On Parseval Frames for Multidirectional Expansions and the Inversion of the Radon Transform

東海大学理学部情報数理学科 藤ノ木 健介 (Kensuke Fujinoki)
Department of Mathematical Sciences, Tokai University
筑波大学大学院数理物質科学研究科 橋本 紘史 (Hirofumi Hashimoto)
Institute of Mathematics, University of Tsukuba
筑波大学大学院数理物質科学研究科 木下 保 (Tamotu Kinoshita)
Institute of Mathematics, University of Tsukuba

1 Introduction

1.1 Polygonal Tiling Frames

The image processing is related to two-dimensional multiresolution analysis. Two-dimensional orthonormal or biorthogonal wavelets have been constructed from suitable two-scale equations or partitions of the frequency domain (see [10], [14], etc.). 12 tapered frame wavelets can provide a refined partition of the frequency domain and are efficient for geometric features with line and curve singularities (see [1]). The curvelets pioneered by Candès and Donoho [3] are multidirectional methods, where the degree of localization in orientation varies with scale (see also [7] and [12]).

We consider concentric regular 2^N-sided polygons ($N \geq 2$) and present multidirectional methods with *polygonal tiling frames* (PTFs) in the simplest possible manner. Let $N \geq 2$ and

$$p_2 = \frac{1}{\sqrt{2}}, \quad p_3 = \frac{1}{\sqrt{2+\sqrt{2}}}, \quad \cdots, \quad p_N = \frac{1}{\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}}.$$

We see that $\cos\frac{\pi}{2^N} = \frac{1}{2p_N}$, i.e., $p_N = (2\cos\frac{\pi}{2^N})^{-1}$, because the double-angle formula gives $1/(2p_N) = 2/(2p_{N+1})^2 - 1$, which means that $1/p_{N+1} = \sqrt{2 + 1/p_N}$.

Then, we get the following:

Theorem 1.1 *Let $N \geq 2$, $J \in \mathbf{Z}$, $k' = \left(k_1 \cot \frac{\pi}{2^N}, k_2\right)$ and the real-valued functions $\Psi^{(N)}_{j,\ell}$ and $\Phi^{(N)}_{j,\ell}$ be defined as*

$$\begin{aligned}
\Psi^{(N)}_{j,\ell}(x) &= \sum_{\pm} \Big\{ \pm \frac{\cos(2^j \pi p_N X^{\pm}_\ell(x)) - \cos(2^{j+1}\pi p_N X^{\pm}_\ell(x))}{2^{j+2}\pi^2 p_N (4p_N^2 - 1)^{1/4} X^{\pm}_\ell(x) R_\ell x \cdot (1,0)} \Big\}, \\
\Phi^{(N)}_{j,\ell}(x) &= \sum_{\pm} \Big\{ \pm \frac{\sin^2(2^j \pi p_N X^{\pm}_\ell(x))}{2^{j+1}\pi^2 p_N (4p_N^2 - 1)^{1/4} X^{\pm}_\ell(x) R_\ell x \cdot (1,0)} \Big\},
\end{aligned}$$

where $X^{\pm}_\ell(x) = x_1 \sin \frac{(2\ell \pm 1)\pi}{2^N} + x_2 \cos \frac{(2\ell \pm 1)\pi}{2^N}$. Then, $f \in L^2(\mathbf{R}^2_x)$ is expanded by PTFs as

$$\begin{aligned}
f(x) &= \sum_{j \geq J+1} \sum_{1 \leq \ell \leq 2^{N-1}} \sum_{k \in \mathbf{Z}^2} \alpha_{j,\ell,k} \Psi^{(N)}_{j,\ell}\left(x - 2^{-j} R_{-\ell} k'\right) \\
&\quad + \sum_{1 \leq \ell \leq 2^{N-1}} \sum_{k \in \mathbf{Z}^2} \beta_{J,\ell,k} \Phi^{(N)}_{J,\ell}\left(x - 2^{-J} R_{-\ell} k'\right), \qquad (1)
\end{aligned}$$

where

$$\alpha_{j,\ell,k} = \int_{\mathbf{R}^2_x} f(x) \Psi^{(N)}_{j,\ell}\left(x - 2^{-j} R_{-\ell} k'\right) dx, \quad \beta_{j,\ell,k} = \int_{\mathbf{R}^2_x} f(x) \Phi^{(N)}_{j,\ell}\left(x - 2^{-j} R_{-\ell} k'\right) dx,$$

and R_ℓ is the operator of anticlockwise rotation by angle $2^{1-N}\ell\pi$.

