ON EVEN-ODD AMICABLE PAIRS

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1. Introduction

Is it possible to have a friendship between a male and a female? — This is a long-standing unsolved problem in daily life asked in an innumerable number of times in literature, TV drama, movie, ... and even in psychology. For Pythagoreans, who associated numbers to real-life objects, males are represented by odd numbers, females are represented by even numbers and friendships are represented by amicable pairs. Mysteriously (too silly to say?), one of the unsolved problems in number theory also asks whether there are any even-odd amicable pairs or not.

An amicable pair is a pair of positive integers (M, N) such that the sum of the proper divisors of each member is equal to the other member. By writing $\sigma(n)$ for the sum of the divisors of n and $s(n) \coloneqq \sigma(n) - n$ for the sum of the proper divisors of n, this definition can be rephrased into the simultaneous equations

$$s(M) = N \quad \text{and} \quad s(N) = M$$

or, equivalently,

$$\sigma(M) = \sigma(N) = M + N.$$

Each member of an amicable pair is called *an amicable number*. The smallest amicable pair $(220, 284)$ is famous to be used as an example of "friendship" by Pythagoras.

Even though they have been known since ancient greek, most of the properties of amicable pairs are unknown. For example, it is not known whether there are infinitely many amicable pairs. For this problem, although it is a little bit ironic, several upper bounds of the number $A(X)$ of amicable numbers up to a given real number X has been obtained. The trivial bound for $A(X)$ is given by, of course, $A(X) \le X$. In 1955, Erdős [3] obtained the first non-trivial result, the asymptotic density of amicable numbers is zero, i.e. $A(X) = o(X)$ as $X \to \infty$. Erdős indicated that his method may imply more quantitative result and such a quantitative result was given by Rieger [14] and by Erdős and Rieger [4]. The method of Erdős and Rieger was sharpened by Pomerance [11] and it yielded that

$$A(X) \ll X \exp(-c(\log\log\log X \log\log\log\log X)^{\frac{1}{2}})$$

with some constant $c > 0$. Then, in 1981, a big leap was given by Pomerance [12]. He introduced a new method to bound $A(X)$ and arrived at

$$A(X) \ll X \exp(-c(\log X \log\log X)^{\frac{1}{3}}) \tag{1}$$

with some constant $c > 0$. In particular, this bound (1) shows that the sum of the reciprocals of amicable numbers converges. Also, the bound (1) settled Erdős' conjecture [3]: $A(X) \ll_k X(\log X)^{-k}$ for every positive integer k. The current best possible bound

$$A(X) \ll X \exp\left(-\left(\frac{1}{2} + o(1)\right)(\log X \log\log\log X)^{\frac{1}{2}}\right) \quad \text{as} \quad X \to \infty \tag{2}$$

is (again!) due to Pomerance [13]. (The triple logarithm on the right-hand side is not a typographical error for the double logarithm.)

We now return to the problem stated at the beginning of this note: even-odd amicable pairs, i.e. amicable pairs (M, N) for which M and N have the opposite parity. One of the reason why they are rare among all amicable pairs is partially explained by Lemma 1 below, which tells us that, roughly speaking, the even-odd amicable pairs are distributed at most quadratically with respect to all amicable pairs. Thus, for the number $B(X)$ of even-odd amicable pairs up to X (or, more clearly, the number of amicable pairs (M, N) with $M \not\equiv N \pmod 2$ and $\min(M, N) \leq X$), the trivial bound is given by $B(X) \ll X^{\frac{1}{2}}$. As far as the author knows, the only known non-trivial bound for $B(X)$ is given by a remark of Pollack [10], which pointed out that the method of Iannucci and Luca [8] may be used to bound $B(X)$. The method of Iannucci and Luca gives

$$B(X) \ll X^{\frac{1}{2}}(\log X)^{-\frac{3}{2}+o(1)} \quad \text{as} \quad X \to \infty.$$

The author announces that he obtained the following non-trivial bound for $B(X)$:

Theorem 1 (S.). *We have*

$$B(X) \ll X^{\frac{1}{2}} \exp(-c(\log X \log\log\log X)^{\frac{1}{2}})$$

with some constant $c > 0$.

Hence, friendships between a male and a female, at least in the world of integers, is infrequent than was known before. Theorem 1 is comparable to the current best possible bound (2) for the usual amicable numbers. The proof of Theorem 1 mainly follows the line of Pomerance's argument [13] with some additional features to deal with the quadratic nature and a bound (see Lemma 3) for the smooth values of $\sigma(n^k)$. Therefore, the method of Pomerance in [13] is extended quadratically in some sense.

In this note, instead of the proof of Theorem 1, we sketch a simpler argument following [12] to obtain a bound comparable to (1):

Theorem 2 (S.). *We have*

$$B(X) \ll X^{\frac{1}{2}} \exp(-c(\log X \log\log X)^{\frac{1}{3}})$$

with some constant $c > 0$.

Unfortunately, we cannot give some details of the proof and, in particular, cannot give the proof of Theorem 1 and Lemma 3 because of the page limitation. Those proofs will be given in the forthcoming preprint of the author. Also, in the proof of Theorem 2, the author couldn't follow the line of Pomerance's argument [12] fully for even-odd amicable pairs and we still appeal to Lemma 3, which was more directly handled in the paper [12, step (v), p.185–187].

