A note on the asymptotic behavior of the twisted Alexander polynomials of 5_2 knot

Airi Aso
Graduate School of Science, Tokyo Metropolitan University

1 Introduction

R. M. Kashaev conjectured that the asymptotics of the Kashaev invariants of a hyperbolic link gives the hyperbolic volume of its compliment [Kas]. H. Murakami and J. Murakami generalized Kashaev's conjecture for the N-dimensional colored Jones polynomial $J_N(K;q)$.

Conjecture 1 (Volume conjecture [MM])**.** *The equality*

$$2\pi \lim_{N\to\infty} \frac{\log |J_N(K;\exp(2\pi\sqrt{-1}/N)|}{N} = \mathrm{Vol}(S^3\backslash K),$$

would hold for any knot K, where $\mathrm{Vol}(S^3\backslash K)$ is the hyperbolic volume of $S^3\backslash K$.

H. Murakami, J. Murakami, M. Okamoto, T. Takata, and Y. Yokota proposed the following generalization of volume conjecture.

Conjecture 2 (Complexification of volume conjecture [MMTOY])**.** *The equality*

$$2\pi \lim_{N\to\infty} \frac{\log J_N(K;\exp(2\pi\sqrt{-1}/N)}{N} = \mathrm{Vol}(S^3\backslash K) + 2\pi^2\sqrt{-1}\mathrm{CS}(S^3\backslash K) \mod \pi^2\sqrt{-1}\mathbb{Z},$$

would hold for any knot K, where $\mathrm{CS}(S^3\backslash K)$ is the Chern-Simons invariant of $S^3\backslash K$ with respect to some representation $\pi_1(S^3\backslash K) \to SL(2;\mathbb{C})$.

On the other hand, H. Goda showed the following relationship between hyperbolic volume and the twisted Alexander polynomials.

Theorem 3 ([Go])**.** *Let K be a hyperbolic knot in the 3-sphere. Then*

$$\lim_{n\to\infty} \frac{4\pi \log |\mathcal{A}_{K,n}(1)|}{n^2} = \mathrm{Vol}(S^3\backslash K),$$

where $\mathcal{A}_{K,2k}(t) := \frac{\Delta_{K,\rho_{2k}}(t)}{\Delta_{K,\rho_2}(t)}$ and $\mathcal{A}_{K,2k+1}(t) := \frac{\Delta_{K,\rho_{2k+1}}(t)}{\Delta_{K,\rho_3}(t)}$.

We would like to study a complexification of the left hand side of the equality in Theorem 3, i.e.

$$\lim_{n\to\infty} \frac{4\pi \log \mathcal{A}_{K,n}(1)}{n^2}.$$

To this end, we obsereve the asymptotic behavior of

$$\frac{4\pi \log \mathcal{A}_{K,n}(1)}{n^2}$$

for 5_2 knot by the help of Mathematica, and conjecture their limit value. At the time of my talk, this observation was in progress and a serious problem was pointed out by some audiences. We could fix the problem by considering a new operation, and we obtained a new nontrivial result for the complexification after the talk.

2 Main result of my talk

Let K be the 5_2 knot. Then the results of the computation of

$$\frac{4\pi \log \mathcal{A}_{K,n}(1)}{n^2}$$

for some natural number n are given as follows:

$n = 4$	$1.785532667455748... + (0.48203336590870277...)i$
$n = 5$	$1.9837366127137959... + (0.4391339851110024...)i$
$n = 6$	$2.455852749246374... + (0.6980605748575335...)i$
$n = 7$	$2.3613041487662985... - (0.7896254033565749...)i$
$n = 8$	$2.566462552049248... - (0.11818019778518644...)i$
$n = 9$	$2.5534559249168067... + (0.2655374217922101...)i$
$n = 10$	$2.682769962219264... - (0.17418347408994717...)i$
$n = 11$	$2.6439403339313885... + (0.18595103941561894...)i$
$n = 12$	$2.719664770570517... - (0.020732353458557916...)i$
$n = 13$	$2.6943772876900085... - (0.17140452705232556...)i$
$n = 14$	$2.7512127179488877... + (0.18457327508837376...)i$
$n = 15$	$2.729097097750846... + (0.09590995554889228...)i$
$n = 16$	$2.7683209978452954... + (0.07791922234246355...)i$
$n = 17$	$2.7501887958616393... + (0.06184125633682704...)i$
$n = 20$	$2.7900248250429374... - (0.04560219657696698...)i$
$n = 30$	$2.8111789937618563... + (0.03684965457608568...)i$

By these computations, we conjectured that

$$\lim_{n\to\infty} \frac{4\pi \log \mathcal{A}_{K,n}(1)}{n^2} = 2.82812... = \mathrm{Vol}(S^3 \backslash K).$$

3 A new result for the complexification

It is pointed out that the results of the above computation is trivial since the imaginary parts of $\log \mathcal{A}_{K,n}(1)$ are given in $[-\pi, \pi]$ by Mathematica and

$$\lim_{n\to\infty} \frac{\mathrm{Im}(\log \mathcal{A}_{K,n}(1))}{n^2} = 0.$$

Hence we tried a new operation to obtain the imaginary part after the talk.

