LEAFWISE COHOMOLOGICAL EXPRESSION OF DYNAMICAL ZETA FUNCTIONS ON FOLIATED DYNAMICAL SYSTEMS

JUNHYEONG KIM

Abstract. A Riemmanian foliated dynamical system of 3-dimension (RFDS^3) is a closed Riemannian 3-manifold with additional structures: foliation, dynamical system. In the context of arithmetic topology, it is a geometric/analytic analogue of an arithmetic scheme with a conjectural dynamical system suggested by C. Deninger. In this paper, we show leafwise cohomological expression of dynamical zeta function on a Riemannian foliated dynamical system.

1. Introduction

In a series of papers (c.f. [1],[4],[6],[7],[8]), C. Deninger considered arithmetic schemes $\overline{\operatorname{spec}\mathcal{O}_K}$ with a conjectural dynamical system for a number field $K/\mathbb{Q}$. He interpreted the completed Dedekind zeta function $\hat{\zeta}_K(s)$ of K in terms of infinite dimensional cohomology groups $H^\bullet_{\mathrm{dyn}}(\overline{\operatorname{spec}\mathcal{O}_K},\mathcal{R})$:

$$\hat{\zeta}_K(s)=\prod_{i=0}^{2}\det{}_\infty\left(\frac{1}{2\pi}(s-\Theta)|H^i_{\mathrm{dyn}}(\overline{\operatorname{spec}\mathcal{O}_K},\mathcal{R})\right)^{(-1)^{i+1}},$$

where $\det_\infty$ denotes the zeta-regularized determinant and Θ denotes an infinitesimal generator of the flow.

This idea is extended to smooth closed 3-manifold M with 1-codimensional foliation structure $\mathcal{F}$, transverse flow ϕ and a bundle-like metric $g_{\mathcal{F}}$ via Arithmetic topology ([9]). We call the manifold with the additional structure a **Riemannian foliated dynamical system** of 3 dimension, simply RFDS^3. It is a geometric/dynamical analogue of the above arithmetic scheme with a conjectural dynamical system, where closed orbits correspond to finite primes. Note that dynamical zeta function corresponds to Dedekind zeta function in this context. The purpose of this paper is to show leafwise cohomological expression of dynamical zeta function on a Riemannian foliated dynamical system. We describe our main results in the following.

Let $(M,\mathcal{F},\phi,g_{\mathcal{F}})$ be a RFDS^3. The additional structures give a **reduced leafwise cohomology** $\bar{H}^\bullet_{\mathcal{F}}(M)$ and an **infinitesimal generator** Θ. Then we

2010 *Mathematics Subject Classification.* Primary 57R30; Secondary 37C30.

*Research Fellow of the Japan Society for the Promotion of Science.

consider infinite series

$$\xi_p(s,z) := \sum_{\rho \in \mathrm{Sp}(\Theta_p)} (s-\rho)^{-z},$$

where Θ_p denotes the operator Θ acting on $\bar{H}^p_{\mathcal{F}}(M)$. We have our first main theorem:

Theorem 1.1. *The following assertions hold:*

(1) *The series $\xi_p(s,z)$ is absolutely convergent on $\mathrm{Re}(z) >> 0$ for any $s \in \mathbb{C}$.*
(2) *It extends to a meromorphic function of $z \in \mathbb{C}$ and $s \in \mathbb{C}$ which is holomorphic at $z = 0$.*

As a consequence of the theorem, the series defines a Hurwitz-type spectral zeta function associated with the infinitesimal generator. It follows that a zeta-regularized determinant can be defined since the spectral zeta function is regular at $z = 0$.

For the second result, we define the dynamical zeta function for RFDS^3.

$$\zeta_{\mathcal{F}}(s) = \prod_{\gamma} (1 - e^{-s \cdot l(\gamma)})^{-\epsilon_\gamma},$$

where γ runs over closed orbits of ϕ and $l(\gamma)$ is the length of γ. Here, ϵ_γ denotes the index of a closed orbit. We give our second main theorem as follows:

Theorem 1.2. *The dynamical zeta function on a Riemmanian foliated dynamical system of 3 dimension has a leafwise cohological expression*

$$\zeta_{\mathcal{F}}(s) = \prod_{i=0}^{2} \det{}_\infty(s - \Theta | \bar{H}^i_{\mathcal{F}}(M))^{(-1)^{i+1}}.$$

The contents of this paper are organized as follows: In section 2, 3, 4, we introduce a Riemannian foliated dynamical system of 3 dimension (RFDS^3) and basic notions: leafwise cohomology and infinitesimal generator. In section 5, we give a proof of the main theorem 1.1. In section 6, 7, we recall the zeta-regularized determinant and dynamical zeta function for RFDS^3. In section 8, we show a leafwise cohomlogical expression of the dynamical zeta function on RFDS^3.