2. Notation and convention

For positive integers d and n, we write $d \parallel n$ if $d \mid n$ and $(d, \frac{n}{d}) = 1$. A positive integer d is said to be *square-free* if $p \mid d \implies p \parallel d$ for every prime p and to be *square-full* if $p \mid d \implies p^2 \mid d$ for every prime p.

We use the following arithmetic functions defined for positive integers n: The function $\sigma(n)$ is the usual sum-of-divisors function, i.e. $\sigma(n)$ is the sum of all positive divisors of n. Let $s(n) := \sigma(n) - n$. The function $\tau(n)$ denotes the number of positive divisors of n. The function $p_{\max}(n)$ is defined by the largest prime factor of n if $n > 1$ and by $p_{\max}(1) = 1$ if $n = 1$. Let

$$\mathrm{sq}^\sharp(n) := \prod_{\substack{p^v \| n\\ v \geq 2}} p^v \quad \text{and} \quad \mathrm{sq}^\flat(n) := \prod_{p \| n} p \tag{3}$$

Note that $\mathrm{sq}^\sharp(n)$ is the largest square-full divisor of n, $\mathrm{sq}^\sharp(n)$ and $\mathrm{sq}^\flat(n)$ are coprime and $n = \mathrm{sq}^\sharp(n)\mathrm{sq}^\flat(n)$.

The letter X, L, K always denote real numbers with $X, L, K \geq 4$. We define

$$u := \frac{\log X}{\log L} \quad \text{and} \quad v := \frac{\log L}{\log K} \tag{4}$$

and let

$$\begin{aligned} \Psi(X, L) &:= \#\{n \leq X \mid p_{\max}(n) \leq L\}, \\ \Sigma_2^\flat(L, K) &:= \#\{n \leq L \mid n\colon \text{square-free},\ p_{\max}(\sigma(n^2)) \leq K\}. \end{aligned} \tag{5}$$

We will define several sets denoted in the calligraphic style $\mathscr{A}, \mathscr{B}, \mathscr{C}$, etc. For such sets, we denote their cardinalities by the corresponding roman style letters A, B, C, etc.

If Theorem, Lemma, Claim, etc. is stated with the phrase "where the implicit constant depends on $a, b, c, \ldots$", then every implicit constant in the corresponding proof also depends at most on $a, b, c, \ldots$ unless otherwise indicated.

3. Preliminary lemmas

Lemma 1. *Let (A, B) be an amicable pair. If A is even and B is odd, then there exist positive integers a, M, N such that*

$$A = 2^a M^2, \quad B = N^2, \quad M, N\colon \text{odd}.$$

Proof. Since (A, B) is an amicable pair, we have

$$\sigma(A) = \sigma(B) = A + B. \tag{6}$$

By the assumption that A is even and B is odd, we find that $A + B$ is odd. By (6), we find that both of $\sigma(A)$ and $\sigma(B)$ are odd. Suppose that p is an odd prime and $p^e \parallel A$. Then, $\sigma(p^e) \mid \sigma(A)$ so $\sigma(p^e)$ should be odd. Since p is odd, this implies

$$1 \equiv \sigma(p^e) \equiv 1 + p + \cdots + p^e \equiv e + 1 \pmod 2,$$

i.e. e is even. This shows that $A = 2^a M^2$ for some positive integer a and odd number M. We can deal with B similarly. This completes the proof. □

Lemma 2. *For real numbers X, L with $X, L \geq 4$ and $L \geq \log X$, we have*

$$\Psi(X, L) \ll X \exp(-u \log(u + e)),$$

where u is given by (4)*, $\Psi(X, L)$ is given by* (5) *and the implicit constant is absolute.*

Proof. This is well-known. For example, combine Theorem 7.6 of [9, p. 207] with Theorem 1 of [6] by using the asymptotic formula (1.8) given in [2]. □

The next lemma corresponds to Theorem 1.1 of [1]. We omit the proof because of the page limitation. This lemma can be proven using the same strategy as in Section 3 of [1] by using Hmyrova's result [7] on smooth values of polynomials at prime numbers. (Actually, the author couldn't follow Hmyrova's argument [7]. In particular, the inequality (24) at p. 117 of its English translation is obtained in a too convenient circumstance, which seems not the case for general $q_\nu^{(1)}$. However, we may apply the method of Friedlander [5] to prove Hmyrova's result.) Note that the admissible range $K \ge \log L$ of the next lemma is narrower than that of Theorem 1.1 of [1], which is $K \ge (\log\log L)^{1+\varepsilon}$ for any $\varepsilon > 0$. This defect is caused since the author couldn't extend the method of Section 2 of [1] to the next lemma.

Lemma 3 (S.)**.** *For real numbers L, K with $L, K \ge 4$ and $K \ge \log L$, we have*

$$\Sigma_2^\flat(L,K) \ll L\exp(-v\log\log(v+e)),$$

where v is given by (4)*, $\Sigma_2^\flat(L,K)$ is given by* (5) *and the implicit constant is absolute.*

The next lemma is a higher-degree analog of the estimate (1) of [13].

Lemma 4. *For $X \ge 1$, a positive integer k and a prime number P,*

$$\sum_{\substack{\varpi^e \le X \\ P \mid \sigma(\varpi^{ke})}} \varpi^{-e} \ll (\log X)^2 P^{-\frac{1}{k}}.$$

where the summation variables ϖ and e runs through prime numbers and positive integers, respectively, and the implicit constant depends only on k.