Remark 1.2 *$\hat{\Psi}^{(N)}_{j,\ell}(\xi)$ satisfies the partition of unity in the following sense:*

$$\tan \frac{\pi}{2^N} \sum_{j \in \mathbf{Z}} \sum_{1 \leq \ell \leq 2^{N-1}} 2^{2j} |\hat{\Psi}^{(N)}_{j,\ell}(\xi)|^2 = 1. \qquad (2)$$

It holds that

$$\|\Psi^{(N)}_{j,\ell}\|^2 = \|\Psi^{(N)}_{j,0}\|^2 = \frac{1}{(2\pi)^2} \|\hat{\Psi}^{(N)}_{j,0}\|^2 = \frac{3}{8} (< 1).$$

Remark 1.3 *From the construction, we see that $\Psi^{(N)}_{j,\ell}$ has an infinite number of vanishing moments as the Shannon wavelet. But we remark that the Fourier transform of the Shannon wavelet has the support $[-2\pi, -\pi] \cup [\pi, 2\pi]$, while the support in ξ_2 of $\hat{\Psi}^{(N)}_{0,0}$ is $[-\pi, -\pi/2] \cup [\pi/2, \pi]$.*

1.2 Wavelet-type Transform

We may rewrite Theorem 1.1 in terms of the generating functions:

$$\Psi^{(N)}(x) \;:=\; 2^{-j}\Psi^{(N)}_{j,0}(2^{-j}x) = \sum_{\pm}\Big\{ \pm \frac{\cos(\pi p_N X_0^{\pm}(x)) - \cos(2\pi p_N X_0^{\pm}(x))}{4\pi^2 p_N (4p_N^2-1)^{1/4} X_0^{\pm}(x) x_1}\Big\},$$

$$\Phi^{(N)}(x) \;:=\; 2^{-j}\Phi^{(N)}_{j,0}(2^{-j}x) = \sum_{\pm}\Big\{ \pm \frac{\sin^2(\pi p_N X_0^{\pm}(x))}{2\pi^2 p_N (4p_N^2-1)^{1/4} X_0^{\pm}(x) x_1}\Big\}.$$

Then, $\Psi^{(N)}_{j,\ell}(x-2^{-j}R_{-\ell}k')$ and $\Phi^{(N)}_{j,\ell}(x-2^{-j}R_{-\ell}k')$ in Theorem 1.1 can be replaced by $2^j\Psi^{(N)}(2^jR_\ell x-k')$ and $2^j\Phi^{(N)}(2^jR_\ell x-k')$ respectively.

Remark 1.4 *The definitions of $\Psi^{(N)}$ and $\Phi^{(N)}$ include removable singularities. For the computer simulations, it will be convenient to rewrite $\Psi^{(N)}$ and $\Phi^{(N)}$ as follows:*

$$\Psi^{(N)} = \begin{cases} \dfrac{\sqrt{\cot\frac{\pi}{2^N}}}{2\pi^2 x_1}\Big\{\dfrac{\cos\{\frac{\pi(x_2+x_1\tan\frac{\pi}{2^N})}{2}\} - \cos\{\pi(x_2+x_1\tan\frac{\pi}{2^N})\}}{x_2+x_1\tan\frac{\pi}{2^N}} \\ \qquad - \dfrac{\cos\{\frac{\pi(x_2-x_1\tan\frac{\pi}{2^N})}{2}\} - \cos\{\pi(x_2-x_1\tan\frac{\pi}{2^N})\}}{x_2-x_1\tan\frac{\pi}{2^N}}\Big\} & \textit{when } x_1\neq 0,\ x_2\neq \pm x_1\tan\dfrac{\pi}{2^N}, \\ \sqrt{\tan\dfrac{\pi}{2^N}}\dfrac{(2\cos\{\pi x_2\}+1)\sin^2\frac{\pi x_2}{2}}{2\pi^2 x_2^2} & \textit{when } x_1\neq 0,\ x_2 = \pm x_1\tan\dfrac{\pi}{2^N}, \\ \dfrac{1}{8}\sqrt{\tan\dfrac{\pi}{2^N}}F'\Big(\dfrac{\pi x_2}{4}\Big) & \textit{when } x_1 = 0,\ x_2\neq 0, \\ \dfrac{3}{8}\sqrt{\tan\dfrac{\pi}{2^N}} & \textit{when } x_1=0,\ x_2=0, \end{cases}$$

