Algebraic independence of certain series related to integral parts of integral multiples of a real number

慶應義塾大学大学院 理工学研究科 田沼優佑 (Yusuke Tanuma)
Graduate School of Science and Technology, Keio University

1 Introduction

We denote by $[x]$ the integral part of real x, namely, the largest integer not exceeding x. Let ω be a real number. The generating function of the sequence $\{[k\omega]\}_{k=1}^{\infty}$ is the Hecke-Mahler series

$$h_\omega(z) = \sum_{k=1}^{\infty} [k\omega] z^k,$$

where z is complex with $|z| < 1$. Hecke [3] proved that, if ω is irrational, then $h_\omega(z)$ has the unit circle $|z| = 1$ as its natural boundary, which implies that $h_\omega(z)$ is transcendental over $\mathbb{C}(z)$. Mahler [7] proved that, if ω is a quadratic irrational number, then the number $h_\omega(\alpha)$ is transcendental, where α is a nonzero algebraic number inside the unit circle. The Hecke-Mahler series can be modified into two variables as follows. For any positive number ω, we define

$$H_\omega(z_1, z_2) = \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{[k_1\omega]} z_1^{k_1} z_2^{k_2}. \tag{1}$$

In the case where ω is a real quadratic irrational number, known are the following theorems on the arithmetic properties of the values of the Hecke-Mahler series.

Theorem 1 (Nishioka [11], see also Nishioka [12]). *Let ω be a positive quadratic irrational number. If α_1, α_2 are algebraic numbers with $0 < |\alpha_1| < 1$ and $0 < |\alpha_1||\alpha_2|^\omega < 1$, then the infinite set of the numbers*

$$\left\{ \frac{\partial^{l+l'} H_\omega}{\partial {z_1}^l \partial {z_2}^{l'}}(\alpha_1, \alpha_2) \;\middle|\; l, l' \geq 0 \right\}$$

is algebraically independent.

In what follows, we denote by $f^{(l)}(z)$ the derivative of $f(z)$ of order l. Letting $\alpha_2 = 1$ and $l' = 0$ in Theorem 1, Nishioka obtained the algebraic independence of the derivatives of the Hecke-Mahler series at any fixed nonzero algebraic number inside the unit circle.

Theorem 2 (Nishioka [11]). *Let ω be a real quadratic irrational number. If α is an algebraic number with $0 < |\alpha| < 1$, then the infinite set of the numbers $\{h_\omega^{(l)}(\alpha) \mid l \geq 0\}$ is algebraically independent.*

On the other hand, Masser proved the algebraic independence of the values of $h_\omega(z)$ at any nonzero distinct algebraic numbers inside the unit circle.

Theorem 3 (Masser [8]). *Let ω be a real quadratic irrational number. Then the infinite set of the numbers $\{h_\omega(\alpha) \mid \alpha \in \overline{\mathbb{Q}},\ 0 < |\alpha| < 1\}$ is algebraically independent.*

We denote by ω' the conjugate of the real quadratic irrational number ω. Tanaka and the author proved the following

Theorem 4 (Tanaka and Tanuma [13]). *Let ω be a real quadratic irrational number. Assume that ω satisfies $|\omega - \omega'| > 2$. Then the infinite set of the numbers $\{h_\omega^{(l)}(\alpha) \mid l \geq 0,\ \alpha \in \overline{\mathbb{Q}},\ 0 < |\alpha| < 1\}$ is algebraically independent.*

In this paper we give a sketch of the proof of the following

Theorem 5 (Tanuma [14]). *Let ω be a positive quadratic irrational number. Then the infinite set of the numbers*

$$\left\{ \frac{\partial^{l+l'} H_\omega}{\partial {z_1}^{l} \partial {z_2}^{l'}}(\alpha, 1) \;\middle|\; l, l' \geq 0,\ \alpha \in \overline{\mathbb{Q}},\ 0 < |\alpha| < 1 \right\}$$

is algebraically independent.

As a corollary, we can remove the assumption of Theorem 4 and obtain the algebraic independence of the "direct product" of the infinite sets treated in Theorems 2 and 3.

Corollary 1. *Let ω be a real quadratic irrational number. Then the infinite set of the numbers $\{h_\omega^{(l)}(\alpha) \mid l \geq 0,\ \alpha \in \overline{\mathbb{Q}},\ 0 < |\alpha| < 1\}$ is algebraically independent.*

For the arithmetic properties of the values of the Hecke-Mahler series for arbitrary irrational ω, the following three theorems are known.