Proof. We first classify the terms according to the value of e as

$$\sum_{\substack{\varpi^e \le X \\ P \mid \sigma(\varpi^{ke})}} \varpi^{-e} = \sum_{e=1}^{O(\log X)} \sum_{\substack{\varpi \le X^{\frac{1}{e}} \\ P \mid \sigma(\varpi^{ke})}} \varpi^{-e} \tag{7}$$

and then we estimate the inner sum. We first observe that

$$P \mid \sigma(\varpi^{ke}) \implies P \le \sigma(\varpi^{ke}) < \varpi^{ke}(1 + \varpi^{-1} + \cdots) \le 2\varpi^{ke}.$$

Thus, we can restrict the range of ϖ as

$$\sum_{\substack{\varpi \le X^{\frac{1}{e}} \\ P \mid \sigma(\varpi^{ke})}} \varpi^{-e} = \sum_{\substack{(P/2)^{\frac{1}{ke}} < \varpi \le X^{\frac{1}{e}} \\ P \mid \sigma(\varpi^{ke})}} \varpi^{-e}.$$

By dissecting the sum dyadically,

$$\begin{aligned}\sum_{\substack{\varpi \le X^{\frac{1}{e}} \\ P \mid \sigma(\varpi^{ke})}} \varpi^{-e} &\ll (\log X^{\frac{1}{e}}) \sup_{(P/2)^{\frac{1}{ke}} \le U \le X^{\frac{1}{e}}} \sum_{\substack{U < \varpi \le 2U \\ P \mid \sigma(\varpi^{ke})}} \varpi^{-e} \\ &\ll (\log X^{\frac{1}{e}}) \sup_{(P/2)^{\frac{1}{ke}} \le U \le X^{\frac{1}{e}}} U^{-e} \sum_{\substack{U < \varpi \le 2U \\ \omega^{ke} + \cdots + 1 \equiv 0 \pmod{P}}} 1.\end{aligned}$$

Since P is prime, the congruence $\omega^{ke} + \cdots + 1 \equiv 0 \pmod{P}$ has only ke solutions at most. Thus, we can continue the above estimate as

$$\begin{aligned}\sum_{\substack{\varpi \le X^{\frac{1}{e}} \\ P \mid \sigma(\varpi^{ke})}} \varpi^{-e} &\ll (\log X^{\frac{1}{e}}) \sup_{(P/2)^{\frac{1}{ke}} \le U \le X^{\frac{1}{e}}} eU^{-e}(UP^{-1} + 1) \\ &= (\log X) \sup_{(P/2)^{\frac{1}{ke}} \le U \le X^{\frac{1}{e}}} (U^{1-e}P^{-1} + U^{-e}).\end{aligned}$$

Since the exponents of U on the right-hand side are non-positive, the supremum is attained at $U=(P/2)^{\frac{1}{ke}}$. Therefore,

$$\sum_{\substack{\varpi\le X^{\frac{1}{e}}\\ P|\sigma(\varpi^{ke})}} \varpi^{-e} \ll (\log X)((P^{\frac{1}{ke}}/P)P^{-\frac{1}{k}}+P^{-\frac{1}{k}}) \ll (\log X)P^{-\frac{1}{k}}.$$

On inserting this estimate into (7), we obtain the lemma. □

Lemma 5. *For $a\in\mathbb{N}$ and odd integers m and n, there are at most four odd prime pairs (p,q) such that the pair $(2^ap^2m^2,q^2n^2)$ is amicable and $(p,m)=(q,n)=1$.*

Proof. Assume that an odd prime pair (p,q) is given such that the pair $(2^ap^2m^2,q^2n^2)$ is amicable and $(p,m)=(q,n)=1$. Since the pair $(2^ap^2m^2,q^2n^2)$ is amicable,

$$\sigma(2^ap^2m^2)=\sigma(q^2n^2) \quad\text{and}\quad \sigma(2^ap^2m^2)-2^ap^2m^2=q^2n^2.$$

By using abbrevations $\mu:=2^am^2$ and $\nu:=n^2$, we may rewrite the last equations to

$$\sigma(p^2\mu)=\sigma(q^2\nu) \quad\text{and}\quad \sigma(p^2\mu)-p^2\mu=q^2\nu.$$

By using $\sigma(p^2)=p^2+p+1$, we can further rewrite these equations to

$$p^2\sigma(\mu)+p\sigma(\mu)+\sigma(\mu)=q^2\sigma(\nu)+q\sigma(\nu)+\sigma(\nu) \tag{8}$$

$$p^2s(\mu)+p\sigma(\mu)+\sigma(\mu)=q^2\nu. \tag{9}$$

By mutiplying (8) and (9) by ν and $\sigma(\nu)$, respectively, and taking the difference,

$$p^2(\sigma(\mu)\nu-s(\mu)\sigma(\nu))-p\sigma(\mu)s(\nu)-\sigma(\mu)s(\nu)=q\sigma(\nu)\nu+\sigma(\nu)\nu,$$

or, equivalently,

$$p^2(\sigma(\mu)\nu-s(\mu)\sigma(\nu))-p\sigma(\mu)s(\nu)-(\sigma(\mu)s(\nu)+\sigma(\nu)\nu)=q\sigma(\nu)\nu.$$

By taking the square,

$$(p^2(\sigma(\mu)\nu-s(\mu)\sigma(\nu))-p\sigma(\mu)s(\nu)-(\sigma(\mu)s(\nu)+\sigma(\nu)\nu))^2=q^2\nu\cdot\sigma(\nu)^2\nu$$