$$\Phi^{(N)} = \begin{cases} \dfrac{\sqrt{\cot\frac{\pi}{2^N}}}{\pi^2 x_1}\Big\{\dfrac{\sin^2\{\frac{\pi(x_2+x_1\tan\frac{\pi}{2^N})}{2}\}}{x_2+x_1\tan\frac{\pi}{2^N}} - \dfrac{\sin^2\{\frac{\pi(x_2-x_1\tan\frac{\pi}{2^N})}{2}\}}{x_2-x_1\tan\frac{\pi}{2^N}}\Big\} & \textit{when } x_1\neq 0,\ x_2\neq \pm x_1\tan\dfrac{\pi}{2^N}, \\ \sqrt{\tan\dfrac{\pi}{2^N}}\dfrac{\sin^2\{\pi x_2\}}{2\pi^2 x_2^2} & \textit{when } x_1\neq 0,\ x_2 = \pm x_1\tan\dfrac{\pi}{2^N}, \\ \dfrac{1}{2}\sqrt{\tan\dfrac{\pi}{2^N}}G'\Big(\dfrac{\pi x_2}{2}\Big) & \textit{when } x_1 = 0,\ x_2\neq 0, \\ \dfrac{1}{2}\sqrt{\tan\dfrac{\pi}{2^N}} & \textit{when } x_1=0,\ x_2=0, \end{cases}$$

where $F(\tau) = \frac{(2\cos 2\tau+1)\sin^2\tau}{\tau}$ *and* $G(\tau) = \frac{\sin^2\tau}{\tau}$.

Similarly, as curvelets based on concentric circles, if the translation part $2^{-j}R_{-\ell}k'$ is replaced by a continuous parameter $b \in \mathbf{R}^2$, then we can consider the semidiscrete transform

$$T_{j,\ell}f(b) := \int_{\mathbf{R}_x^2} \Psi_{j,\ell}^{(N)}(x-b)f(x)dx = \int_{\mathbf{R}_x^2} 2^j\Psi^{(N)}\Big(R_{-\ell}2^j(x-b)\Big)f(x)dx.$$

Hence, we also reduce this to the following two-dimensional continuous wavelet-type transform:

$$W_{\Psi^{(N)}}f(b,a,A) := \int_{\mathbf{R}_x^2} \frac{1}{a}\Psi^{(N)}\Big(A^{-1}\big(\frac{x-b}{a}\big)\Big)f(x)dx$$

for $(b,a,A) \in \mathbf{R}^2 \times \mathbf{R}^+ \times SO(2)$. It holds that

$$W_{\Psi^{(N)}}f(b,a,A) = \frac{a}{(2\pi)^2}\int_{\mathbf{R}_\xi^2} e^{ib\cdot\xi}\overline{\hat{\Psi}^{(N)}(aA^{-1}\xi)}\hat{f}(\xi)d\xi = a\mathcal{F}^{-1}\Big[\overline{\hat{\Psi}^{(N)}(aA^{-1}\xi)}\hat{f}(\xi)\Big](b).$$

Putting

$$c_\psi := \int_{\mathbf{R}_\xi^2} \frac{|\hat{\Psi}^{(N)}(\xi)|^2}{|\xi|^2}d\xi,$$

we also obtain Parseval's identity in the following sense:

$$\begin{aligned}
&\int_0^{2\pi}\int_0^\infty\int_{\mathbf{R}_b^2}\Big|W_{\Psi^{(N)}}f(b,a,A)\Big|^2 db\frac{da}{a^3}d\theta \\
=\ &\int_0^{2\pi}\int_0^\infty\Big\{\int_{\mathbf{R}_b^2}\Big|\mathcal{F}^{-1}\Big[\overline{\hat{\Psi}^{(N)}(aA^{-1}\xi)}\hat{f}(\xi)\Big](b)\Big|^2 db\Big\}\frac{da}{a}d\theta \\
=\ &\int_0^{2\pi}\int_0^\infty\Big\{\int_{\mathbf{R}_\xi^2}\Big|\overline{\hat{\Psi}^{(N)}(aA^{-1}\xi)}\hat{f}(\xi)\Big|^2 d\xi\Big\}\frac{da}{a}d\theta \\
=\ &\int_{\mathbf{R}_\xi^2}\Big\{\int_0^{2\pi}\int_0^\infty\Big|\hat{\Psi}^{(N)}\Big(rA^{-1}\frac{\xi}{|\xi|}\Big)\Big|^2\frac{dr}{r}d\theta\Big\}|\hat{f}(\xi)|^2 d\xi \\
=\ &(2\pi)^2 c_\psi\int_{\mathbf{R}_x^2}|f(x)|^2dx,
\end{aligned}$$

where for all $\xi \in \mathbf{R}_\xi^2$ we used

$$\int_0^{2\pi}\int_0^\infty\Big|\hat{\Psi}^{(N)}\Big(rA^{-1}\frac{\xi}{|\xi|}\Big)\Big|^2\frac{dr}{r}d\theta = \int_0^{2\pi}\int_0^\infty\frac{\Big|\hat{\Psi}^{(N)}\Big(r(\cos\theta,\sin\theta)\Big)\Big|^2}{r^2}rdrd\theta = c_\psi.$$