Theorem 6 (Loxton and van der Poorten [5]). *Let ω be a real irrational number. Then the number $h_\omega(\alpha)$ is transcendental for any algebraic number α with $0 < |\alpha| < 1$.*

Theorem 7 (Flicker [2]). *For any real irrational number ω, there exists a real number λ, depending on ω, for which the following property holds: If $\alpha_1, \ldots, \alpha_n$ are algebraic numbers with $0 < |\alpha_i| < 1$ $(1 \leq i \leq n)$ such that $\log|\alpha_1|, \ldots, \log|\alpha_n|$ are linearly independent over $\mathbb{Q} + \lambda\mathbb{Q}$, then the numbers $h_\omega(\alpha_1), \ldots, h_\omega(\alpha_n)$ are algebraically independent.*

Let ω be expanded in the continued fraction

$$\omega = [a_0; a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}.$$

Theorem 8 (Nishioka [10]). *Let ω be a real irrational number. Suppose that the sequence $\{a_k\}_{k\geq 0}$ of partial quotients in the continued fraction expansion $[a_0; a_1, a_2, \ldots]$ of ω is unbounded. Let $\alpha_1, \ldots, \alpha_n$ be algebraic numbers with $0 < |\alpha_i| < 1$ $(1 \leq i \leq n)$. If α_i/α_j is not a root of unity for any i, j $(1 \leq i < j \leq n)$, then the numbers $h_\omega(\alpha_1), \ldots, h_\omega(\alpha_n)$ are algebraically independent.*

As mentioned in [2], although Loxton and van der Poorten [5] also stated that the number $\sum_{k=1}^{\infty} p([k\omega])\alpha^k$, where $p(X)$ is a non-constant polynomial with algebraic coefficients, is transcendental for any real irrational number ω and for any nonzero algebraic number α inside the unit circle, the proof is not valid. On the other hand, Theorem 5 implies the following

Theorem 9. *Let ω be a real quadratic irrational number and $S \subset \overline{\mathbb{Q}}[X]$ a set of non-constant polynomials with algebraic coefficients such that $S \cup \{1\}$ is linearly independent over $\overline{\mathbb{Q}}$. Put $f_p(z) = \sum_{k=1}^{\infty} p([k\omega])z^k$ $(p(X) \in S)$. Then the infinite set of the numbers $\{f_p^{(l)}(\alpha) \mid p(X) \in S,\ l \geq 0,\ \alpha \in \overline{\mathbb{Q}},\ 0 < |\alpha| < 1\}$ is algebraically independent.*

For the case where the sequence of partial quotients is unbounded, Theorem 8 implies Theorem 7. The case of quadratic irrational is a special case of the bounded case. Thus we have Table 1.

(Partial quotients of) ω / Arithmetic properties		unbounded	bounded	
			quadratic	not quadratic
transcendence		Loxton and van der Poorten		
Algebraic independence	values	Nishioka	Masser	Flicker
	derivatives	?	Theorem 5	?

Table 1: Known results for arithmetic properties of the values of Hecke-Mahler series

Not only the Hecke-Mahler series but also the generating function of the difference sequence $\{[(k+1)\omega] - [k\omega]\}_{k=0}^{\infty}$ is also studied by many authors. If $b - 2 < \omega < b - 1$ for some integer b greater than 1, then $[(k+1)\omega] - [k\omega] \in \{b-2, b-1\}$ for any $k \geq 0$. Hence $\sum_{k=0}^{\infty}([(k+1)\omega] - [k\omega])b^{-k}$ gives the b-ary expansion of a real number. We also consider exponential type Hecke-Mahler series

$$g_\omega(z) = \sum_{k=1}^{\infty} z^{[k\omega]}$$

for any positive number ω. If ω is a positive irrational number, then the generating function of $\{[(k+1)\omega] - [k\omega]\}_{k=0}^{\infty}$ coincides with the exponential type Hecke-Mahler series for $1/\omega$. Indeed, for any integers k and l, $[l/\omega] = k$ if and only if $k\omega \leq l < (k+1)\omega$. Hence $\#\{l \in \mathbb{Z}_{\geq 0} \mid l/\omega = k\} = [(k+1)\omega] - [k\omega]$ for any $k \geq 0$. Therefore $g_{1/\omega}(z) = \sum_{k=0}^{\infty}([(k+1)\omega] - [k\omega])z^k$.

Borwein and Borwein [1] modified the exponential type Hecke-Mahler series into two variables:

$$G_\omega(z_1, z_2) = \sum_{k=1}^{\infty} z_1^{[k\omega]} z_2^k.$$

If ω is a positive irrational number, then the two-variable Hecke-Mahler series and the two-variable exponential type Hecke-Mahler series satisfy a duality relation:

$$(1 - z_1)H_\omega(z_1, z_2) = z_1 G_{1/\omega}(z_1, z_2). \tag{2}$$

This relation is deduced from the following observation. Let $\{s_k\}_{k\geq 1}$ be a nondecreasing sequence of nonnegative integers with $\lim_{k\to\infty} s_k = +\infty$. Let $\{c_k\}_{k\geq 1}$ be a sequence in $\mathbb{C}$ with $|c_k| = o(1/s_k)$. Suppose that the sum $\sum_{k=1}^{\infty} |c_k|$ converges. Let

$$\varphi(y) = \sum_{k=1}^{\infty} \sum_{l=1}^{s_k} y^l (c_k - c_{k+1}).$$

Then we see that $\varphi(y) = \sum_{l=1}^{\infty} \sum_{k=t_l}^{\infty} (c_k - c_{k+1}) y^l$, where $t_l = \min\{k \in \mathbb{Z}_{\geq 0} \mid s_k \geq l\}$ $(l \geq 1)$, and so

$$\varphi(y) = \sum_{l=1}^{\infty} c_{t_l} y^l.$$

If $s_k = [k\omega]$, then $t_l = [l/\omega] + 1$. Letting $c_k = z_1^k$ and $y = z_2$, we have (2).