2. Riemannian foliated dynamical system (RFDS^3)

We consider a smooth, compact, orientable, closed 3-manifold M with additional structure: foliation $\mathcal{F}$, transverse flow ϕ.

2.1. **Foliation.** A foliation $\mathcal{F}$ of d-codimension is a partition of sub-manifolds of d-codimension. Let M be a smooth, connected, closed and oriented manifold of n-dimension. It is equipped with a foliation of d-codimension as follows: let

$(U_i, \phi_i)_{i\in I}$ be an atlas. The transition maps $\phi_{ij} := \phi_j \circ \phi_i^{-1}$ which are defined over $U_i \cap U_j$ take forms

$$\phi_{ij}(x_1, \cdots, x_d, y_{d+1}, \cdots, y_n) = (\phi_{ij}^1(x_1, \cdots, x_d), \cdots, \phi_{ij}^d(x_1, \cdots, x_d), \\ \phi_{ij}^{d+1}(x_1, \cdots, x_d, y_{d+1}, \cdots, y_n), \phi_{ij}^n(x_1, \cdots, x_d, y_{d+1}, \cdots, y_n))$$

for $i, j \in I$. By piecing together the stripes, where $(x_1, \cdots, x_d)$ are constant, from chart to chart, we obtain a maximal immersed sub-manifold $\mathcal{L}$ whose first d local coordinates are constant on each U_i. We call the sub-manifold a *leaf* of the foliation. The foliation consists of the disjoint union of leaves.

2.2. **Transversal flow.** A transverse flow ϕ is a smooth $\mathbb{R}$-action on a manifold M

$$\phi : \mathbb{R} \times M \to M$$

which maps leaves of a foliation to leaves. For any two points x and y in a same leaf $\mathcal{L}$, there is a leaf $\mathcal{L}'$ containing $\phi(t, x)$ and $\phi(t, y)$ for any $t \in \mathbb{R}$. Let $\dot{\phi}$ be the vector field giving the velocity vector at a point. We denote by ω_ϕ the dual 1-form of the vector field $\dot{\phi}$.

2.3. **Bundle-like metric.** A Riemannian metric $g_{\mathcal{F}}$ on $(M, \mathcal{F}, \phi)$ is called a **bundle-like metric** whose geodesics are perpendicular to all leaves whenever they are perpendicular to one leaf. Note that any 1-codimensional foliation without singularities is Riemannian.

We consider 3-manifolds with additional structures as follows:

Definition 2.1. *We define a foliated dynamical system on a 3-manifold by a triple $(M, \mathcal{F}, \phi)$, where*

1. *M is a smooth, compact, orientable 3-manifold,*
2. *$\mathcal{F}$ is a 1-codimensional foliation on M,*
3. *ϕ is a smooth $\mathbb{R}$-action acting on M such that*
   (a) *The flow is transverse to the leaves of the foliation up to a finite number of compact leaves;*
   (b) *The $\mathbb{R}$-action maps leaves to leaves.*

The manifold and the flow may have boundaries and fixed-points. For this paper, we assume that the manifold is closed and the flow has no fixed-point. It is known that only mapping torus allows such a foliated dynamical system on itself.

3. Leafwise cohomology

3.1. **Leafwise de Rham complex.** For the triple $(M, \mathcal{F}, \phi)$, let $T\mathcal{F}$ be a sub-bundle of the tangent bundle TM which is tangent to the leaves of the foliation. The restriction of $T\mathcal{F}$ on a leaf $\mathcal{L}$ is identified with the tangent bundle $T\mathcal{L}$ of the leaf. We call $T\mathcal{F}$ the *leafwise tangent bundle*.

We define the space of leafwise i-forms by

$$\Omega^i_{\mathcal{F}}(M) := \Gamma(M, \wedge^i T^*\mathcal{F}) \subset \Omega^i(M).$$

Let $d_{\mathcal{F}}$ (resp. d_0) be the exterior derivative acting only along leaves (resp. the flow). Then the de Rham complex $(\Omega^i(M), d^i)$ has a decomposition:

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \Omega^i_{\mathcal{F}}(M) & \xrightarrow{d^i_{\mathcal{F}}} & \Omega^{i+1}_{\mathcal{F}}(M) & \longrightarrow & \cdots \\
 & \searrow & \downarrow \oplus & \searrow^{d^i_0} & \downarrow \oplus & \searrow & \\
\cdots & \longrightarrow & \Omega^i_0(M) & \xrightarrow{d^i_{\mathcal{F}}} & \Omega^{i+1}_0(M) & \longrightarrow & \cdots
\end{array}$$

where $\Omega^i_0(M)$ is the complement of $\Omega^i_{\mathcal{F}}(M)$.