1.3 Simpler PTFs

If we do not expect less redundancy, it is possible to get simpler PTFs with a small modification. We do not need $(4p_N^2-1)^{1/4}$ in the definitions of $\Psi_{j,\ell}^{(N)}$ and $\Phi_{j,\ell}^{(N)}$. Although k (instead of k') is not optimal for our case from the point of view of the sampling theorem, we can obtain the following simpler results with $\mathbf{\Psi}$ and $\mathbf{\Phi}$:

Theorem 1.5 *Let $N \geq 2$, $J \in \mathbf{Z}$ and the real-valued functions $\mathbf{\Psi}_{j,\ell}^{(N)}$ and $\mathbf{\Phi}_{j,\ell}^{(N)}$ be defined as*

$$\begin{aligned}\mathbf{\Psi}_{j,\ell}^{(N)}(x) &= \sum_{\pm}\Big\{ \pm \frac{\cos(2^j\pi p_N X_\ell^{\pm}(x)) - \cos(2^{j+1}\pi p_N X_\ell^{\pm}(x))}{2^{j+2}\pi^2 p_N X_\ell^{\pm}(x) R_\ell x\cdot(1,0)}\Big\},\\ \mathbf{\Phi}_{j,\ell}^{(N)}(x) &= \sum_{\pm}\Big\{ \pm \frac{\sin^2(2^j\pi p_N X_\ell^{\pm}(x))}{2^{j+1}\pi^2 p_N X_\ell^{\pm}(x) R_\ell x\cdot(1,0)}\Big\},\end{aligned}$$

where $X_\ell^{\pm}(x) = x_1 \sin\frac{(2\ell\pm1)\pi}{2^N} + x_2\cos\frac{(2\ell\pm1)\pi}{2^N}$. Then, $f \in L^2(\mathbf{R}_x^2)$ is expanded by PTFs as

$$\begin{aligned}f(x) &= \sum_{j\geq J+1}\sum_{1\leq\ell\leq 2^{N-1}}\sum_{k\in\mathbf{Z}^2}\alpha_{j,\ell,k}\mathbf{\Psi}_{j,\ell}^{(N)}\Big(x-2^{-j}R_{-\ell}k\Big)\\ &\quad+\sum_{1\leq\ell\leq 2^{N-1}}\sum_{k\in\mathbf{Z}^2}\beta_{J,\ell,k}\mathbf{\Phi}_{J,\ell}^{(N)}\Big(x-2^{-J}R_{-\ell}k\Big),\end{aligned}$$

where

$$\alpha_{j,\ell,k} = \int_{\mathbf{R}_x^2} f(x)\mathbf{\Psi}_{j,\ell}^{(N)}\Big(x-2^{-j}R_{-\ell}k\Big)dx, \quad \beta_{j,\ell,k} = \int_{\mathbf{R}_x^2} f(x)\mathbf{\Phi}_{j,\ell}^{(N)}\Big(x-2^{-j}R_{-\ell}k\Big)dx.$$

Remark 1.6 *Similarly as Remark 1.2, we have*

$$\|\mathbf{\Psi}_{j,\ell}^{(N)}(x)\|^2 = \frac{3}{8}\tan\frac{\pi}{2^N}(<1).$$

As N increases, they become smaller. The frame expansion has more redundancy.

More details and results of frames having Lipschitz continuous Fourier transforms will be reported in our forthcoming paper [9].

2 Applications

2.1 Application to Radon transform

Computed tomography (CT) is a medical imaging technique for clinical use. During the CT process, X-rays are transmitted through an object. As a mathematical model, the X-ray transform (two-dimensional Radon transform) is defined as

$$\mathcal{R}(f)(r,\gamma) := \int_{x\in\mathcal{L}_{r,\gamma}} f(x)dx = \int_{\mathbf{R}} f\Big(x_1, \frac{r-x_1\gamma_1}{\gamma_2}\Big)\frac{dx_1}{|\gamma_2|},$$

where $\mathcal{L}_{r,\gamma}$ denotes a line with the normal direction $\gamma = (\gamma_1,\gamma_2) = (\cos\theta, \sin\theta) \in S^1$ and the signed distance from the origin $r \in \mathbf{R}$. For $\varphi(r,\gamma) \in L^1(\mathbf{R}\times S^1)$ the dual Radon transform $\mathcal{R}^*$ is given as

$$\mathcal{R}^*(\varphi)(x) := \int_{S^1} \varphi(x\cdot\gamma,\gamma)d\sigma(\gamma) = \int_0^{2\pi} \varphi\Big(x_1\cos\theta + x_2\sin\theta, (\cos\theta,\sin\theta)\Big)d\theta.$$