By substituting (9) again here,

$$\begin{aligned}&(p^2(\sigma(\mu)\nu-s(\mu)\sigma(\nu))-p\sigma(\mu)s(\nu)-(\sigma(\mu)s(\nu)+\sigma(\nu)\nu))^2\\ &\qquad\qquad-(p^2s(\mu)+p\sigma(\mu)+\sigma(\mu))\sigma(\nu)^2\nu=0.\end{aligned} \tag{10}$$

If we regard this equation as the quartic equation of the indeterminate p, then the linear term coefficient on the left-hand side is

$$2\sigma(\mu)s(\nu)(\sigma(\mu)s(\nu)+\sigma(\nu)\nu)-\sigma(\mu)\sigma(\nu)^2\nu. \tag{11}$$

This is odd since $\nu=n^2$ is odd and $\sigma(\mu),\sigma(\nu)$ are odd by $\sigma(\mu)=\sigma(2^am^2)$ and $\sigma(\nu)=\sigma(n^2)$. In particular, the coefficient (11) is non-zero. Thus, the equation (10) is not trivial. This implies that for a given (a,m,n), there are at most four possible values for p. For each of those p, the value of q is uniquely determined by (9). This completes the proof. □

Lemma 6. *For $X,K\ge 4$, we have*

$$\sum_{P>K^6}\frac{1}{P^{\frac{1}{2}}}\sum_{\substack{K^2<Q\le X^{\frac{1}{2}}\\ P|\sigma(Q^2)}}\frac{1}{Q}\ll\frac{\log X}{K},$$

where P,Q runs through prime numbers and the implicit constant is absolute.

Proof. We first decompose the sum as

$$\sum_{P>K^6}\frac{1}{P^{\frac12}}\sum_{\substack{K^2<Q\le X^{\frac12}\\ P\mid\sigma(Q^2)}}\frac{1}{Q}=\sum\nolimits_1+\sum\nolimits_2, \tag{12}$$

where

$$\sum\nolimits_1 := \sum_{P>K^6}\frac{1}{P^{\frac12}}\sum_{\substack{K^2<Q\le \min(P,X^{\frac12})\\ P\mid\sigma(Q^2)}}\frac{1}{Q},\quad \sum\nolimits_2 := \sum_{P>K^6}\frac{1}{P^{\frac12}}\sum_{\substack{P<Q\le X^{\frac12}\\ P\mid\sigma(Q^2)}}\frac{1}{Q}.$$

The sum $\sum_1$ can be estimated by using $\omega(n)\ll 1+\log n$ as

$$\sum\nolimits_1 \le \sum_{K^2<Q\le X^{\frac12}}\frac{1}{Q}\sum_{\substack{P\mid\sigma(Q^2)\\ Q\le P}}\frac{1}{P^{\frac12}} \ll \sum_{K^2<Q\le X^{\frac12}}\frac{\omega(\sigma(Q^2))}{Q^{\frac32}}\ll\frac{\log X}{K}. \tag{13}$$

On the other hand, for the sum $\sum_2$, we bound the inner sum as

$$\sum_{\substack{P<Q\le X^{\frac12}\\ P\mid\sigma(Q^2)}}\frac{1}{Q}\ll(\log X)\sup_{P\le U\le X^{\frac12}}U^{-1}\sum_{\substack{U<Q\le 2U\\ Q^2+Q+1\equiv 0\ (\mathrm{mod}\ P)}}1\ll\frac{\log X}{P}$$

since $U/P\ge 1$ in the above supremum. Therefore, $\sum_2$ can be bounded by

$$\sum\nolimits_2 \ll \sum_{P>K^6}\frac{(\log X)}{P^{\frac32}}\ll\frac{\log X}{K^3}. \tag{14}$$

By combining (12), (13) and (14), we obtain the lemma. □

4. Proof of the main theorem

Let $X\ge 4$ be a large real number, a be a fixed positive integer and

$$\begin{aligned}\mathscr{B}&=\mathscr{B}(X,a)\\ &:=\{(M,N)\mid(2^aM^2,N^2)\colon \text{amicable}, \max(2^aM^2,N^2)\le X,\ M,N\colon \text{odd}\}.\end{aligned} \tag{15}$$

We discard several parts of this set to introduce useful restrictions to the variables M,N and estimate the size of the discarded parts.

We first want to introduce the restriction

$$\min(p_{\max}(M),p_{\max}(N))>L^8, \tag{16}$$

for some real number $L\ge 4$ chosen later, We decompose $\mathscr{B}$ as $\mathscr{B}=\mathscr{B}^{(1)}\sqcup\mathscr{E}^{(1)}$, where

$$\mathscr{B}^{(1)}:=\{(M,N)\in\mathscr{B}\mid \min(p_{\max}(M),p_{\max}(N))>L^8\},$$
$$\mathscr{E}^{(1)}:=\{(M,N)\in\mathscr{B}\mid \min(p_{\max}(M),p_{\max}(N))\le L^8\}.$$

In the subsequent argument, we denote the part of $\mathscr{B}$ remaining after the i-th step of our discarding process by $\mathscr{B}^{(i)}$ and the part of $\mathscr{B}$ discarded at the i-th step by $\mathscr{E}^{(i)}$. We refer to each lemma for the i-th step of the discarding process by "Claim i".

Claim 1. *We have*

$$E^{(1)}\ll X^{\frac12}\exp\left(-\frac{1}{16}u\log u+O(u)\right),$$

where the implicit constant is absolute.