We simply denote the restriction $d^i|_{\Omega^i_{\mathcal{F}}(M)}$ by $d^i_{\mathcal{F}}$. Since we have $d^{i+1}_{\mathcal{F}} \circ d^i_{\mathcal{F}} = 0$ on $\Omega^i_{\mathcal{F}}(M)$, the pairs $\{(\Omega^i_{\mathcal{F}}(M), d^i_{\mathcal{F}})\}_i$ form a cochain complex:

$$0 \to \Omega^0_{\mathcal{F}}(M) \overset{d^0_{\mathcal{F}}}{\to} \Omega^1_{\mathcal{F}}(M) \overset{d^1_{\mathcal{F}}}{\to} \Omega^2_{\mathcal{F}}(M) \overset{d^2_{\mathcal{F}}}{\to} 0.$$

We call the complex *leafwise de Rham complex*.

3.2. **Leafwise cohomology.** We denote the kernel of $d^i_{\mathcal{F}}$ by $Z^i_{\mathcal{F}}(M)$ and the image of $d^i_{\mathcal{F}}$ by $B^{i+1}_{\mathcal{F}}(M)$. Note that a leafwise i-th form in $Z^i_{\mathcal{F}}(M)$ (resp. $B^{i+1}_{\mathcal{F}}(M)$) is called leafwise closed i-th form (resp. leafwise exact i-th form).

Definition 3.1 (Leafwise cohomology)**.** *We define the i-th leafwise cohomology group by*

$$H^i_{\mathcal{F}}(M) := Z^i_{\mathcal{F}}(M)/B^i_{\mathcal{F}}(M).$$

The leafwise cohomology group is trivial for $i > 2$.

Unfortunately, the leafwise cohomology group is of infinite dimension in general and not a Hausdorff space. We modify it by taking a quotient with respect to the closure in the smooth topology.

$$\bar{H}^i_{\mathcal{F}}(M) := Z^i_{\mathcal{F}}(M)/\overline{B^i_{\mathcal{F}}(M)}.$$

We call it the reduced leafwise cohomology group.

3.3. **Leafwise Hodge theorem.** For a bundle-like metric $g_{\mathcal{F}}$, we have the Hodge $*$-operator. If we denote by δ the adjoint operator of the exterior derivative d, it has a decomposition into $\delta_{\mathcal{F}}$ and δ_0

$$\begin{array}{ccccccc}
\cdots & \longleftarrow & \Omega^i_{\mathcal{F}}(M) & \xleftarrow{\delta^i_{\mathcal{F}}} & \Omega^{i+1}_{\mathcal{F}}(M) & \longleftarrow & \cdots \\
 & \nwarrow & \downarrow \oplus & \nwarrow^{\delta^i_0} & \downarrow \oplus & \nwarrow & \\
\cdots & \longleftarrow & \Omega^i_0(M) & \xleftarrow{\delta^i_{\mathcal{F}}} & \Omega^{i+1}_0(M) & \longleftarrow & \cdots .
\end{array}$$

Definition 3.2 (Leafwise Laplacian)**.** *An operator defined by*

$$\Delta_{\mathcal{F}} := d_{\mathcal{F}}\delta_{\mathcal{F}} + \delta_{\mathcal{F}} d_{\mathcal{F}} \text{ on } \Omega^i_{\mathcal{F}}(M)$$

is called the ***leafwise Laplacian****. A leafwise form* $\omega \in \ker \Delta_{\mathcal{F}}$ *is called a leafwise harmonic form.*

If we define Δ_0 on $\Omega^i_{\mathcal{F}}(M)$ by $\delta_0 d_0$, the restriction of the Laplacian Δ on $\Omega^i_{\mathcal{F}}(M)$ can be represented by

$$\Delta|_{\Omega^i_{\mathcal{F}}(M)} = \Delta_{\mathcal{F}} + \Delta_0.$$

We have a significant proposition by Álvarez López and Kordyukov ([3]):

Proposition 3.3 (Leafwise Hodge theorem)**.** *Given a bundle-like metric, An leafwise cohomology class can be uniquely represented by a leafwise harmonic form. We have an isomorphism*

$$\bar{H}^i_{\mathcal{F}}(M) \cong \ker(\Delta^i_{\mathcal{F}}).$$

4. Infinitesimal generator

Assume that the $\mathbb{R}$-action ϕ is conformal on $\Omega^i_{\mathcal{F}}(M)$ with respect to the bundle-like metric $g_{\mathcal{F}}$, i.e.

