Irrationality exponents of certain alternating serries

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1 Introduction

Davison and Shallit [2] introduced the sequence $\{q_n\}$ of positive integers defined by the recurrence

$$q_0 = 1, \quad q_1 = w_0, \quad q_{n+1} = q_{n-1}(w_n q_n + 1) \quad (n \geq 1),$$

where $\{w_n\}$ is any sequence of positive integers. They gave the following regular cotinued fraction representing alternating series0

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{q_n q_{n-1}} = [0; w_0, w_1 q_0, w_2 q_1, w_3 q_2, \dots]$$

and proved its transecendence by using Roth's theorem. As a spcial case, transcendence of Cahen's constant

$$C = \sum_{n=0}^{\infty} \frac{(-1)^n}{S_n - 1},$$

where $S_0 = 2,\ S_{n+1} = S_n^2 - S_n + 1\ (n \geq 0)$ is Sylvester's sequence (cf.[7]), was established. Finch [5, Section 6.7] asked what can be said about the number

$$\sum_{n=0}^{\infty} \frac{1}{S_n - 1}.$$

Recently, Duverney, Kurosawa, and the author of this paper proved the following (see [3, Example 1.5]): For a positive integer l and algebraic numbers $a \neq 0$ and ρ with $S_n \neq \rho$ for all $n \geq 0$, the number

$$\sum_{n=0}^{\infty} \frac{a^n}{(S_n - \rho)^l}$$

is transcendental except when $l = a = 1$ and $\rho = 0$, and if so

$$\sum_{n=0}^{\infty} \frac{1}{S_n} = \frac{1}{2}.$$

For a sequence $\{w_n\}$ of positive integers and a sequence $\{y_n\}$ with $y_1 > 0$ of *nonzero* integers, we define

$$q_0 = 1, \;\; q_1 = w_0, \;\; q_{n+1} = q_{n-1}(w_n q_n^m + y_n) \quad (n \geq 1) \tag{1}$$

where m is a positive integer. We assume that

$$w_n + \frac{y_n}{q_n^m} > 1 \quad (n \geq 2), \tag{2}$$

so that $\{q_n\}_{n\geq 1}$ is an increasing sequence of positive integers. Moreover, since $\log q_{n+1} > m \log_n + \log_{n-1}$, we have $\log q_n > P_n$ for all $n \geq 2$, where $P_1 = 1, \; P_2 = m$ and $P_{n+1} = mP_n + P_{n-1}$ $(n \geq 2)$. Hence, there exists a constant $\gamma > 1$ such that

$$q_n > \gamma^{\alpha^n} \quad (n \geq 2), \tag{3}$$

where $\alpha \geq (1+\sqrt{5})/2$ and $\beta = -1/\alpha$ are the roots of the equation $X^2 - mX - 1 = 0$. We define the series

$$\xi = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{y_1 y_2 \cdots y_n}{q_n q_{n-1}}. \tag{4}$$

In this talk, we give exact value of the number ξ(cf.[6]), where the irrationality exponent $\mu(\alpha)$ of a real number α is defined by the supremum of the set of numbers μ for which the inequality

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^\mu}$$

has infinitely many rational solutions p/q. Every irrational number α satisfies $\mu(\alpha) \geq 2$. If $\mu(\alpha) > 2$, then α is transcendental by Roth's theorem. If $\mu(\alpha) = \infty$, then α is called a Liouville number.

We first expand the number ξ in the irregular continued fraction:

Lemma 1. *Let $\{q_n\}$ be the sequence defined by (1). Assume that the series (4) is convergent. Then we have*

$$\xi = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{y_1 y_2 \cdots y_n}{q_n q_{n-1}} = \frac{y_1}{w_0 +} \frac{y_2}{w_1 q_0 q_1^{m-1} +} \frac{y_3}{w_2 q_1 q_2^{m-1} +} \cdots .$$

We then apply the next lemma to the above continued fraction.

Lemma 2 ([4, Corollary 4])**.** *Let an infinite continued fraction*

$$\xi = \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \cdots \frac{a_n}{+ b_n +} \cdots$$

be convergent, where a_n and b_n are non-zero integers. Assume that

$$\sum_{n=0}^{\infty} \left| \frac{a_{n+1}}{b_n b_{n+1}} \right| < \infty$$

and

$$\lim_{n\to\infty} \frac{\log |a_n|}{\log |b_n|} = 0.$$

Then

$$\mu(\xi) = 2 + \limsup_{n\to\infty} \frac{\log |b_{n+1}|}{\log |b_1 b_2 \cdots b_n|}. \tag{5}$$

In this way, we find the following formula.