In particular, in the two-dimensional case, the following inversion formula holds for some suitable f (see [13]):

$$f(x) = \frac{1}{4\pi}(-\Delta)^{1/2}\mathcal{R}^*\Big(\mathcal{R}(f)\Big)(x). \tag{3}$$

It would be convenient if $\alpha_{j,\ell,k}$ of the expansion of f are derived directly from $\mathcal{R}(f)$ without use of the reconstruction formula including the nonlocal operator $(-\Delta)^{1/2}$. Berenstein and Walnut [2] used the theory of the continuous wavelet transform to derive inversion formulas for the Radon transform. Candès and Donoho [4] applied the curvelets to consider the problem of noisy Radon inversion (see also [8]). Colonna, Easley, Guo, and Labate [5] gave the Radon transform inversion via the shearlet representation (see also [6]). For this purpose, we shall utilize the simplest frame in Theorem 1.1.

Considering (3), we suppose that f satisfies

$$\begin{aligned}
\alpha_{j,\ell,k} &:= \int_{\mathbf{R}^2} f(x)\overline{\Psi^{(N)}_{j,\ell}(x-2^{-j}R_{-\ell}k')}dx \\
&= \frac{1}{4\pi}\int_{\mathbf{R}^2}(-\Delta)^{1/2}\mathcal{R}^*\Big(\mathcal{R}(f)\Big)(x)\overline{\Psi^{(N)}_{j,\ell}(x-2^{-j}R_{-\ell}k')}dx \\
&= \frac{1}{4\pi}\int_0^{2\pi}\int_{\mathbf{R}_r}\mathcal{R}(f)(r,\gamma)\overline{\mathcal{R}\Big((-\Delta)^{1/2}\Psi^{(N)}_{j,\ell}(x-2^{-j}R_{-\ell}k')\Big)(r,\gamma)}drd\theta, \quad (4)
\end{aligned}$$

$$\begin{aligned}
\beta_{j,\ell,k} &:= \int_{\mathbf{R}^2} f(x)\overline{\Phi^{(N)}_{j,\ell}(x-2^{-j}R_{-\ell}k')}dx \\
&= \frac{1}{4\pi}\int_{\mathbf{R}^2}(-\Delta)^{1/2}\mathcal{R}^*\Big(\mathcal{R}(f)\Big)(x)\overline{\Phi^{(N)}_{j,\ell}(x-2^{-j}R_{-\ell}k')}dx \\
&= \frac{1}{4\pi}\int_0^{2\pi}\int_{\mathbf{R}_r}\mathcal{R}(f)(r,\gamma)\overline{\mathcal{R}\Big((-\Delta)^{1/2}\Phi^{(N)}_{j,\ell}(x-2^{-j}R_{-\ell}k')\Big)(r,\gamma)}drd\theta. \quad (5)
\end{aligned}$$

Now, we put

$$U^{(k)}_{j,\ell}(r,\gamma) := \frac{1}{4\pi}\mathcal{R}\Big((-\Delta)^{1/2}\Psi^{(N)}_{j,\ell}(x-2^{-j}R_{-\ell}k')\Big)(r,\gamma).$$

By the Fourier slice theorem $\mathcal{R}(g)(r,\gamma)=\frac{1}{2\pi}\int_{\mathbf{R}_\rho}e^{i\rho r}\mathcal{F}_x[g](\rho\gamma)d\rho$, we obtain