Using the relation (2), we obtain the following

Theorem 10 (Tanuma [14])**.** *Let ω be a positive quadratic irrational number. Then the infinite set of the numbers*

$$\left\{ \frac{\partial^{l+l'} G_\omega}{\partial {z_1}^l \partial {z_2}^{l'}}(\alpha, 1) \;\middle|\; l, l' \geq 0,\ \alpha \in \overline{\mathbb{Q}},\ 0 < |\alpha| < 1 \right\}$$

is algebraically independent.

Corollary 2. *Let ω be a positive quadratic irrational number. Then the infinite set of the numbers $\{g_\omega^{(l)}(\alpha) \mid l \geq 0,\ \alpha \in \overline{\mathbb{Q}},\ 0 < |\alpha| < 1\}$ is algebraically independent.*

Remark 1. By (2), we have $z g_\omega(z) = (1 - z) h_{1/\omega}(z)$ for any positive irrational number ω. Hence we see that $g_\omega(z)$ has the unit circle as its natural boundary if ω is positive irrational.

Some similar results to Corollary 2 are known for the power series $f(z) = \sum_{k=1}^{\infty} z^{e_k}$, where $\{e_k\}_{k=1}^{\infty}$ is an increasing sequence of nonnegative integers. For example, Nishioka proved the following

Theorem 11 (Nishioka [9])**.** *Let $f(z) = \sum_{k=1}^{\infty} z^{k!+k}$. Then the infinite set of the numbers $\{f^{(l)}(\alpha) \mid l \geq 0,\ \alpha \in \overline{\mathbb{Q}},\ 0 < |\alpha| < 1\}$ is algebraically independent.*

On the other hand, in the case where $\lim_{k\to\infty} e_{k+1}/e_k = 1$, Kaneko [4] proved the following

Theorem 12 (Kaneko [4])**.** *Let* $f_\varepsilon(z) = \sum_{k=1}^{\infty} z^{[k^{(\log k)^\varepsilon}]}$ *with* ε *positive. Let* $\beta > 1$ *be a Pisot or Salem number. Then the continuum set* $\{f_\varepsilon(\beta^{-1}) \mid \varepsilon \in \mathbb{R},\ \varepsilon \geq 1\}$ *is algebraically independent.*

However, in contrast with Corollary 2, it is difficult in general to treat the values at any algebraic number α with $0 < |\alpha| < 1$ in the case where $\lim_{k\to\infty} e_{k+1}/e_k = 1$. As far as the author knows, in the case where $\lim_{k\to\infty} e_{k+1}/e_k = 1$, Corollary 2 is the first result treating the algebraic independence of the values and the derivatives at any distinct algebraic numbers.

2 Lemmas

We denote by $R[z_1, \ldots, z_n]$ and by $R[\![z_1, \ldots, z_n]\!]$ the ring of polynomials and that of formal power series in the variables $z_1, \ldots, z_n$ with coefficients in a ring R, respectively. Let K be a field. We denote by $K(z_1, \ldots, z_n)$ the field of rational functions in the variables $z_1, \ldots, z_n$ with coefficients in K.

Let $\Omega = (\omega_{ij})$ be an $n \times n$ matrix with nonnegative integer entries. For $\boldsymbol{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$, we define a multiplicative transformation $\Omega : \mathbb{C}^n \to \mathbb{C}^n$ by

$$\Omega \boldsymbol{z} = \left(\prod_{j=1}^{n} z_j^{\omega_{1j}}, \prod_{j=1}^{n} z_j^{\omega_{2j}}, \ldots, \prod_{j=1}^{n} z_j^{\omega_{nj}} \right). \tag{3}$$

In the proof of Theorem 5 we use the following lemmas.