$$(\phi^{t*}\omega, \phi^{t*}\eta) = (\omega, \eta) \text{ for } \forall t \in \mathbb{R}.$$

It is easy to check that ϕ^{t*} on $\bar{H}^i_{\mathcal{F}}(M)$ is surjective and strongly continuous, i.e.

$$\lim_{t \to t_0} \phi^{t*}[h] = \phi^{t_0 *}[h] \quad \text{for } \forall t_0 \in \mathbb{R}, [h] \in \bar{H}^i_{\mathcal{F}}(M).$$

Then the following lemma follows from the Stone's theorem

Lemma 4.1 (Stone's theorem)**.** *We define the infinitesimal generator of* $(\phi^{t*})_{t\in\mathbb{R}}$ *by*

$$\Theta := \lim_{t \to 0} \frac{\phi^{t*} - \mathrm{id}}{t}.$$

Since $(\phi^{t*})_{t\in\mathbb{R}}$ *is the strongly continuous one-parameter unitary group on the Hilbert space* $\bar{H}^i_{\mathcal{F}}(M)$*, then* $A := -i\Theta$ *is self-adjoint on* $\bar{H}^i_{\mathcal{F}}(M)$ *and we have*

$$\phi^{t*} = e^{itA} = e^{t\Theta} \text{ for } \forall t \in \mathbb{R}.$$

The infinitesimal generator is a first-order differential operator along the transversal flow. Then the exterior derivative d_0 along the flow (resp. adjoint operator δ_0) can be represented by

$$d_0\omega = \Theta\omega \wedge \omega_\phi,$$
$$\delta_0(\omega \wedge \omega_\phi) = -\Theta\omega.$$

It leads the following lemma.

Lemma 4.2. *The negative square of the infinitesimal generator on a leafwise cohomology group coincide with the Laplacian on a space of leafwise harmonic forms*

$$\begin{aligned} -\Theta^2|_{\bar{H}^i_{\mathcal{F}}(M)} &= \Delta_0|_{\ker(\Delta_{\mathcal{F}})} \\ &= \Delta|_{\ker(\Delta_{\mathcal{F}})}. \end{aligned}$$

Since M is compact and closed, the Laplacian has pure point spectrum which consists of non-negative eigenvalues with finite multiplicity. Hence the infinitesimal generator has pure imaginary eigenvalues with finite multiplicity.

For a leafwise harmonic form $\omega_{\mathcal{F}} \in \ker \Delta_{\mathcal{F}}$, a fundamental solution of the heat equation whose initial value is $\omega_{\mathcal{F}}$, i.e.

$$\begin{cases} \left(\frac{\partial}{\partial t} + \Delta_0\right) \omega_{\mathcal{F}}^t = 0, \\ \omega_{\mathcal{F}}^0 = \omega_{\mathcal{F}}, \end{cases}$$

is given by

$$\omega_{\mathcal{F}}^t(x,y,s) = \begin{cases} \int_{S^1} K(t,s,s')\omega_{\mathcal{F}}(x,y,s')ds' & \text{if } (x,y,s) \text{ is periodic,} \\ \int_{\mathbb{R}} K(t,s,s')\omega_{\mathcal{F}}(x,y,s')ds' & \text{otherwise} \end{cases}$$

where $K(t,s,s')$ is a factor of the heat kernel of the Laplacian Δ. Hence we have an asymptotic expansion for the heat kernel around $t = 0$

$$\operatorname{tr}(e^{-\Delta_0 t}|\ker \Delta_{\mathcal{F}}) \overset{t\downarrow 0}{\sim} t^{-\frac{1}{2}}(a_0 + a_1 t + a_2 t^2 + \cdots).$$

Then the spectral zeta function $\zeta_{\Delta_0}(s)$ associated with Δ_0 has only simple poles at $s = \frac{1}{2} - n$ $(n = 0, 1, 2, \cdots)$.