$$\begin{aligned}
U^{(k)}_{j,\ell}(r,\gamma) &= \frac{1}{8\pi^2}\int_{\mathbf{R}_\rho}e^{i\rho r}\mathcal{F}_x\Big[(-\Delta)^{1/2}\Psi^{(N)}_{j,\ell}(x-2^{-j}R_{-\ell}k')\Big](\rho\gamma)d\rho \\
&= \frac{1}{8\pi^2}\int_{\mathbf{R}_\rho}e^{i\rho r}\mathcal{F}_x\Big[\mathcal{F}^{-1}_\xi\Big[|\xi|\mathcal{F}_y[\Psi^{(N)}_{j,\ell}(y-2^{-j}R_{-\ell}k')]\Big]\Big](\rho\gamma)d\rho \\
&= \frac{1}{8\pi^2}\int_{\mathbf{R}_\rho}e^{i\rho r}|\rho\gamma|\int e^{-iy\cdot\rho\gamma}\Psi^{(N)}_{j,\ell}(y-2^{-j}R_{-\ell}k')dyd\rho \\
&= \frac{1}{8\pi^2}\int_{\mathbf{R}_\rho}e^{i\rho(r-2^{-j}R_{-\ell}k'\cdot\gamma)}|\rho|\hat{\Psi}^{(N)}_{j,\ell}(\rho\gamma)d\rho \\
&= \frac{1}{8\pi^2}\int_{\mathbf{R}_\rho}e^{i\rho(r-2^{-j}R_{-\ell}k'\cdot\gamma)}|\rho|\hat{\Psi}^{(N)}_{j,0}(\rho R_\ell\gamma)d\rho \\
&= \frac{1}{2^{j+3}\pi^2(4p_N^2-1)^{1/4}}\int_{\mathbf{R}_\rho}e^{i\rho(r-2^{-j}R_{-\ell}k'\cdot\gamma)}|\rho|\chi_{\mathbf{S}^{(N)}_j}(\rho R_\ell\gamma)d\rho,
\end{aligned}$$

where

$$\mathbf{S}^{(N)}_j=\Big[-|\xi_2|\tan\frac{\pi}{2^N},|\xi_2|\tan\frac{\pi}{2^N}\Big]\times\Big\{[-2^j,-2^{j-1}]\cup[2^{j-1},2^j]\Big\}.$$

Thus, we get the following:

Lemma 2.1 *Let $N\geq 2$, $k'=\Big(k_1(4p_N^2-1)^{-1/2},k_2\Big)$ and $\gamma=(\cos\theta,\sin\theta)$. For $f\in\mathcal{S}(\mathbf{R}^2_x)$ satisfying (4) and (5), $U^{(k)}_{j,\ell}(r,\gamma)$ such that*

$$\alpha_{j,\ell,k} := \int_{\mathbf{R}^2} f(x)\overline{\Psi^{(N)}_{j,\ell}(x-2^{-j}R_{-\ell}k')}dx = \int_0^{2\pi}\int_{\mathbf{R}_r}\mathcal{R}(f)(r,\gamma)\overline{U^{(k)}_{j,\ell}(r,\gamma)}drd\theta,$$

is represented as

$$U^{(k)}_{j,\ell}(r,\gamma)=\frac{1}{2^{j+3}\pi^2(4p_N^2-1)^{1/4}}\int_{\mathbf{R}_\rho}e^{i\rho(r-2^{-j}R_{-\ell}k'\cdot\gamma)}|\rho|\chi_{\mathbf{S}^{(N)}_j}(\rho R_\ell\gamma)d\rho,$$

where

$$\mathbf{S}_j^{(N)} = \Big[-|\xi_2|\tan\frac{\pi}{2^N}, |\xi_2|\tan\frac{\pi}{2^N}\Big] \times \Big\{[-2^j, -2^{j-1}] \cup [2^{j-1}, 2^j]\Big\}.$$

Furthermore, in order to compute the coefficients $\alpha_{j,\ell,k}$ from a given function $\mathcal{R}(f)(r,\gamma)$, by changes of variables we have

$$\begin{aligned}\alpha_{j,\ell,k} &= \int_0^{2\pi}\int_{\mathbf{R}_r} \mathcal{R}(f)\Big(\frac{r+k'\cdot\gamma}{2^j}, R_{-\ell}\gamma\Big)\overline{2^{-j}U_{j,\ell}^{(k)}\Big(\frac{r+k'\cdot\gamma}{2^j}, R_{-\ell}\gamma\Big)}drd\theta \\ &= \sum_{\pm}\int_{-\frac{\pi}{2^N}\pm\frac{\pi}{2}}^{\frac{\pi}{2^N}\pm\frac{\pi}{2}}\int_{\mathbf{R}_r} \mathcal{R}(f)\Big(\frac{r+k'\cdot\gamma}{2^j}, R_{-\ell}\gamma\Big)\overline{\mathbf{U}(r,\theta)}drd\theta,\end{aligned}$$

with the following function $\mathbf{U}$ such that supp $\mathbf{U} \subset \mathbf{R}_r \times [-\frac{\pi}{2^N}+\frac{\pi}{2}, \frac{\pi}{2^N}+\frac{\pi}{2}] \cup [-\frac{\pi}{2^N}-\frac{\pi}{2}, \frac{\pi}{2^N}-\frac{\pi}{2}\Big]$:

$$\begin{aligned}\mathbf{U}(r,\theta) &:= 2^{-j}U_{j,\ell}^{(k)}\Big(\frac{r+k'\cdot\gamma}{2^j}, R_{-\ell}\gamma\Big) = \frac{2^{-j}}{2^{j+3}\pi^2(4p_N^2-1)^{1/4}}\int_{\mathbf{R}_\rho} e^{i2^{-j}\rho r}|\rho|\chi_{\mathbf{S}_j^{(N)}}(\rho\gamma)d\rho \\ &= \frac{2^{-j}}{2^{j+2}\pi^2(4p_N^2-1)^{1/4}}\int_0^{\infty}\cos(2^{-j}\rho r)\rho\chi_{\mathbf{S}_j^{(N)}}(\rho\gamma)d\rho \\ &= \frac{2^{-j}}{2^{j+2}\pi^2(4p_N^2-1)^{1/4}}\int_{\frac{2^{j-1}\pi}{|\sin\theta|}}^{\frac{2^j\pi}{|\sin\theta|}}\cos(2^{-j}\rho r)\rho d\rho = \frac{1}{(4p_N^2-1)^{1/4}(2\pi r)^2}\int_{|\frac{r\pi}{2\sin\theta}|}^{|\frac{r\pi}{\sin\theta}|}\cos(\rho)\rho d\rho \\ &= \frac{|\frac{r\pi}{\sin\theta}|\sin|\frac{r\pi}{\sin\theta}| + \cos|\frac{r\pi}{\sin\theta}| - |\frac{r\pi}{2\sin\theta}|\sin|\frac{r\pi}{2\sin\theta}| - \cos|\frac{r\pi}{2\sin\theta}|}{(4p_N^2-1)^{1/4}(2\pi r)^2},\end{aligned}$$

for $\theta \in [-\frac{\pi}{2^N}+\frac{\pi}{2}, \frac{\pi}{2^N}+\frac{\pi}{2}] \cup [-\frac{\pi}{2^N}-\frac{\pi}{2}, \frac{\pi}{2^N}-\frac{\pi}{2}]$, otherwise $\mathbf{U}(r,\theta) \equiv 0$. Therefore, noting that $\mathcal{R}(f)(\frac{\pm r - k'\cdot\gamma}{2^j}, -R_{-\ell}\gamma) = \mathcal{R}(f)(\frac{\mp r + k'\cdot\gamma}{2^j}, R_{-\ell}\gamma)$, we obtain

$$\begin{aligned}&\alpha_{j,\ell,k} \\ &= \sum_{\pm}\int_{-\frac{\pi}{2^N}\pm\frac{\pi}{2}}^{\frac{\pi}{2^N}\pm\frac{\pi}{2}}\int_{\mathbf{R}_r}\frac{|\frac{r\pi}{\sin\theta}|\sin|\frac{r\pi}{\sin\theta}| + \cos|\frac{r\pi}{\sin\theta}| - |\frac{r\pi}{2\sin\theta}|\sin|\frac{r\pi}{2\sin\theta}| - \cos|\frac{r\pi}{2\sin\theta}|}{(4p_N^2-1)^{1/4}(2\pi r)^2} \\ &\quad \times\mathcal{R}(f)\Big(\frac{r+k'\cdot\gamma}{2^j}, R_{-\ell}\gamma\Big)drd\theta \\ &= \sum_{\pm}\int_{-\frac{\pi}{2^N}\pm\frac{\pi}{2}}^{\frac{\pi}{2^N}\pm\frac{\pi}{2}}\int_0^{\infty}\frac{\frac{r\pi}{|\sin\theta|}\sin\frac{r\pi}{|\sin\theta|} + \cos\frac{r\pi}{|\sin\theta|} - \frac{r\pi}{2|\sin\theta|}\sin\frac{r\pi}{2|\sin\theta|} - \cos\frac{r\pi}{2|\sin\theta|}}{(4p_N^2-1)^{1/4}(2\pi r)^2} \\ &\quad \times\Big\{\mathcal{R}(f)\Big(\frac{r+k'\cdot\gamma}{2^j}, R_{-\ell}\gamma\Big) + \mathcal{R}(f)\Big(\frac{-r+k'\cdot\gamma}{2^j}, R_{-\ell}\gamma\Big)\Big\}drd\theta \qquad (6)\end{aligned}$$