Proof. It suffices to bound the sizes of the sets

$$\{(M,N)\in\mathscr{B}\mid p_{\max}(M)\le L^8\},\quad \{(M,N)\in\mathscr{B}\mid p_{\max}(N)\le L^8\} \tag{17}$$

since these sets cover $\mathscr{E}^{(1)}$. For $(M,N)\in\mathscr{B}$, the value of M is determined uniquely by N and the value of N is determined uniquely by M. Also, recall that the variables M,N are bounded as $M,N\le X^{\frac{1}{2}}$ in (15). Thus, the sizes of (17) are bounded as

$$\le \#\{n\le X^{\frac{1}{2}}\mid p_{\max}(n)\le L^8\}=\Psi(X^{\frac{1}{2}},L^8).$$

Then, the claim follows by Lemma 2. □

We then want to introduce the restriction

$$\max(\mathrm{sq}^\sharp(M),\mathrm{sq}^\sharp(N))\le K^2 \tag{18}$$

for some real number $K\ge 4$ chosen later, where $\mathrm{sq}^\sharp(n)$ is defined as in (3). To this end, we introduce a decomposition $\mathscr{B}^{(1)}=\mathscr{B}^{(2)}\sqcup\mathscr{E}^{(2)}$, where

$$\mathscr{B}^{(2)}:=\{(M,N)\in\mathscr{B}^{(1)}\mid \max(\mathrm{sq}^\sharp(M),\mathrm{sq}^\sharp(N))\le K^2\},$$
$$\mathscr{E}^{(2)}:=\{(M,N)\in\mathscr{B}^{(1)}\mid \max(\mathrm{sq}^\sharp(M),\mathrm{sq}^\sharp(N))> K^2\}.$$

Claim 2. *We have*

$$E^{(2)}\ll X^{\frac{1}{2}}K^{-1},$$

where the implicit constant is absolute.

Proof. Note that every square-full number d can be written as $d=e^2f^3$ with positive integers e,f. Thus, for real number $z>0$, we have

$$\sum_{\substack{d>z\\ d\colon\text{square-full}}}\frac{1}{d}=\sum_{e=1}^{\infty}\frac{1}{e^3}\sum_{d^2>z/e^3}\frac{1}{d^2}\ll\sum_{e=1}^{\infty}\frac{1}{e^3}\left(\frac{e^3}{z}\right)^{\frac{1}{2}}\ll\frac{1}{z^{\frac{1}{2}}}.$$

Therefore, similarly to the proof of Claim 1, the assertion is reduced to the bound

$$\#\{n\le X^{\frac{1}{2}}\mid \mathrm{sq}^\sharp(n)>K^2\}\le\sum_{\substack{d>K^2\\ d\colon\text{square-full}}}\sum_{\substack{n\le X^{\frac{1}{2}}\\ d\mid n}}1\le\sum_{\substack{d>K^2\\ d\colon\text{square-full}}}\frac{X^{\frac{1}{2}}}{d}\ll X^{\frac{1}{2}}K^{-1}.$$

This completes the proof. □

We next introduce the restriction

$$p_{\max}(M,N)\le K^2, \tag{19}$$

where $p_{\max}(M,N):=p_{\max}((M,N))$. (We denote the greatest common divisor of M and N by (M,N).) We then decompose $\mathscr{B}^{(2)}$ as $\mathscr{B}^{(2)}=\mathscr{B}^{(3)}\sqcup\mathscr{E}^{(3)}$, where

$$\mathscr{B}^{(3)}:=\{(M,N)\in\mathscr{B}^{(2)}\mid p_{\max}(M,N)\le K^2\},$$
$$\mathscr{E}^{(3)}:=\{(M,N)\in\mathscr{B}^{(2)}\mid p_{\max}(M,N)> K^2\}.$$

Claim 3. *We have*

$$E^{(3)}\ll X^{\frac{1}{2}}(\log X)^2K^{-1},$$

where the implicit constant is absolute.

Proof. For $(M,N)\in\mathscr{B}^{(2)}$, since $(2^aM^2,N^2)$ is amicable, we have

$$(N^2,\sigma(N^2))=(N^2,s(N^2))=(N^2,2^aM^2)$$

so that

$$p_{\max}(M,N)>K^2 \implies p_{\max}(N^2,\sigma(N^2))>K^2.$$

Therefore, since N determines M uniquely, we have

$$E^{(3)}\le \#\{N\le X^{\frac12}\mid p_{\max}(N^2,\sigma(N^2))>K^2\}.$$

On writing $p_{\max}(N^2,\sigma(N^2))=P$, this quantity can be bounded by

$$=\sum_{K^2<P\le X^{\frac12}}\ \sum_{\substack{N\le X^{\frac12}\\ p_{\max}(N^2,\sigma(N^2))=P}}1\le \sum_{K^2<P\le X^{\frac12}}\ \sum_{\substack{N\le X^{\frac12}\\ P\mid(N^2,\sigma(N^2))}}1.$$