Lemma 1 (Nishioka [12])**.** *Let* K *be an algebraic number field. Suppose that* $f_1(\boldsymbol{z}), \ldots, f_m(\boldsymbol{z}) \in K[\![z_1, \ldots, z_n]\!]$ *converge in an* n*-polydisc* U *around the origin of* $\mathbb{C}^n$ *and satisfy the system of functional equations of the form*

$$\begin{pmatrix} f_1(\boldsymbol{z}) \\ \vdots \\ f_m(\boldsymbol{z}) \end{pmatrix} = A \begin{pmatrix} f_1(\Omega\boldsymbol{z}) \\ \vdots \\ f_m(\Omega\boldsymbol{z}) \end{pmatrix} + \begin{pmatrix} b_1(\boldsymbol{z}) \\ \vdots \\ b_m(\boldsymbol{z}) \end{pmatrix},$$

where A *is an* $m \times m$ *matrix with entries in* K *and* $b_i(\boldsymbol{z}) \in K(z_1, \ldots, z_n)$ $(1 \leq i \leq m)$*. Let* $\boldsymbol{\alpha}$ *be a point in* U *whose components are nonzero algebraic numbers. Assume that* Ω *and* $\boldsymbol{\alpha}$ *satisfy suitable conditions. Then, if* $f_1(\boldsymbol{z}), \ldots, f_r(\boldsymbol{z})$ $(r \leq m)$ *are linearly independent over* K *modulo* $K(z_1, \ldots, z_n)$*, then* $f_1(\boldsymbol{\alpha}), \ldots, f_r(\boldsymbol{\alpha})$ *are algebraically independent.*

Lemma 2 (Loxton and van der Poorten [6])**.** *Let* $\alpha_1, \ldots, \alpha_n$ *be algebraic numbers with* $0 < |\alpha_i| < 1$ $(1 \leq i \leq n)$*. Then there exist multiplicatively independent algebraic* *numbers* $\beta_1, \ldots, \beta_m$ *with* $0 < |\beta_j| < 1$ $(1 \le j \le m)$ *such that*

$$\alpha_i = \zeta_i \prod_{j=1}^{m} \beta_j^{\ell_{ij}} \qquad (1 \le i \le n),$$

where ζ_i $(1 \le i \le n)$ *are roots of unity and* ℓ_{ij} $(1 \le i \le n,\ 1 \le j \le m)$ *are nonnegative integers.*

Remark 2. In Lemma 2, at least one of $\ell_{i1}, \ldots, \ell_{im}$ is positive for any i.

The following lemma is a key to the proof of Theorem 5. We denote by $\{x\}$ the fractional part of a real number x.

Lemma 3 (Tanuma [14])**.** *Let* $Q_1(X), \ldots, Q_r(X) \in \mathbb{C}[X]$ *be not all constant and let* $Q(X_1, \ldots, X_r) = Q_1(X_1) + \cdots + Q_r(X_r) \in \mathbb{C}[X_1, \ldots, X_r]$. *Let* $a_1, \ldots, a_r$ *be real numbers with* $a_1 > a_2 > \cdots > a_r > 0$. *Then the function*

$$f(\tau) = Q(\{a_1\tau\}, \ldots, \{a_r\tau\})$$

is not constant on $\mathbb{R}$.

Proof. Let i_0 be the smallest integer such that $Q_{i_0}(X)$ is not constant and set $I_0 = [0, 1/a_{i_0})$. Let

$$\widehat{f}(\tau) = Q(a_1\tau, \ldots, a_r\tau) = Q_1(a_1\tau) + \cdots + Q_{i_0}(a_{i_0}\tau) + \cdots + Q_r(a_r\tau).$$

Then $\widehat{f}(\tau)$ is a polynomial in τ, and $f(\tau) = \widehat{f}(\tau)$ on the interval I_0 by the choice of i_0 and the fact that $\{a_i\tau\} = a_i\tau$ $(i_0 \le i \le r)$ for any $\tau \in I_0$. If $\hat{f}(\tau)$ is not constant, then $f(\tau)$ takes a value different from c when τ varies on I_0.

If $\widehat{f}(\tau)$ is constant, then $\widehat{f}(\tau) = f(\tau) = f(0)$ on the interval I_0. Letting $I_1 = [1/a_{i_0}, 1/a_{i_0+1}) \cap [1/a_{i_0}, 2/a_{i_0})$, where $a_{r+1} = a_r/2$, we see that $\widehat{f}(\tau) = f(0)$ also on the interval I_1 since $\widehat{f}(\tau)$ is a polynomial in τ. For any $\tau \in I_1$, we have

$$(\{a_{i_0}\tau\}, \{a_{i_0+1}\tau\}, \ldots, \{a_r\tau\}) = (a_{i_0}\tau - 1, a_{i_0+1}\tau, \ldots, a_r\tau)$$

and hence, by the choice of i_0, we have $f(\tau) - \widehat{f}(\tau) = Q_{i_0}(a_{i_0}\tau - 1) - Q_{i_0}(a_{i_0}\tau)$. Therefore

$$f(\tau) = Q_{i_0}(a_{i_0}\tau - 1) - Q_{i_0}(a_{i_0}\tau) + f(0)$$

on the interval I_1. Since $Q_{i_0}(a_{i_0}\tau - 1)$ and $Q_{i_0}(a_{i_0}\tau)$ are distinct as polynomials in τ, $f(\tau)$ takes at least two values when τ varies on the interval $[0, 1/a_{r+1}) \supset I_0 \cup I_1$. $\square$

Lemma 4 (Tanaka and Tanuma [13])**.** *Let* ω *be a positive irrational number and let* $s_1, \ldots, s_r$ *be positive integers. Then, for any real number* τ, *there exists an increasing sequence* $\{k_\nu\}_{\nu \ge 0}$ *of positive integers such that*

$$\lim_{\nu \to \infty} (\{s_1 k_\nu \omega\}, \ldots, \{s_r k_\nu \omega\}) = (\{s_1\tau\}, \ldots, \{s_r\tau\}),$$

where each component of the left-hand side approaches the corresponding component of the right-hand side from the right.