Fixing a positive number $T > 0$, we consider a series

$$V^p(t) = \sum_{\operatorname{Im}(\rho) > T} e^{\rho i t}$$

where ρ runs over the spectrum of Θ on $\bar{H}^p_{\mathcal{F}}(M)$. It follows from the lemma 4.2 that it is a partial sum of $\operatorname{tr}(e^{-\sqrt{\Delta_0}t}|\ker \Delta_{\mathcal{F}})$. Since $\operatorname{tr}(e^{-\sqrt{\Delta_0}t}|\ker \Delta_{\mathcal{F}})$ is the inverse Mellin transform of $\Gamma(s)\zeta_{\Delta_0}(\frac{s}{2})$ with simple poles at $s = 1, -2n$ and double poles at $s = -2n - 1$ $(n = 0, 1, 2, \cdots)$, we deduce that the series $V^p(t)$ converges absolutely and has an asymptotic expansion around $t = 0$ as follows:

$$V^p(t) \overset{t\downarrow 0}{\sim} at^{-1} + \sum_{k=0}^{N}(b_k + c_k t \log t)t^{2k} + \mathcal{O}_1(t^{N-1}) + \mathcal{O}_2(t^{N-1})t\log t.$$

5. Proof of theorem 1.1

Proof. We fix a positive number $T > 0$. We consider the 2 series for $s \in \mathbb{C}$ such that $|\operatorname{Im}(s)| < T$

$$\theta_p^+(t) = V^p(t)e^{-sit},$$
$$\theta_p^-(t) = V^p(t)e^{sit}.$$

They play a role like a partition function.

We take the Mellin transform for the series and define the following functions

$$\xi_p^+(s,z) = \frac{e^{\frac{\pi}{2}iz}}{\Gamma(z)} \int_0^\infty \theta_p^+(t) t^{z-1} dt,$$
$$\xi_p^-(s,z) = \frac{e^{-\frac{\pi}{2}iz}}{\Gamma(z)} \int_0^\infty \theta_p^-(t) t^{z-1} dt.$$

Since $V^p(t)$ is convergent, we have for $\mathrm{Re}(z) > 1$

$$\xi_p^+(s,z) = \sum_{\mathrm{Im}(\rho)>T} (s-\rho)^{-z},$$
$$\xi_p^-(s,z) = \sum_{\mathrm{Im}(\rho)<-T} (s-\rho)^{-z}.$$

Next, we consider

$$\begin{aligned}
\xi_p^+(s,z) &= \frac{e^{\frac{\pi}{2}iz}}{\Gamma(z)} \int_0^\infty \theta_p^+(t) t^{z-1} dt \\
&= \frac{e^{\frac{\pi}{2}iz}}{\Gamma(z)} \left(\int_0^1 \theta_p^+(t) t^{z-1} dt + \int_1^\infty \theta_p^+(t) t^{z-1} dt \right), \\
\xi_p^-(s,z) &= \frac{e^{-\frac{\pi}{2}iz}}{\Gamma(z)} \int_0^\infty \theta_p^-(t) t^{z-1} dt \\
&= \frac{e^{-\frac{\pi}{2}iz}}{\Gamma(z)} \left(\int_0^1 \theta_p^-(t) t^{z-1} dt + \int_1^\infty \theta_p^-(t) t^{z-1} dt \right).
\end{aligned}$$

Since $V^p(t)$ is of rapid decay at infinity, the second terms are convergent for any $z \in \mathbb{C}$. Hence we have

$$\begin{aligned}
\xi_p^+(s,z) &= \frac{e^{\frac{\pi}{2}iz}}{\Gamma(z)} \left(\int_0^1 \theta_p^+(t) t^{z-1} dt + \int_1^\infty \theta_p^+(t) t^{z-1} dt \right) \\
&= \frac{e^{\frac{\pi}{2}iz}}{\Gamma(z)} \left(\frac{a}{z-1} - \frac{asi + b_0}{z} - \frac{b_0 si}{z+1} - \frac{c_0}{(z+1)^2} + \cdots \right) \\
&= \frac{ai}{z-1} + \eta_p^+(s,z),
\end{aligned}$$

where $\eta_p^+(s,z)$ is a meromorphic function of (s,z) for $|\mathrm{Im}(s)| < T$ and $z \in \mathbb{C}$ and is regular at $z = 0$. The meromorphic function $\eta_p^+(s,z)$ has only simple poles at $z = -2n-1$ $(n = 0, 1, 2, \cdots)$. The same result holds for $\xi_p^-(s,z)$ by the same argument:

$$\xi_p^-(s,z) = \frac{-ai}{z-1} + \eta_p^-(s,z).$$

Note that $\eta_p^-(s,z)$ is meromorphic in $|\mathrm{Im}(s)| < T$ and $z \in \mathbb{C}$ and is regular at $z = 0$. Since $\xi_p(s,z)$ differs from $\xi_p^+(s,z) + \xi_p^-(s,z)$ by the sum of the finite terms, we have that $\xi_p(s,z)$ is a meromorphic function for all $|\mathrm{Im}(s)| < T$ and all $z \in \mathbb{C}$ and is regular at $z = 0$. □