For a positive integer N, the condition $P\mid\sigma(N^2)$ implies the existence of some prime power ϖ^e such that $\varpi^e\parallel N$ and $P\mid\sigma(\varpi^{2e})$. Note that in this case, $\sigma(\varpi^{2e})$ is coprime to ϖ so $(\varpi^e,P)=1$, which implies $\varpi^eP\mid N$. Therefore, the last quantity is further bounded by using Lemma 4 with $k=2$ as

$$\begin{aligned}&\le\sum_{K^2<P\le X^{\frac12}}\ \sum_{\substack{\varpi^e\le X^{\frac12}/P\\ P\mid\sigma(\varpi^{2e})}}\ \sum_{\substack{N\le X^{\frac12}\\ \varpi^eP\mid N}}1\le X^{\frac12}\sum_{K^2<P\le X^{\frac12}}P^{-1}\sum_{\substack{\varpi^e\le X^{\frac12}\\ P\mid\sigma(\varpi^{2e})}}\varpi^{-e}\\ &\ll X^{\frac12}(\log X)^2\sum_{K^2<P\le X^{\frac12}}P^{-\frac32}\ll X^{\frac12}(\log X)^2K^{-1}.\end{aligned}$$

This completes the proof. □

In the following argument, we always assume

$$\textbf{(LK)}\qquad L>K.$$

Then, by (16), (18) and (**LK**), for every $(M,N)\in\mathscr{B}^{(3)}$, we have $p_{\max}(m)\parallel m$ and $p_{\max}(n)\parallel n$ and so we can write (M,N) as

$$\begin{aligned}&M=pm,\quad N=qn,\quad p:=p_{\max}(M),\quad q:=p_{\max}(N),\\ &\text{with conditions}\quad \max(p_{\max}(m),L^8)<p\quad\text{and}\quad \max(p_{\max}(n),L^8)<q.\end{aligned}\tag{20}$$

Note that this factorization is clearly unique and we have $(2p,m)=(2q,n)=1$ in this factorization. Also, if $(M,N)\in\mathscr{B}^{(3)}$, then we have

$$\max(m,n)\le X^{\frac12}\min(p,q)^{-1}\le X^{\frac12}L^{-8}.\tag{21}$$

We next introduce the restriction

$$\min(m,n)>L^4\tag{22}$$

under the notation (20). Write $\mathscr{B}^{(3)}=\mathscr{B}^{(4)}\sqcup\mathscr{E}^{(4)}$, where

$$\begin{aligned}\mathscr{B}^{(4)}&:=\{(M,N)\in\mathscr{B}^{(3)}\mid \min(m,n)>L^4\},\\ \mathscr{E}^{(4)}&:=\{(M,N)\in\mathscr{B}^{(3)}\mid \min(m,n)\le L^4\}.\end{aligned}$$

Claim 4. *We have*

$$E^{(4)}\ll X^{\frac12}L^{-4},$$

where the implicit constant is absolute.

Proof. By (21), it suffices to bound the sizes of

$$\{(M,N)\in\mathscr{B}^{(3)}\mid m\le L^4,\ n\le X^{\frac12}L^{-8}\},\ \{(M,N)\in\mathscr{B}^{(3)}\mid m\le X^{\frac12}L^{-8},\ n\le L^4\}.$$

By Lemma 5, the sizes of these sets are

$$\le 4\#\{(m,n)\in\mathbb{N}^2\mid m\le L^4,\ n\le X^{\frac12}L^{-8}\}\ll X^{\frac12}L^{-4}.$$

This completes the proof. □

We further introduce the restriction

$$\max(p_{\max}(\sigma(\mathrm{sq}^\flat(m)^2)),p_{\max}(\sigma(\mathrm{sq}^\flat(n)^2))))>K^6 \tag{23}$$

under the notation (20). Let $\mathscr{B}^{(4)}=\mathscr{B}^{(5)}\sqcup\mathscr{E}^{(5)}$, where

$$\mathscr{B}^{(5)}:=\{(M,N)\in\mathscr{B}^{(4)}\mid\ \max(p_{\max}(\sigma(\mathrm{sq}^\flat(m)^2)),p_{\max}(\sigma(\mathrm{sq}^\flat(n)^2))))>K^6\},$$
$$\mathscr{E}^{(5)}:=\{(M,N)\in\mathscr{B}^{(4)}\mid\ \max(p_{\max}(\sigma(\mathrm{sq}^\flat(m)^2)),p_{\max}(\sigma(\mathrm{sq}^\flat(n)^2))))\le K^6\}.$$

Claim 5. *We have*

$$E^{(5)}\ll X^{\frac12}(\log X)^2\exp\left(-\frac13 v\log\log v+O(v)\right),$$

where the implicit constant is absolute.

Proof. It suffices to bound the sizes of

$$\{(M,N)\in\mathscr{B}^{(4)}\mid p_{\max}(\sigma(\mathrm{sq}^\flat(m)^2))\le K^6\},$$
$$\{(M,N)\in\mathscr{B}^{(4)}\mid p_{\max}(\sigma(\mathrm{sq}^\flat(n)^2))\le K^6\}.$$

By writing

$$d:=\mathrm{sq}^\sharp(m),\quad \mu:=\mathrm{sq}^\flat(m),\quad e:=\mathrm{sq}^\sharp(n),\quad \nu:=\mathrm{sq}^\flat(n),$$

we can uniquely write $M=pd\mu$ and $N=qe\nu$ with p,q given in (20), square-full integers d,e and square-free numbers μ,ν with $(pd,\mu)=(qe,\nu)=1$. By recalling $\max(M,N)\le X^{\frac12}$, (18), (22), (**LK**) and that M and N determines their values each other, we find that it suffices to bound

$$\#\{(p,d,m)\mid pd\mu\le X^{\frac12},\ d\le K^2,\ d\mu>L^4,\ \mu\text{: square-free},\ p_{\max}(\sigma(\mu^2))\le K^6\}$$
$$\le\#\{(p,d,m)\mid pd\mu\le X^{\frac12},\ \mu>L^2,\ \mu\text{: square-free},\ p_{\max}(\sigma(\mu^2))\le K^6\}.$$