3 Sketch of the proof of Theorem 5

Let $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $C(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ for any positive integer a. Define $\begin{pmatrix} b & c \\ d & e \end{pmatrix} \eta = (b\eta + c)/(d\eta + e)$, where b, c, d, e are nonnegative integers. For any positive irrational number η, we have

$$H_{B\eta}(z_1, z_2) \equiv -H_\eta(B(z_1, z_2)) \pmod{\mathbb{Q}(z_1, z_2)} \tag{4}$$

and

$$H_{C(a)\eta}(z_1, z_2) \equiv H_\eta(C(a)(z_1, z_2)) \pmod{\mathbb{Q}(z_1, z_2)}, \tag{5}$$

where $B(z_1, z_2)$ and $C(a)(z_1, z_2)$ are defined by (3) (cf. Masser [8]).

Suppose that the positive quadratic irrational ω is expanded in the continued fraction $\omega = [a_0; a_1, a_2, \ldots]$, where a_0 is a nonnegative integer and $\{a_k\}_{k\geq 1}$ is an eventually periodic sequence of positive integers. Then there exist even positive integers μ and ν such that $\{a_k\}_{k\geq \mu}$ is purely periodic with period ν. Let $\chi = [0; a_{\mu+1}, a_{\mu+2}, \ldots]$. Then we see that

$$\omega = [a_0; a_1, a_2, \ldots, a_\mu, \chi] = C(a_0)BC(a_1)BC(a_2)\cdots BC(a_\mu)\chi$$

and

$$\chi = [0; a_{\mu+1}, a_{\mu+2}, \ldots, a_{\mu+\nu}, \chi] = BC(a_{\mu+1})\cdots BC(a_{\mu+\nu})\chi.$$

Let

$$S = \begin{pmatrix} p & q \\ r & s \end{pmatrix} = C(a_\mu)BC(a_{\mu-1})B\cdots C(a_1)BC(a_0)$$

and

$$T^{(1)} = C(a_{\mu+\nu})BC(a_{\mu+\nu-1})B\cdots C(a_{\mu+1})B.$$

Then, by (4) and (5), we see that

$$H_\omega(z_1, z_2) \equiv H_\chi(S(z_1, z_2)) \qquad \pmod{\mathbb{Q}(z_1, z_2)}$$

and

$$H_\chi(z_1, z_2) \equiv H_\chi(T^{(1)}(z_1, z_2)) \qquad \pmod{\mathbb{Q}(z_1, z_2)}.$$

For any positive integer m, we define

$$T^{(m)} = \operatorname{diag}\big(\underbrace{T^{(1)}, \ldots, T^{(1)}}_{m}\big).$$

Let $\beta_1, \ldots, \beta_m$ be multiplicatively independent algebraic numbers with $0 < |\beta_j| < 1$ $(1 \leq j \leq m)$ and let

$$\boldsymbol{z}_0 = (\beta_1^p, \beta_1^r, \beta_2^p, \beta_2^r, \ldots, \beta_m^p, \beta_m^r). \tag{6}$$

Let $\boldsymbol{x} = (x_1, \ldots, x_m)$, $\boldsymbol{y} = (y_1, \ldots, y_m)$ be variables. Let $\zeta_1, \ldots, \zeta_n$ be roots of unity and $M_1, \ldots, M_n$ non-constant monomials in m variables. Define

$$F_i(\boldsymbol{z}) = H_\chi(\zeta_i^p M_i(\boldsymbol{x}), \zeta_i^r M_i(\boldsymbol{y})) = \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{[k_1\chi]} (\zeta_i^p M_i(\boldsymbol{x}))^{k_1} (\zeta_i^r M_i(\boldsymbol{y}))^{k_2} \quad (1 \le i \le n), \quad (7)$$

where $\boldsymbol{z} = (x_1, y_1, x_2, y_2, \ldots, x_m, y_m)$.

Lemma 5 (Masser [8, Lemma 3.3])**.** *There exists a positive power T of $T^{(m)}$ such that*

$$F_i(\boldsymbol{z}) \equiv F_i(T\boldsymbol{z}) \qquad (\mathrm{mod}\ \overline{\mathbb{Q}}(\boldsymbol{z})) \quad (8)$$

for any i $(1 \le i \le n)$.

