle| \begin{matrix} j_1 < \cdots < j_k, \\ j_k > k,\ j_i \in J_{\mathrm{B}} \end{matrix} \right\} & \text{if both (1) and (2) hold,} \end{cases}$$

$$\mathrm{Tab}_{\mathrm{B},\iota,\mathrm{n}}[\lambda] := \begin{cases} \{-x_{1,n} + \langle \lambda, h_n \rangle\} & \text{if } \iota^{(n)} < \iota^{(n-1)}, \\ \left\{[j_1, \cdots, j_n]_{-P(n-1)} + \langle \lambda, h_n \rangle \middle| \begin{matrix} j_1 < \cdots < j_n, \\ j_n > n,\ j_i \in J_{\mathrm{B}} \\ |j_l| \neq |j_m| \text{ if } l \neq m \end{matrix} \right\} & \text{if } \iota^{(n)} > \iota^{(n-1)}. \end{cases}$$

For $k \in \{1, \cdots, n-1\}$,

$$\mathrm{Tab}_{\mathrm{C},\iota,\mathrm{k}}[\lambda] := \begin{cases} \{-x_{1,k} + \langle \lambda, h_k \rangle\} & \text{if } (1), (2) \text{ do not hold,} \\ \{\boxed{t}_{1-P(k+1)} + \langle \lambda, h_k \rangle | k+1 \leq t \leq \overline{1}\} & \text{if only (1) holds,} \\ \{\boxed{t}_{-P(k-1)-n+k} + \langle \lambda, h_k \rangle | \overline{k} \leq t \leq \overline{1}\} & \text{if only (2) holds,} \\ \left\{ \begin{matrix} [j_1, \cdots, j_k]_{-P(k-1)} \\ +\langle \lambda, h_k \rangle \end{matrix} \middle| \begin{matrix} j_1 < \cdots < j_k, \\ j_k > k,\ j_i \in J_{\mathrm{C}} \end{matrix} \right\} & \text{if both (1) and (2) hold,} \end{cases}$$

$$
\mathrm{Tab}_{\mathrm{C},\iota,\mathrm{n}}[\lambda] := \begin{cases} \{-x_{1,n} + \langle\lambda, h_n\rangle\} & \text{if } \ \iota^{(n)} < \iota^{(n-1)}, \\ \left\{ [\overline{n+1}, j_2, \cdots, j_s]_{-P(n-1)} + \langle\lambda, h_n\rangle \,\middle|\, \begin{array}{l} 2 \le s \le n+1, \\ \overline{n} \le j_2 < \cdots \\ \cdots < j_s \le \overline{1} \end{array} \right\} & \text{if } \iota^{(n)} > \iota^{(n-1)}. \end{cases}
$$

For $k \in \{1, 2, \cdots, n-3\}$, we set

$$
\mathrm{Tab}_{\mathrm{D},\iota,\mathrm{k}}[\lambda] := \begin{cases} \{-x_{1,k} + \langle\lambda, h_k\rangle\} & \text{if } (1),(2) \text{ do not hold,} \\ \{\boxed{t}_{1-P(k+1)} + \langle\lambda, h_k\rangle | k+1 \le t \le \overline{1}\} & \text{if only (1) holds,} \\ \{\boxed{t}_{1-P(k-1)-n+k} + \langle\lambda, h_k\rangle | \overline{k} \le t \le \overline{1}\} & \text{if only (2) holds} \\ \left\{ [j_1, \cdots, j_k]_{1-P(k+1)} + \langle\lambda, h_k\rangle \,\middle|\, \begin{array}{l} j_i \in \mathrm{J_D}, \ k < j_k, \\ j_1 \not\geq \cdots \not\geq j_k \end{array} \right\} & \text{if both (1) and (2) hold.} \end{cases}
$$

For $t \in \{n-3, n-1, n\}$, we consider the following conditions:

$C_t : \iota^{(t)} < \iota^{(n-2)}$ holds, $\quad \overline{C}_t : \iota^{(t)} > \iota^{(n-2)}$ holds.