By Lemma 3, this cardinality can be bounded as

$$\le\sum_{p\le X^{\frac12}}\sum_{d\le X^{\frac12}/pL^2}\Sigma_2^\flat\left(\frac{X^{\frac12}}{pd},K^6\right)\le\sum_{p\le X^{\frac12}}\sum_{d\le X^{\frac12}/pL^2}\frac{X^{\frac12}}{pd}\exp\left(-\frac{v}{3}\log\log\frac{v}{3}\right)$$
$$\ll X^{\frac12}(\log X)^2\exp\left(-\frac13 v\log\log v+O(v)\right).$$

This completes the proof. □

We finally estimate the size of the whole remaining part $\mathscr{B}^{(5)}$:

Claim 6. *We have*

$$B^{(5)}\ll X^{\frac12}(\log X)^4K^{-1},$$

where the implicit constant is absolute.

Proof. We decompose the remaining part into two parts:

$$\mathscr{B}_{\mathsf{p}}^{(5)} := \{(M,N) \in \mathscr{B}^{(5)} \mid p > q\}, \quad \mathscr{B}_{\mathsf{q}}^{(5)} := \{(M,N) \in \mathscr{B}^{(5)} \mid p < q\},$$

where we used the notation (20) and we do not have $p = q$ by (16), (19) and (**LK**).

We only consider the part $\mathscr{B}_{\mathsf{p}}^{(5)}$ since the part $\mathscr{B}_{\mathsf{q}}^{(5)}$ can be estimated in a similar manner. Take $(M,N) \in \mathscr{B}_{\mathsf{p}}^{(5)}$ arbitrarily. By (23), we can find a prime number P with $P > K^6$ such that $P \mid \sigma(\mathrm{sq}^\flat(m)^2)$. Then, since $\mathrm{sq}^\flat(m)$ is square-free, there is an odd prime number Q such that $Q \mid \mathrm{sq}^\flat(m)$ and $P \mid \sigma(Q^2) = Q^2+Q+1$. This also implies $Q^2+Q+1 \ge P > K^6 > K^4+K^2+1$ so that $Q > K^2$. Also, since $\mathrm{sq}^\flat(m) \parallel m$ and $(2p,m)=1$, we have $P \mid \sigma(\mathrm{sq}^\flat(m)^2) \mid \sigma(m^2) \mid \sigma(2^aM^2)$. Recalling that $(2^aM^2, N^2)$ is amicable, we have $P \mid \sigma(2^aM^2) = \sigma(N^2)$. Therefore, there is an odd prime power R^e such that $R^e \parallel N$ and $P \mid \sigma(R^{2e})$. Note that $R \le p_{\max}(N) = q < p$. If $e \ge 2$, then by (18), $K^6 < P \le \sigma(R^{2e}) \le 2R^{2e} \le 2\mathrm{sq}^\sharp(n)^2 \le 2K^4$, which cannot hold, so that $R \parallel N$. This, in turn, implies $P \mid \sigma(R^2) = R^2+R+1$. This also implies $R^2+R+1 \ge P > K^6 > K^4+K^2+1$ so that $R > K^2$. We then recall $M = pm$ and then the definition of amicable pairs implies

$$N^2 = \sigma(2^aM^2) - 2^aM^2 = s(2^am^2)p^2 + \sigma(2^am^2)p + \sigma(2^am^2)$$

so by taking (mod R) reduction,

$$s(2^am^2)p^2 + \sigma(2^am^2)p + \sigma(2^am^2) \equiv 0 \pmod{R}. \tag{24}$$

Let $\mathscr{X}(a,m,R) \subset \mathbb{F}_R$ be the set of solutions of this congruence (24). Define $f \in \{0,1\}$ by $R^f := (s(2^am^2), \sigma(2^am^2), R)$. Since $R^f \mid \sigma(2^am^2) - s(2^am^2) = 2^am^2$ and R is odd, we have $R^f \mid m^2$. If $f = 1$, then $R \mid M$ and $R \mid N$, which contradicts $R > K^2$ and (19) so that $f = 0$. Therefore, (24) is a quadratic or linear equation in the finite field $\mathbb{F}_R$ and so we have

$$X(a,m,R) \le 2. \tag{25}$$

By combining the above arguments, we find an injective correspondence $M \mapsto (P,Q,R,m,p)$ with the restriction

$$M = pm, \quad P > K^6, \quad K^2 < Q \le X^{\frac{1}{2}}, \quad K^2 < R < p \le X^{\frac{1}{2}},$$
$$P \mid \sigma(Q^2), \quad P \mid \sigma(R^2), \quad Q \mid m, \quad p \pmod{R} \in \mathscr{X}(a,m,R)$$

Therefore, we have

$$B_{\mathsf{p}}^{(5)} \le \sum_{P>K^6} \sum_{\substack{K^2<Q\le X^{\frac{1}{2}} \\ P\mid\sigma(Q^2)}} \sum_{\substack{K^2<R\le X^{\frac{1}{2}} \\ P\mid\sigma(R^2)}} \sum_{\substack{m\le X^{\frac{1}{2}} \\ Q\mid m}} \sum_{\substack{R<p\le X^{\frac{1}{2}}/m \\ p \,(\mathrm{mod}\ R)\in\mathscr{X}(a,m,R)}} 1. \tag{26}$$