The matrix T in Lemma 5 can be written as

$$T = \mathrm{diag}\Bigg(\underbrace{\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}, \ldots, \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}}_{m}\Bigg).$$

Let $D_j = x_j \partial/\partial x_j$ and $D'_j = y_j \partial/\partial y_j$ $(1 \le j \le m)$. Differentiating both sides of (8), we see that $D_1^{l_1} {D'_1}^{l'_1} \cdots D_m^{l_m} {D'_m}^{l'_m} F_i(\boldsymbol{z})$ $(0 \le l_1 + l'_1 + \cdots + l_m + l'_m \le L,\ 1 \le i \le n)$ satisfy a system of functional equations as in Lemma 1 for any $L \ge 0$. Moreover, We can show that the matrix T and the point $\boldsymbol{z}_0$ satisfy required conditions in Lemma 1 (cf. Masser [8]).

Sketch of the proof of Theorem 5. Let $\alpha_1, \ldots, \alpha_n$ be any nonzero distinct algebraic numbers with $|\alpha_i| < 1$ $(1 \le i \le n)$. Since the algebraic independency of $\{\partial^{l+l'} H_\omega / \partial {z_1}^l \partial {z_2}^{l'} (\alpha_i, 1) \mid 1 \le i \le n,\ 0 \le l, l' \le L\}$ is equivalent to that of $\{D^l {D'}^{l'} H_\chi(S(\alpha_i, 1)) \mid 1 \le i \le n,\ 0 \le l, l' \le L\}$, it is sufficient to show that $\{D^l {D'}^{l'} H_\chi(S(\alpha_i, 1)) \mid 1 \le i \le n,\ 0 \le l, l' \le L\}$ is algebraically independent for any sufficiently large L. For the $\alpha_1, \ldots, \alpha_n$, let $\beta_1, \ldots, \beta_m$, $\zeta_1, \ldots, \zeta_n$, and ℓ_{ij} $(1 \le i \le n,\ 1 \le j \le m)$ be as in Lemma 2 and let $\boldsymbol{z}_0$ be defined by (6). Let $M_i(\boldsymbol{x}) = x_1^{\ell_{i1}} \cdots x_m^{\ell_{im}}$. By (7) and Remark 2 after Lemma 2 we have

$$D_{j_i}^l {D'_{j_i}}^{l'} F_i(\boldsymbol{z}_0) = \ell_{ij_i}^{l+l'} D^l {D'}^{l'} H_\chi(S(\alpha_i, 1)),$$

where $\ell_{ij_i} > 0$, $D = z_1 \partial/\partial z_1$, and $D' = z_2 \partial/\partial z_2$. Hence the algebraic independency of $\{D^l {D'}^{l'} H_\chi(S(\alpha_i, 1)) \mid 1 \le i \le n,\ 0 \le l, l' \le L\}$ is equivalent to that of $\{D_{j_i}^l {D'_{j_i}}^{l'} F_i(\boldsymbol{z}_0) \mid 1 \le i \le n,\ 0 \le l, l' \le L\}$. By Lemma 1, it is enough to show that $\{D_{j_i}^l {D'_{j_i}}^{l'} F_i(\boldsymbol{z}) \mid 1 \le i \le n,\ 0 \le l, l' \le L\}$ is linearly independent over $\overline{\mathbb{Q}}$ modulo $\overline{\mathbb{Q}}(\boldsymbol{z})$.

On the contrary, we assume that $\{D_{j_i}^l {D'_{j_i}}^{l'} F_i(\boldsymbol{z}) \mid 1 \le i \le n,\ 0 \le l, l' \le L\}$ is linearly dependent over $\overline{\mathbb{Q}}$ modulo $\overline{\mathbb{Q}}(\boldsymbol{z})$. Then there exist algebraic numbers $\lambda_{ill'}$ $(1 \le i \le n,\ 0 \le l, l' \le L)$, not all zero, and a rational function $R(\boldsymbol{z}) \in \overline{\mathbb{Q}}(\boldsymbol{z})$ such that

$$\sum_{i=1}^{n} \sum_{l=0}^{L} \sum_{l'=0}^{L} \lambda_{ill'} D_{j_i}^l {D'_{j_i}}^{l'} F_i(\boldsymbol{z}) = R(\boldsymbol{z}). \quad (9)$$

We take a sufficiently large positive integer t and attempt a specialization of the form

$$\boldsymbol{x} = (z^{pt}, z^{pt^2}, \ldots, z^{pt^m}), \quad \boldsymbol{y} = (z^{rt}, z^{rt^2}, \ldots, z^{rt^m})$$

for a single variable z. Let $t_i = \sum_{j=1}^{m} \ell_{ij} t^j$ $(1 \le i \le n)$. Then $M_i(\boldsymbol{x}) = z^{pt_i}$ and $M_i(\boldsymbol{y}) = z^{rt_i}$. We take t so large that, if $M_i \neq M_j$, then $t_i \neq t_j$ $(1 \le i < j \le n)$. Then (9) is specialized to a relation