6. Zeta-regularized determinant

We recall the notion of the zeta-regularized determinant. Let $\Theta : V \to V$ be a linear operator acting on a complex vector space V of countable dimension. We assume that V is the direct sum of finite-dimensional Θ-invariant sub-spaces. Let $\mathrm{Sp}(\Theta)$ be the set of eigenvalues of Θ. The spectral zeta function associated with the operator Θ is defined by the analytic continuation of Dirichlet series

$$\zeta_\Theta(s) = \sum_{\lambda \neq 0 \in \mathrm{Sp}(\Theta)} \lambda^{-s} \text{ with } \lambda^{-s} = |\lambda|^{-s} e^{-is(\mathrm{Arg}\lambda)}, -\pi < \mathrm{Arg}\lambda \leq \pi.$$

We assume that the Dirichlet series converges absolutely on some right-half plane and has an analytic continuation to the half plane $\mathrm{Re}(s) > -\epsilon$ for some $\epsilon > 0$ which is holomorphic at $s = 0$. Under these conditions, we define a zeta-regularized determinant by

$$\det\nolimits_\infty(\Theta|V) := \exp\left(-\partial_s \zeta_\Theta(0)\right).$$

7. Dynamical zeta function on RFDS^3

Let $(M, \mathcal{F}, \phi, g_\mathcal{F})$ be the foliated dynamical system with a bundle-like metric which we discussed above. We define the dynamical zeta function for a RFDS^3 by the analytic continuation of the infinite product

$$\zeta_\mathcal{F}(s) = \prod_\gamma (1 - e^{-s \cdot l(\gamma)})^{-\epsilon_\gamma},$$

where γ runs over periodic orbits of ϕ and $l(\gamma)$ is the length of γ. Here, ϵ_γ is the index of a closed orbit.

7.1. **Index of a closed orbit.**

For a closed orbit γ of ϕ, we set an index

$$\epsilon_\gamma := \mathrm{sgn}\ \det(1 - T_x \phi^{l(\gamma)} | T_x \mathcal{F}) \quad \text{for } x \in \gamma,$$

where $T_x \phi^t : T_x \mathcal{F} \to T_{\phi^t(x)} \mathcal{F}$ is the differential of ϕ^t. It does not depend on the choice of the point $x \in \gamma$. We call a closed orbit γ **non-degenerate** in a sense that ϵ_γ is non-zero.

7.2. **Absolute convergent condition.**

It is known that the infinite product converges absolutely on $\mathrm{Re}(s) > h(\phi)$ where $h(\phi)$ is **topological entropy** and only if it is finite. Note that the topological entropy $h(\phi)$ is defined by

$$h(\phi) := \lim_{T \to +\infty} \frac{1}{T} \log N(T) \geq 0,$$

where $N(T)$ denotes the cardinality of orbits whose length is less than or equal to T, i.e. $N(T) = \mathrm{Card}\{\gamma | l(\gamma) \leq T\}$. We assume that the topological entropy $h(\phi)$ of a foliated dynamical system $(M, \mathcal{F}, \phi)$ is finite so that $\zeta(s)$ converges absolutely on the right-half plane.

8. Proof of theorem 1.2

8.1. **Dynamical Lefschetz trace formula.** For the foliated dynamical system $(M, \mathcal{F}, \phi, g_{\mathcal{F}})$ whose closed orbits are all non-degenerate, Alvarez-Lopez and Kordyukov developed the **dynamical Lefschetz trace formula**:

Proposition 8.1 ([5]). *For every test function $\varphi \in \mathcal{D}(\mathbb{R}) = C_0^\infty(\mathbb{R})$, the operator*

$$A_\varphi = \int_{\mathbb{R}} \varphi(t)\phi^{t*}dt$$

on $\bar{H}^i_{\mathcal{F}}(M)$ is of trace class. Setting:

$$\mathrm{Tr}(\phi^{t*}|\bar{H}^i_{\mathcal{F}}(M))(\varphi) = \mathrm{tr}A_\varphi$$

defines a distribution on $\mathbb{R}$. The following formula holds in $\mathcal{D}'(\mathbb{R})$:

$$\sum_{i=0}^{\dim \mathcal{F}} (-1)^i \mathrm{Tr}(\phi^{t*}|\bar{H}^i_{\mathcal{F}}(M)) = \chi_{\mathrm{Co}}(\mathcal{F}, g_{\mathcal{F}})\delta_0 + \sum_\gamma l(\gamma) \sum_{k\in\mathbb{Z}\setminus 0} \epsilon_\gamma \delta_{kl(\gamma)}.$$

Here $\chi_{\mathrm{Co}}(\mathcal{F}, \mu)$ denotes Connes' Euler characteristic of the foliation with respect to the bundle-like metric (c.f. [2]) and δ_τ is the Dirac delta function in $\mathcal{D}'(\mathbb{R})$ which is non-zero at τ.