By taking care of the condition $R < p$ and using (25), we find that

$$\sum_{\substack{m\le X^{\frac{1}{2}} \\ Q\mid m}} \sum_{\substack{R<p\le X^{\frac{1}{2}}/m \\ p \,(\mathrm{mod}\ R)\in\mathscr{X}(a,m,R)}} 1 \ll \sum_{\substack{m\le X^{\frac{1}{2}} \\ Q\mid m}} \frac{X^{\frac{1}{2}}}{mR} \ll \frac{X^{\frac{1}{2}}\log X}{QR}.$$

On inserting this estimate into (26), we obtain

$$B_{\mathsf{p}}^{(5)} \ll \sum_{P>K^6} \sum_{\substack{K^2<Q\le X^{\frac{1}{2}} \\ P\mid\sigma(Q^2)}} \sum_{\substack{K^2<R\le X^{\frac{1}{2}} \\ P\mid\sigma(R^2)}} \frac{X^{\frac{1}{2}}\log X}{QR}$$

By using the $e=1$ part of Lemma 4 to the sum over R and then using Lemma 6,

$$B_{\mathsf{p}}^{(5)} \ll X^{\frac{1}{2}}(\log X)^3 \sum_{P>K^6} \frac{1}{P^{\frac{1}{2}}} \sum_{\substack{K^2<Q\le X^{\frac{1}{2}} \\ P\mid\sigma(Q^2)}} \frac{1}{Q} \ll \frac{X^{\frac{1}{2}}(\log X)^4}{K}.$$

This completes the bound for $B_{\mathsf{p}}^{(5)}$. □

We can now combine the above arguments to obtain Theorem 2:

Proof of Theorem 2. It suffices to prove

$$\begin{aligned}&\#\{(A,B)\in\mathbb{N}^2 \mid (A,B)\colon \text{amicable},\ A\colon \text{even},\ B\colon \text{odd},\ \max(A,B)\le X\}\\ &\ll X^{\frac{1}{2}}\exp(-c(\log X\log\log\log X)^{\frac{1}{3}})\end{aligned}$$

with some $c>0$ since if (A,B) is an amicable pair and $\min(A,B)\le X$, then

$$\max(A,B)=s(\min(A,B))\le\sigma(\min(A,B))\ll X\log\log X.$$

By using Lemma 1 and recalling (15), we have

$$\begin{aligned}&\#\{(A,B)\in\mathbb{N}^2 \mid (A,B)\colon \text{amicable},\ A\colon \text{even},\ B\colon \text{odd},\ \max(A,B)\le X\}\\ &=\sum_{a=1}^{O(\log X)} B(X,a)=\sum_{1\le a\le\frac{\log X}{\log\log X}} B(X,a)+\sum_{\frac{\log X}{\log\log X}<a\ll\log X} B(X,a).\end{aligned}$$

Since for $(M,N)\in\mathscr{B}(X,a)$, N is uniquely determined by a and M, we have

$$B(X,a)\le\#\{M\in\mathbb{N}\mid M\le 2^{-\frac{a}{2}}X^{\frac{1}{2}}\}\le 2^{-\frac{a}{2}}X^{\frac{1}{2}}$$

so that

$$\sum_{\frac{\log X}{\log\log X}<a\ll\log X} B(X,a)\le X^{\frac{1}{2}}\sum_{\frac{\log X}{\log\log X}<a\ll\log X} 2^{-\frac{a}{2}}\ll X^{\frac{1}{2}}\exp\left(-c\left(\frac{\log X}{\log\log X}\right)\right)$$

with some $c>0$. Thus, it suffices to show

$$\sum_{1\le a\le\frac{\log X}{\log\log X}} B(X,a)\ll X^{\frac{1}{2}}\exp(-c(\log X\log\log\log X)^{\frac{1}{3}}) \tag{27}$$

by using our preceding arguments. Choose the parameters L and K by

$$L:=\exp((\log X\log\log X)^{\frac{2}{3}})\quad\text{and}\quad K:=\exp((\log L)^{\frac{1}{2}})=\exp((\log X\log\log X)^{\frac{1}{3}})$$

Then, for sufficiently large X, this choice satisfies (**LK**) so this choice is available in our preceding arguments. Also, we have

$$u\log u\ge\frac{1}{4}(\log X\log\log X)^{\frac{1}{3}}\quad\text{and}\quad v\log\log v\ge(\log X\log\log X)^{\frac{1}{3}} \tag{28}$$

for large X. By definitions of sets $\mathscr{B}^{(i)}$ and $\mathscr{E}^{(i)}$, we have $B(X,a)=\sum_{i=1}^{5}E^{(i)}+B^{(5)}$. By Claim 2, Claim 3, Claim 4, Claim 6, we have

$$E^{(2)},E^{(3)},E^{(4)},B^{(5)}\ll X^{\frac{1}{2}}(\log X)^4\exp(-(\log X\log\log X)^{\frac{1}{3}}).$$

By Claim 1, Claim 5, (28), we have

$$E^{(1)} \ll X^{\frac{1}{2}} \exp\left(-\frac{1}{128}(\log X \log\log X)^{\frac{1}{3}}\right),$$
$$E^{(5)} \ll X^{\frac{1}{2}} \exp\left(-\frac{1}{4}(\log X \log\log X)^{\frac{1}{3}}\right)$$

for large X. Combining the above estimates, we arrive at (27) with $c = \frac{1}{128}$. $\square$

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