$$\sum_{i=1}^{n} \sum_{l=0}^{L} \sum_{l'=0}^{L} \lambda_{ill'} \ell_{ij_i}^{l+l'} D^l D'^{l'} H_\chi(S(\zeta_i z^{t_i}, 1)) = R'(z),$$

where $R'(z) = R(z^{pt}, z^{rt}, \ldots, z^{pt^m}, z^{rt^m})$. Since $\lambda_{ill'}$ $(1 \le i \le n,\ 0 \le l, l' \le L)$ are not all zero, $\{D^l D'^{l'} H_\chi(S(\zeta_i z^{t_i}, 1)) \mid 1 \le i \le n,\ 0 \le l, l' \le L\}$ is linearly dependent over $\overline{\mathbb{Q}}$ modulo $\overline{\mathbb{Q}}(z)$. Hence we see that $\{D^l D'^{l'} H_\omega(\zeta_i z^{t_i}, 1) \mid 1 \le i \le n,\ 0 \le l, l' \le L\}$ is linearly dependent over $\overline{\mathbb{Q}}$ modulo $\overline{\mathbb{Q}}(z)$. Since the $\overline{\mathbb{Q}}$-vector spaces generated respectively by $\{D^l D'^{l'} H_\omega(\zeta_i z^{t_i}, 1) \mid 1 \le i \le n,\ 0 \le l, l' \le L\}$ and by $\{\sum_{k=1}^{\infty} k^l [k\omega]^{l'} (\zeta_i z^{t_i})^k \mid 1 \le i \le n,\ 0 \le l \le L,\ 1 \le l' \le L+1\}$ are equal, there exist algebraic integers $\lambda'_{ill'}$ $(1 \le i \le n,\ 0 \le l \le L,\ 1 \le l' \le L+1)$, not all zero, and a rational function $R^*(z) \in \overline{\mathbb{Q}}(z)$ such that

$$\sum_{i=1}^{n} \sum_{l=0}^{L} \sum_{l'=1}^{L+1} \lambda'_{ill'} \sum_{k=1}^{\infty} k^l [k\omega]^{l'} (\zeta_i z^{t_i})^k = \sum_{k=1}^{\infty} a_k z^k = R^*(z), \tag{10}$$

where

$$a_k = \sum_{\substack{1 \le i \le n \\ t_i | k}} \sum_{l=0}^{L} \sum_{l'=1}^{L+1} \lambda'_{ill'} \left(\frac{k}{t_i}\right)^l \left(\frac{k\omega}{t_i} - \left\{\frac{k\omega}{t_i}\right\}\right)^{l'} \zeta_i^{k/t_i}$$

and $\{x\}$ denotes the fractional part of real x. Let N be a positive integer such that $\zeta_1^N = \cdots = \zeta_n^N = 1$. Let $\{t'_1, \ldots, t'_r\}$ be the maximum subset of $\{t_1, \ldots, t_n\}$ such that $t'_1 < t'_2 < \cdots < t'_r$. Let $T_i = \{j \mid t_j = t'_i\}$ $(1 \le i \le r)$. Then ζ_j $(j \in T_i)$ are distinct for each i, since $\alpha_1, \ldots, \alpha_n$ are distinct. Put $s = t'_1 \cdots t'_r N$ and $s_i = s/t'_i$ $(1 \le i \le r)$. Then $s_1 > s_2 > \cdots > s_r$. Noting that $\{1, \ldots, n\}$ is a disjoint union of $T_1, \ldots, T_r$, for any $k \ge 0$ and for any fixed nonnegative integer h, we see that

$$a_{sk+h} = \sum_{i=1}^{r} \sum_{l=0}^{L} \sum_{l'=1}^{L+1} \lambda_{ill'}^{(h)} \left(\frac{sk+h}{t'_i}\right)^l \left(\frac{(sk+h)\omega}{t'_i} - \left\{\frac{(sk+h)\omega}{t'_i}\right\}\right)^{l'}, \tag{11}$$

where

$$\lambda_{ill'}^{(h)} = \begin{cases} \sum_{j \in T_i} \lambda'_{jll'} \zeta_j^{h/t'_i}, & \text{if } t'_i | h, \\ 0, & \text{otherwise,} \end{cases}$$

for $1 \le i \le r$, $0 \le l \le L$, and $1 \le l' \le L+1$.

We assert that $\lambda_{ill'}^{(h)}=0$ $(1\le i\le r,\ 0\le l\le L,\ 1\le l'\le L+1)$ for any $h\ge 0$. If this is the case, we can deduce a contradiction as follows. For each i with $1\le i\le r$, let $h=t_i'k$ with $k\ge 0$. Then, for any $0\le l\le L,\ 1\le l'\le L+1$, we have

$$\sum_{j\in T_i}\lambda'_{jll'}\zeta_j^{h/t_i'}=\sum_{j\in T_i}\lambda'_{jll'}\zeta_j^{k}=0$$

for any $k\ge 0$. Since ζ_j $(j\in T_i)$ are distinct, by the non-vanishing of Vandermonde determinant, we see that $\lambda'_{ill'}=0$ for any $1\le i\le n,\ 0\le l\le L$, and $1\le l'\le L+1$, which is a contradiction.