If there exist real numbers α and β such that

$$\lim_{n\to\infty} \frac{4\pi \log \mathcal{A}_{K,n}(1)}{n^2} = \alpha + i\beta,$$

then we should have

$$\lim_{n\to\infty} \frac{\pi}{2} \log \frac{\mathcal{A}_{K,n-2}(1)\mathcal{A}_{K,n+2}(1)}{(\mathcal{A}_{K,n}(1))^2} = \alpha + i\beta.$$

Hence, we tried to compute

$$\frac{\pi}{2} \log \frac{\mathcal{A}_{K,n-2}(1)\mathcal{A}_{K,n+2}(1)}{(\mathcal{A}_{K,n}(1))^2}$$

for some natural numbers n, and we obtained the following computations:

$n = 6$	$2.00009... + (3.60568...)i$
$n = 7$	$3.12694... + (3.86417...)i$
$n = 8$	$3.52256... + (2.85486...)i$
$n = 9$	$2.7451... + (2.46852...)i$
$n = 10$	$2.41642... + (3.03596...)i$
$n = 11$	$2.79327... + (3.31223...)i$
$n = 12$	$3.03141... + (3.09112...)i$
$n = 13$	$2.90802... + (2.88221...)i$
$n = 14$	$2.73082... + (2.94575...)i$
$n = 15$	$2.75758... + (3.08777...)i$

By the above results, now we conjecture that

$$\begin{aligned}\lim_{n\to\infty} \frac{4\pi \log \mathcal{A}_{K,n}(1)}{n^2} &= 2.82812... + (3.02413...)i \\ &= \mathrm{Vol}(S^3 \backslash K) + 2\pi^2\sqrt{-1}\mathrm{CS}(S^3 \backslash K) \qquad \mod \pi^2\sqrt{-1}\mathbb{Z}.\end{aligned}$$

4 How to calculate

For the 5_2 knot K, its knot group $G(K) = \pi_1(S^3 \backslash K)$ is given by

$$G(K) = \langle a, b \mid b[a,b][a,b] = [a,b]a \rangle.$$

Then its holonomy representation $\rho : G(K) \to SL(2;\mathbb{C})$ is given by

$$\rho(a) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \ \rho(b) = \begin{pmatrix} 1 & -\frac{1}{x^2} \\ 0 & 1 \end{pmatrix}$$

where $x^3 - x - 1 = 0$.

4.1 Computation of ρ_n

It is known that the pair of the vector space

$$V_n = \mathrm{span}_{\mathbb{C}}\langle x^{n-1}, x^{n-2}y, \ldots, xy^{n-2}, y^{n-1}\rangle \subset \mathbb{C}[x,y]$$

and the the action of $A \in SL(2;\mathbb{C})$ expressed as

$$A \cdot p\begin{pmatrix} x \\ y \end{pmatrix} := p\left(A^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right)$$

gives an n-dimensional irreducible representation $\sigma_n \colon SL(2;\mathbb{C}) \to SL(n;\mathbb{C})$, where $p\begin{pmatrix} x \\ y \end{pmatrix}$ is a homogeneous polynomial of degree $n-1$. Then, composing the holonomy representation ρ with σ_n, we obtain the representation

$$\rho_n \colon G(K) \to SL(n;\mathbb{C}).$$

Example 4. *By* $\rho(a) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$*, we have*

$$(\rho_3(a))\left(p\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot p\begin{pmatrix} x \\ y \end{pmatrix} = p\begin{pmatrix} x \\ y-x \end{pmatrix}.$$

If we write $p\begin{pmatrix} x \\ y \end{pmatrix} = \alpha x^2 + \beta xy + \gamma y^2$ $(\alpha, \beta, \gamma \in \mathbb{C})$*, then we have*

$$(\rho_3(a))\left(p\begin{pmatrix} x \\ y \end{pmatrix}\right) = (\alpha - \beta + \gamma)x^2 + (\beta - 2\gamma)xy + \gamma y^2.$$

Hence we have

$$\rho_3(a) = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Lemma 5. *Write* $\rho(s)^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ *for some* $\alpha, \beta, \gamma, \delta \in \mathbb{C}$*. Then*

$$\rho_n(s) = [s_{ij}] = \left[\sum_k \frac{(n-j)!(j-1)!\alpha^k \beta^{n-j-k} \gamma^{n-i-k} \delta^{i+j-n+k-1}}{k!(n-j-k)!(n-i-k)!(i+j-n+k-1)!}\right]$$

where

$$\begin{cases} 0 \le k \le n-j & i \le j, n+1 \le i+j, \\ 0 \le k \le n-i & j \le i, n+1 \le i+j, \\ n-i-j+1 \le k \le n-j & i \le j, i+j \le n+1, \\ n-i-j+1 \le k \le n-i & j \le i, i+j \le n+1. \end{cases}$$