The lemma 4.1 (Stone's theorem) leads to the corollary:

Corollary 8.1.1. *The following equality holds in $\mathcal{D}'(\mathbb{R}_{>0})$:*

$$\sum_{i=0}^{2}(-1)^i \mathrm{Tr}(\phi^{t*}|\bar{H}^i_{\mathcal{F}}(M)) = \sum_{i=0}^{2}(-1)^i \sum_{\rho\in \mathrm{Sp}(\Theta_i)} e^{\rho t}.$$

where Θ_i denotes the operator Θ acting on $\bar{H}^i_{\mathcal{F}}(M)$.

It is enough to show that the zeta-regularized determinant coincides with the infinite product of periodic orbits on some right-half plane of the topological entropy, *i.e.*,

$$\begin{aligned}\zeta_{\mathcal{F}}(s) &= \prod_\gamma (1 - e^{-sl(\gamma)})^{-\epsilon_\gamma} \\ &= \prod_{i=0}^{2} \det_\infty(s - \Theta|\bar{H}^i_{\mathcal{F}}(M))^{(-1)^{i+1}} \quad \text{for } \mathrm{Re}(s) > P,\end{aligned}$$

where P is a sufficiently large number $P > h(\phi)$. Then the assertion follows from the uniqueness of analytic continuation.

We apply the Laplace transform for the dynamical Lefschetz trace formula. We have

$$\mathcal{L}\left[\sum_{i=0}^{2}(-1)^i \sum_{\rho\in \mathrm{Sp}(\Theta_i)} t^{z-1}e^{\rho t}\right](s) = \Gamma(z)\sum_{i=0}^{2}(-1)^i \sum_{\rho\in \mathrm{Sp}(\Theta_i)} (s-\rho)^{-z} \qquad (1)$$

for the left hand side, and

$$\mathcal{L}\left[\sum_{\gamma}\sum_{n\in\mathbb{N}} l(\gamma)\epsilon_{\gamma}\delta_{nl(\gamma)}t^{z-1}\right](s) = \sum_{\gamma}\sum_{n\in\mathbb{N}}\frac{l(\gamma)\epsilon_{\gamma}}{e^{nl(\gamma)s}}(nl(\gamma))^{z-1} \tag{2}$$

for the right hand side. Both sides are defined for $\mathrm{Re}(z) > 1$ over where the former infinite series (1) is defined from the proof of theorem 1.1, and $\mathrm{Re}(s) > h(\phi)$ over where the latter infinite series (2) is defined. We denote by P a sufficiently large number bigger than $h(\phi)$.

Let $\mathrm{L}_{\delta-}$ be a contour consisting of the lower edge of the cut from $-\infty$ to $-\delta$, the circle $t = \delta e^{i\phi}$ for $-\pi \leq \phi \leq \pi$ and the upper edge of the cut from $-\delta$ to $-\infty$.

$$\int_{\mathrm{L}_{\delta-}} e^{\lambda t}t^{-z}dt = 2i\sin(z\pi)\int_{\delta}^{\infty} e^{-v}v^{-z}dv + I$$

where I denotes the integral along the circle $t = |\delta|$. Since I tends to zero as $\delta \to 0$, we have

$$\begin{aligned}\lim_{\delta\to 0}\int_{\mathrm{L}_{\delta-}} e^{\lambda t}t^{-z}dt &= 2i\sin(z\pi)\Gamma(1-z)\\ &= \frac{2\pi i}{\Gamma(z)}\end{aligned}$$

Hence we have the formula for $\lambda > 0$

$$\frac{\lambda^{z-1}}{\Gamma(z)} = \frac{1}{2\pi i}\lim_{\delta\to 0}\int_{\mathrm{L}_{\delta-}} e^{\lambda t}t^{-z}dt.$$

By applying the formula for the series (2), we get

$$\begin{aligned}\frac{1}{\Gamma(z)}\sum_{\gamma}\sum_{n\in\mathbb{N}}\frac{l(\gamma)\epsilon_{\gamma}}{e^{nl(\gamma)s}}(nl(\gamma))^{z-1} &= \frac{1}{2\pi i}\lim_{\delta\to 0}\int_{\mathrm{L}_{\delta-}}\left(\sum_{\gamma}\sum_{n\in\mathbb{N}} l(\gamma)\epsilon_{\gamma}e^{-nl(\gamma)(s-t)}\right)t^{-z}dt\\ &= \frac{-1}{2\pi i}\lim_{\delta\to 0}\int_{\mathrm{L}_{\delta-}}\frac{\zeta_{\mathcal{F}}^{'}}{\zeta_{\mathcal{F}}}(s-t)t^{-z}dt.\end{aligned}$$