On the contrary, we assume that $\lambda_{ill'}^{(h_0)}$ $(1\le i\le r,\ 0\le l\le L,\ 1\le l'\le L+1)$ are not all zero for some nonnegative integer h_0. For each i with $1\le i\le r$ we see that

$$\begin{aligned}&\sum_{l=0}^{L}\sum_{l'=1}^{L+1}\lambda_{ill'}^{(h_0)}\left(\frac{sk+h_0}{t_i'}\right)^{l}\left(\frac{(sk+h_0)\omega}{t_i'}-\left\{\frac{(sk+h_0)\omega}{t_i'}\right\}\right)^{l'}\\ =&Q_{d_ii}(\{(sk+h_0)\omega/t_i'\})k^{d_i}+\cdots+Q_{0i}(\{(sk+h_0)\omega/t_i'\}),\end{aligned}$$

where $Q_{d_ii}(X),\dots,Q_{0i}(X)\in\overline{\mathbb{Q}}[X]$ with $Q_{d_ii}(X)\ne 0$. In addition, at least one of $Q_{d_ii}(X),\dots,Q_{0i}(X)$ is not constant for any i such that $\lambda_{ill'}^{(h_0)}$ $(0\le l\le L,\ 1\le l'\le L+1)$ are not all zero. Then by (11) we see that

$$\begin{aligned}a_{sk+h_0}=&\sum_{i=1}^{r}\big(Q_{d_ii}(\{(sk+h_0)\omega/t_i'\})k^{d_i}+\cdots+Q_{0i}(\{(sk+h_0)\omega/t_i'\})\big)\\ =&Q_d(\{(sk+h_0)\omega/t_1'\},\dots,\{(sk+h_0)\omega/t_r'\})k^d+\cdots\\ &+Q_0(\{(sk+h_0)\omega/t_1'\},\dots,\{(sk+h_0)\omega/t_r'\}),\end{aligned}\tag{12}$$

where $Q_d(X_1,\dots,X_r),\dots,Q_0(X_1,\dots,X_r)\in\overline{\mathbb{Q}}[X_1,\dots,X_r]$ are not all constant polynomials of the form

$$Q_j(X_1,\dots,X_r)=Q_{j1}(X_1)+\cdots+Q_{jr}(X_r)$$

with $Q_d(X_1,\dots,X_r)\ne 0$. On the other hand, by (10), we can write

$$a_{sk+h_0}=c_dk^d+\cdots+c_0\tag{13}$$

for all sufficiently large k, where $c_d,\dots,c_0$ are algebraic numbers. Let j_0 be the largest integer such that $Q_{j_0}(X_1,\dots,X_r)$ is not constant. From Lemma 3 there exists a real number τ_0 such that

$$Q_{j_0}(\{s_1\tau_0\},\dots,\{s_r\tau_0\})\ne c_{j_0}.\tag{14}$$

From Lemma 4 there exists an increasing sequence $\{k_\nu\}_{\nu\ge 0}$ of positive integers such that

$$\lim_{\nu\to\infty}(\{s_1k_\nu\omega\},\dots,\{s_rk_\nu\omega\})=(\{s_1\tau_0'\},\dots,\{s_r\tau_0'\}),\tag{15}$$

where $\tau_0' = \tau_0 - h_0\omega/s$ and each component of the left-hand side approaches the corresponding component of the right-hand side from the right. By (15) we see that

$$\lim_{\nu\to\infty} (\{(sk_\nu + h_0)\omega/t_1'\}, \ldots, \{(sk_\nu + h_0)\omega/t_r'\}) = (\{s_1\tau_0\}, \ldots, \{s_r\tau_0\}).$$

If $j_0 = d$, then $\lim_{\nu\to\infty} a_{sk_\nu+h_0}/k_\nu^d = Q_d(\{s_1\tau_0\}, \ldots, \{s_r\tau_0\}) \neq c_d$ by (12) and (14). On the other hand $\lim_{k\to\infty} a_{sk+h_0}/k^d = c_d$ by (13), which is a contradiction. Hence we see that the polynomial $Q_d(X_1, \ldots, X_r)$ is equal to the constant c_d identically. Then, since

$$\lim_{\nu\to\infty} \frac{a_{sk_\nu+h_0} - c_d k_\nu^d}{k_\nu^{d-1}} = Q_{d-1}(\{s_1\tau_0\}, \ldots, \{s_r\tau_0\})$$

by (12) and

$$\lim_{k\to\infty} \frac{a_{sk+h_0} - c_d k^d}{k^{d-1}} = c_{d-1}$$

by (13), we see that $j_0 < d - 1$. Continuing this process, we obtain a contradiction. This concludes the proof of Theorem 5. □

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