$\mathrm{Tab}_{\mathrm{D},\iota,\mathrm{n-2}}[\lambda] :=$

$$
\begin{cases} \{-x_{1,n-2} + \langle\lambda, h_{n-2}\rangle\} & \text{if } \overline{C}_{n-3}, \overline{C}_{n-1}, \overline{C}_n, \\ \{\lambda^{(n-2)}, \ -x_{2,n-1} + \langle\lambda, h_{n-2}\rangle\} & \text{if } \overline{C}_{n-3}, C_{n-1}, \overline{C}_n, \\ \{\lambda^{(n-2)}, \ -x_{2,n} + \langle\lambda, h_{n-2}\rangle\} & \text{if } \overline{C}_{n-3}, \overline{C}_{n-1}, C_n, \\ \{\boxed{t}_{-1-P(n-3)} + \langle\lambda, h_{n-2}\rangle | \overline{n-2} \le t \le \overline{1}\} & \text{if } C_{n-3}, \overline{C}_{n-1}, \overline{C}_n, \\ \{\boxed{t}_{-P(n-2)} + \langle\lambda, h_{n-2}\rangle | n-1 \le t \le \overline{1}, \ t \in \mathrm{J}_D\} & \text{if } \overline{C}_{n-3}, C_{n-1}, C_n, \\ \left\{ [\overline{n+1}, j_2, \cdots, j_s]_{-1-P(n-2)} + \langle\lambda, h_{n-2}\rangle \,\middle|\, \begin{array}{l} 3 \le s \le n+1, \ s \text{ is odd,} \\ \text{if } s = 3 \text{ then } j_3 \ge \overline{n-2}, \\ \overline{n} \le j_2 < \cdots < j_s \le \overline{1} \end{array} \right\} & \text{if } C_{n-3}, C_{n-1}, \overline{C}_n, \\ \left\{ [\overline{n+1}, j_2, \cdots, j_s]_{-1-P(n-2)} + \langle\lambda, h_{n-2}\rangle \,\middle|\, \begin{array}{l} 2 \le s \le n+1, \ s \text{ is even,} \\ \text{if } s = 2 \text{ then } j_2 \ge \overline{n-2}, \\ \overline{n} \le j_2 < \cdots < j_s \le \overline{1} \end{array} \right\} & \text{if } C_{n-3}, \overline{C}_{n-1}, C_n, \\ \left\{ [j_1, \cdots, j_{n-2}]_{-P(n-2)} + \langle\lambda, h_{n-2}\rangle \,\middle|\, \begin{array}{l} j_1, \cdots, j_{n-2} \in \mathrm{J}_D, \\ j_1 \not\geq \cdots \not\geq j_{n-2}, \\ j_{n-2} \ge n-1, \end{array} \right\} & \text{if } C_{n-3}, C_{n-1}, C_n, \end{cases}
$$

$$
\mathrm{Tab}_{\mathrm{D},\iota,\mathrm{n-1}}[\lambda] := \begin{cases} \{-x_{1,n-1} + \langle\lambda, h_{n-1}\rangle\} & \text{if } C_{n-1}, \\ \left\{ [\overline{n+1}, j_2, \cdots, j_s]_{-P(n-2)} + \langle\lambda, h_{n-1}\rangle \,\middle|\, \begin{array}{l} 2 \le s \le n+1, \ s \text{ is even,} \\ j_2, \cdots, j_s \in \mathrm{J}_D, \\ \overline{n} \le j_2 < \cdots < j_s \le \overline{1}, \\ \text{if } s = 2 \text{ then } j_2 \ge \overline{n-1} \end{array} \right\} & \text{if } \ \overline{C}_{n-1}, \end{cases}
$$

$$
\mathrm{Tab}_{\mathrm{D},\iota,\mathrm{n}}[\lambda] := \begin{cases} \{-x_{1,n} + \langle\lambda, h_n\rangle\} & \text{if } C_n, \\ \left\{ [\overline{n+1}, j_2, \cdots, j_s]_{-P(n-2)} + \langle\lambda, h_n\rangle \,\middle|\, \begin{array}{l} 3 \le s \le n+1, \ s \text{ is odd,} \\ j_2, \cdots, j_s \in \mathrm{J}_D, \\ \overline{n} \le j_2 < \cdots < j_s \le \overline{1} \end{array} \right\} & \text{if } \overline{C}_n. \end{cases}
$$