Since the series (1) has a meromorphic extension and is holomorphic at $z = 0$ from theorem 1.1, we obtain the two equalities for $-\pi \leq \arg(t) \leq \pi$

$$\begin{aligned}\sum_{i=0}^{2}(-1)^i\xi_i(s,z) &= \frac{-1}{2\pi i}\lim_{\delta\to 0}\int_{\mathrm{L}_{\delta-}}\frac{\zeta_{\mathcal{F}}^{'}}{\zeta_{\mathcal{F}}}(s-t)t^{-z}dt,\\ \sum_{i=0}^{2}(-1)^i\partial_z\xi_i(s,0) &= \frac{1}{2\pi i}\lim_{\delta\to 0}\int_{\mathrm{L}_{\delta-}}\frac{\zeta_{\mathcal{F}}^{'}}{\zeta_{\mathcal{F}}}(s-t)\log(|t|e^{\arg(t)i})dt\end{aligned}$$

It remains to see the following:

$$\begin{aligned}
&\frac{1}{2\pi i}\lim_{\delta\to 0}\int_{\mathrm{L}_{\delta-}}\frac{\zeta_{\mathcal{F}}^{'}}{\zeta_{\mathcal{F}}}(s-t)\log(|t|e^{\arg(t)i})dt\\
&=\frac{1}{2\pi i}\int_{-\infty}^{0}\frac{\zeta_{\mathcal{F}}^{'}}{\zeta_{\mathcal{F}}}(s-t)(\log(|t|)-\pi i)dt+\frac{1}{2\pi i}\int_{0}^{-\infty}\frac{\zeta_{\mathcal{F}}^{'}}{\zeta_{\mathcal{F}}}(s-t)(\log(|t|)+\pi i)dt\\
&=\int_{0}^{-\infty}\frac{\zeta_{\mathcal{F}}^{'}}{\zeta_{\mathcal{F}}}(s-t)dt=-\int_{0}^{\infty}\frac{\zeta_{\mathcal{F}}^{'}}{\zeta_{\mathcal{F}}}(s+t)dt\\
&=\log(\zeta_{\mathcal{F}}(s))
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\zeta_{\mathcal{F}}(s)&=\prod_{i=0}^{2}\exp(-\partial_z\xi(s,0))^{(-1)^{i+1}}\\
&=\prod_{i=0}^{2}\det{}_{\infty}(s-\Theta|\bar{H}^{i}_{\mathcal{F}}(M))^{(-1)^{i+1}}
\end{aligned}$$

for $\mathrm{Re}(s)>P\geq h(\phi)$. Hence the theorem 1.2 follows from the uniqueness of the analytic continuation.

References

[1] C. Deninger. "Some analogies between number theory and dynamical systems on foliated spaces". In: *Doc. Math. J. DMV. Extra* 901 (1998), pp. 23–46.

[2] A. Connes. "Noncommutative geometry year 2000". In: *Visions in Mathematics*. Springer, 2000, pp. 481–559.

[3] Jesús A Álvarez López and Yuri A Kordyukov. "Long time behaviour of leafwise heat flow for Riemannian foliations". In: *Compositio Mathematica* 125.2 (2001), pp. 129–153.

[4] C. Deninger. "Number theory and dynamical systems on foliated spaces". In: *arXiv preprint math/0204110* (2002).

[5] Jesús A Álvarez López and Yuri A Kordyukov. "Distributional Betti numbers of transitive foliations of codimension one". In: *Foliations: Geometry and Dynamics*. World Scientific, 2002, pp. 159–183.

[6] C. Deninger. "Arithmetic geometry and analysis on foliated spaces". In: *arXiv preprint math/0505354* (2005).

[7] C. Deninger. "A dynamical systems analogue of Lichtenbaum's conjectures on special values of Hasse-Weil zeta functions (Analytic Number Theory)". In: (2006).

[8] C. Deninger. "Analogies between analysis on foliated spaces and arithmetic geometry". In: *Groups and Analysis* (2008), pp. 174–190.

[9] Masanori Morishita. *Knots and primes: an introduction to arithmetic topology*. Springer Science & Business Media, 2011.

Graduate School of mathematics
Kyushu University
744 Motooka, Nishi-ku,
Fukuoka 819-0395, Japan
E-mail address: j-kim@math.kyushu-u.ac.jp