Since we have $\rho(a) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$,

$$A_n := \rho_n(a) = [a_{ij}],$$

where

$$a_{ij} = \begin{cases} \frac{(j-1)!}{(i-1)!(j-i)!}(-1)^{j-i} & i \le j, \\ 0 & i > j. \end{cases}$$

Similarly, since $\rho(b) = \begin{pmatrix} 1 & -\frac{1}{x^2} \\ 0 & 1 \end{pmatrix}$, we have

$$B_n := \rho_n(b) = [b_{ij}],$$

where

$$b_{ij} = \begin{cases} 0 & i < j, \\ \frac{(n-j)!}{(n-i)!(i-j)!}x^{-2(i-j)} & i \ge j. \end{cases}$$

Then, for simplicity, we put

$$AB_n := [A_n, B_n] = A_n B_n A_n^{-1} B_n^{-1}$$

4.2 Computation of $\mathcal{A}_{K,n}(t)$

Recall that

$$\mathcal{A}_{K,2k}(t) := \frac{\Delta_{K,\rho_{2k}}(t)}{\Delta_{K,\rho_2}(t)},$$
$$\mathcal{A}_{K,2k+1}(t) := \frac{\Delta_{K,\rho_{2k+1}}(t)}{\Delta_{K,\rho_3}(t)}.$$

Definition 6. Let K be a knot in S^3 and

$$G(K) = \langle x_1, \dots, x_n \,|\, r_1, \dots, r_{n-1} \rangle$$

the knot group of K. Then, the twisted Alexander polynomial of K associated to a representation $\rho : G(K) \to SL_n(\mathbb{F})$ is given by

$$\Delta_{K,\rho}(t) = \frac{\det A_{\rho,k}}{\det[(\rho \otimes \mathfrak{a}) \circ \phi(x_k - 1)]},$$

where $\mathfrak{a} : \mathbb{Z}G(K) \to \mathbb{Z}[t, t^{-1}]$ is the abelianization of the group ring $\mathbb{Z}G(K)$ and $\phi : \mathbb{Z}\Gamma \to \mathbb{Z}G(K)$ is the natural ring homomorphism of the free group Γ generated by $x_1, \cdots, x_n$. We put

$$A_{i,j} = (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r_i}{\partial x_j} \right),$$

where $\dfrac{\partial}{\partial x_j} : \mathbb{Z}\Gamma \to \mathbb{Z}\Gamma$ denotes the Fox derivative with respect to x_j. Then, $A_{\rho,k}$ is the $d(n-1) \times d(n-1)$ matrix defined by

$$\begin{pmatrix} A_{1,1} & \cdots & A_{1,k-1} & A_{1,k+1} & \cdots & A_{1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{n-1,1} & \cdots & A_{n-1,k-1} & A_{n-1,k+1} & \cdots & A_{n-1,n} \end{pmatrix}.$$

Since

$$G(K) = \langle a, b \mid b[a,b][a,b] = [a,b]a \rangle,$$

we have

$$\Delta_{K,\rho_n}(t) = \frac{N_n}{D_n},$$

where

$$N_n := |-t^{-1}B_n^{-1}(AB_n + E) + t^0(B_n^{-1}AB_nB_n + E + AB_n) - t^1(AB_n + E)AB_nB_n|,$$
$$D_n := |tA_n - E|.$$

Computing N_n and D_n by the help of Mathematica, we obtained $\mathcal{A}_{K,n}(t)$ for $n = 4, 5, \cdots, 17, 20, 30$.

5 Acknowledgement

The author would like to thank Professor H. Akiyoshi and Professor J. Murakami for pointing out a serious problem of the computation in Section 2.

References

[Kas] R. M. Kashaev, The hyperbolic volume of knots from the quantum dilogarithm, Lett. Math. Phys., **39**(1997), 269–275.

[MM] H. Murakami and J. Murakami, The colored Jones polynomials and the simplicial volume of a knot, Acta Math., **186** (2001), 85–104.

[MMOTY] H. Murakami, J. Murakami, M. Okamoto, T. Takata and Y. Yokota, Kashaev's conjecture and the Chern-Simons invariants of knots and links, Experiment. Math., **11** (2002), 427–435.

[Go] H. Goda, Twisted Alexander invariants and hyperbolic volume, Proc. Japan Acad. Ser. A Math. Sci. **93** (2017), no. 7, 61–66.

Depertment of Mathematical sciences
Graduate School of Science
Tokyo Metropolitan University
1-1 Minamiohsawa, Hachioji-shi, Tokyo, 192-0397 JAPAN
E-mail address: aso-airi@ed.tmu.ac.jp

東京都立大学 理学研究科　阿蘇 愛理