For $\mathrm{X} = \mathrm{A}, \mathrm{B}, \mathrm{C}$ or D, we set

$$\mathrm{Tab}_{\mathrm{X},\iota}[\lambda] := \left(\bigcup_{k \in I} \mathrm{Tab}_{\mathrm{X},\iota,\mathrm{k}}[\lambda] \right) \cup \{0\}.$$

Theorem 3.11. *Let $\mathfrak{g}$ be of type A,B,C or D. If ι is adapted to the Cartan matrix of $\mathfrak{g}$ then ι satisfies the ample condition and we have*

$$\Xi_\iota[\lambda] = \mathrm{Tab}_{\mathrm{X},\iota}[\lambda] \cup \mathrm{Tab}_{\mathrm{X},\iota}.$$

Corollary 3.12.

$$\mathrm{Im}(\Psi_\iota^{(\lambda)}) = \{\boldsymbol{x} \in \mathbb{Z}^\infty | \varphi(\boldsymbol{x}) \geq 0, \forall \varphi \in \mathrm{Tab}_{\mathrm{X},\iota}[\lambda] \cup \mathrm{Tab}^n_{\mathrm{X},\iota},\ x_{m,i} = 0\ (\forall i \in I, m > n)\}.$$

References

[1] D.Hernandez, B.Leclerc, Quantum Grothendieck rings and derived Hall algebras, J. Reine Angew. Math. 701, 77-126 (2015).

[2] A.Hoshino, Polyhedral realizations of crystal bases for quantum algebras of finite types, J. Math. Phys. 46, no. 11, 113514, 31 pp, (2005).

[3] A.Hoshino, Polyhedral realizations of crystal bases for quantum algebras of classical affine types, J. Math. Phys. 54, no. 5, 053511, 28 pp (2013).

[4] V. G. Kac, *Infinite-dimensional Lie algebras*, third edition. Cambridge University Press, Cambridge, xxii+400 pp, (1990).

[5] M.Kashiwara, Crystalling the q-analogue of universal enveloping algebras, Comm. Math. Phys., *133*, 249–260 (1990).

[6] M.Kashiwara, On crystal bases of the q-analogue of universal enveloping algebras, Duke Math. J., *63* (2), 465–516 (1991).

[7] M.Kashiwara, The crystal base and Littelmann's refined Demazure character formula, Duke Math. J., 71, no 3, 839–858 (1993).

[8] G.Lusztig, Canonical bases arising from quantized enveloping algebras, J. Amer. Math. Soc. 3, no. 2, 447–498 (1990).

[9] H. Nakajima, t-analogs of q-characters of quantum affine algebras of type A_n, D_n, Combinatorial and geometric representation theory (Seoul, 2001), Contemp. Math, 325, AMS, Providence, RI, 141–160 (2003).

[10] W.Nakai, T.Nakanishi, Paths, tableaux and q-characters of quantum affine algebras: The C_n case, J. Phys. A: Math. Gen. 39, no. 9, 2083-2115, (2006).

[11] T.Nakashima, Polyhedral realizations of crystal bases for integrable highest weight modules, J. Algebra, vol.219, no. 2, 571-597, (1999).

[12] T.Nakashima, A.Zelevinsky, Polyhedral realizations of crystal bases for quantized Kac-Moody algebras, Adv. Math. **131**, no. 1, 253–278, (1997).