Theorem 1. *Let ξ be as in (4). Assume that*

$$\log |y_n| = o(\alpha^n). \tag{6}$$

Then we have

$$\mu(\xi) = 1 + \limsup_{n\to\infty} \frac{\log q_{n+1}}{\log q_n}.$$

Furthermore, we show an expression of $\log q_n$. Let P_n be the linear requrrent sequence defined by

$$P_1 = 1,\ P_2 = m,\ P_{n+1} = mP_n + P_{n-1} \quad (n \geq 2),$$

or equivalently,

$$P_n = \frac{\alpha^n - \beta^n}{\sqrt{D}} \quad (n \geq 0), \tag{7}$$

where

$$\alpha = \frac{m + \sqrt{D}}{2}, \quad \beta = \frac{m - \sqrt{D}}{2}$$

with $D = m^2 + 4$ are the roots of the equation $X^2 - mX - 1 = 0$.

Lemma 3. *Let $\{q_n\}$ be defined by (1). Then we have*

$$\log q_n = P_n \log w_0 + \sum_{k=1}^{n-1} P_{n-k} \log \left(w_k + \frac{y_k}{q_k^m} \right) \quad (n \geq 1). \tag{8}$$

Using this formula, we obtain the explicit value of the number ξ.

Theorem 2. *Make the same assumptions as in Theorem 1. Then we have*

$$\mu(\xi) = \begin{cases} 1+\alpha & \text{if } \displaystyle\sum_{k=0}^{\infty} \frac{\log\, w_k}{\alpha^k} < \infty, \\ 1+\alpha+\displaystyle\limsup_{n\to\infty} \frac{\displaystyle\sum_{k=0}^{n} \beta^{n-k} \log\, w_k}{\displaystyle\sum_{k=0}^{n-1} P_{n-k} \log\, w_k} & \text{otherwise.} \end{cases}$$

Corollary 1. *Every number ξ as in Theorem 1 is transcendental.*

Finally, we give few examples.

Example 1. *For any sequence $\{\epsilon_n\}$ of 1 or -1 with $\epsilon_1 = 1$, we define the sequence $\{q_n\}$ by*

$$q_0 = 1,\ \ q_1 = w_0,\ \ q_{n+1} = q_{n-1}(w_n q_n^m + \delta_n) \quad (n \geq 1),$$

where $\{w_n\}$ be any sequence of positive integers satisfying

$$\sum_{k=0}^{\infty} \frac{\log\, w_k}{\alpha^k} = +\infty$$

and $\delta_n = \epsilon_n/\delta_1 \cdots \delta_{n-1}$. Then we have by Theorem 2

$$\mu\left(\sum_{n=1}^{\infty} \frac{\epsilon_n}{q_n q_{n-1}}\right) = 1+\alpha.$$

As $\{w_n\}$, we can take for example any one of the following sequences;

$$\{n!\},\ \{f(n)\},\ \{a^{f(n)}\},\ \{\lfloor b^{\lambda^n} \rfloor\},$$

where $b > 1$ is an integer, $1 < \lambda < \alpha$, and $f(x)$ is a polynomial of x, possibly a constant, taking positive integral values at any positive integers.

Example 2. *For any positive integer a, we put $w_0 = a,\ w_n = q_{n-1}$ $(n \geq 1)$ and $y_n = a$ $(n \geq 1)$. We have by (1) with $m = 1$*

$$q_0 = 1,\ \ q_1 = a,\ \ q_{n+1} = q_{n-1}(q_{n-1} q_n + a)\ (n \geq 1), \tag{9}$$

The assumption (2) is automatically satisfied. Define the number ξ *by (4). We set* $s_n = q_{n+1}q_n + a$ $(n \geq 1)$. *Since* $q_{n+1}q_{n+2} = q_n q_{n+1}(q_n q_{n+1} + a)$ $(n \geq 0)$, *we find*

$$s_0 = 2a, \ \ s_{n+1} = s_n^2 - as_n + a \quad (n \geq 0).$$

Taking logarithm of both sides of (9) and using the resulting formula repeatedly, we have

$$\log q_n = c_5 2^n + o(2^n).$$

Appying Theorem 1, we obtain

$$\mu(\left(\sum_{n=0}^{\infty} \frac{a^n}{s_n - a}\right) = 3.$$

In the case of $a = 1$, *we have* $\mu(C) = 3$.

We note that, for any real λ with $1 + \alpha \leq \lambda \leq \infty$, we can construct uncountably many numbers ξ as in Theorem 1 having the irrationality exponent λ.

References

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