$$
\begin{aligned}
&= \int_{-\frac{\pi}{2^N}+\frac{\pi}{2}}^{\frac{\pi}{2^N}+\frac{\pi}{2}} \int_0^\infty \frac{\frac{r\pi}{\sin\theta}\sin\frac{r\pi}{\sin\theta}+\cos\frac{r\pi}{\sin\theta}-\frac{r\pi}{2\sin\theta}\sin\frac{r\pi}{2\sin\theta}-\cos\frac{r\pi}{2\sin\theta}}{(4p_N^2-1)^{1/4}(2\pi r)^2} \\
&\quad \times\Big\{2\mathcal{R}(f)\Big(\frac{r+k'\cdot\gamma}{2^j}, R_{-\ell}\gamma\Big)+2\mathcal{R}(f)\Big(\frac{-r+k'\cdot\gamma}{2^j}, R_{-\ell}\gamma\Big)\Big\}drd\theta \\
&= \int_{-\frac{\pi}{2^N}+\frac{\pi}{2}}^{\frac{\pi}{2^N}+\frac{\pi}{2}} \int_0^\infty \frac{\tau\sin\tau+\cos\tau-\frac{\tau}{2}\sin\frac{\tau}{2}-\cos\frac{\tau}{2}}{2\pi(4p_N^2-1)^{1/4}\tau^2\sin\theta} \\
&\quad \times\Big\{\sum_{\pm}\mathcal{R}(f)\Big(\frac{k'\cdot\gamma}{2^j}\pm\frac{\tau\sin\theta}{2^j\pi}, R_{-\ell}\gamma\Big)\Big\}d\tau d\theta.
\end{aligned}
$$

Replacing $-\frac{\tau}{2}\sin\frac{\tau}{2}-\cos\frac{\tau}{2}$ by -1, we can obtain $\beta_{j,\ell,k}$ of the scaling function $\Phi_{j,\ell}^{(N)}\big(x-2^{-j}R_{-\ell}k'\big)$. Consequently, we get the following:

Proposition 2.2 *Let $N\geq 2$ and $J\in\mathbf{Z}$. For $f\in\mathcal{S}(\mathbf{R}_x^2)$ satisfying (4) and (5), the coefficients $\alpha_{j,\ell,k}$ and $\beta_{j,\ell,k}$ of the expansion*

$$
\begin{aligned}
f(x) &= \sum_{j\geq J+1}\sum_{1\leq\ell\leq 2^{N-1}}\sum_{k\in\mathbf{Z}^2}\alpha_{j,\ell,k}\Psi_{j,\ell}^{(N)}\big(x-2^{-j}R_{-\ell}k'\big) \\
&\quad +\sum_{1\leq\ell\leq 2^{N-1}}\sum_{k\in\mathbf{Z}^2}\beta_{J,\ell,k}\Phi_{J,\ell}^{(N)}\big(x-2^{-J}R_{-\ell}k'\big)
\end{aligned}
$$

are given by

$$
\begin{aligned}
\alpha_{j,\ell,k} &= \int_{-\frac{\pi}{2^N}+\frac{\pi}{2}}^{\frac{\pi}{2^N}+\frac{\pi}{2}} \int_0^\infty \frac{\tau\sin\tau+\cos\tau-\frac{\tau}{2}\sin\frac{\tau}{2}-\cos\frac{\tau}{2}}{2\pi(4p_N^2-1)^{1/4}\tau^2\sin\theta} \\
&\quad \times\Big\{\sum_{\pm}\mathcal{R}(f)\Big(\frac{k'\cdot\gamma}{2^j}\pm\frac{\tau\sin\theta}{2^j\pi}, R_{-\ell}\gamma\Big)\Big\}d\tau d\theta, \\
\beta_{j,\ell,k} &= \int_{-\frac{\pi}{2^N}+\frac{\pi}{2}}^{\frac{\pi}{2^N}+\frac{\pi}{2}} \int_0^\infty \frac{\tau\sin\tau+\cos\tau-1}{2\pi(4p_N^2-1)^{1/4}\tau^2\sin\theta} \\
&\quad \times\Big\{\sum_{\pm}\mathcal{R}(f)\Big(\frac{k'\cdot\gamma}{2^j}\pm\frac{\tau\sin\theta}{2^j\pi}, R_{-\ell}\gamma\Big)\Big\}d\tau d\theta.
\end{aligned}
$$

Remark 2.3 *$\tau=0$ in the kernel functions is a removable singularity, because*

$$
\begin{aligned}
\frac{\tau\sin\tau+\cos\tau-\frac{\tau}{2}\sin\frac{\tau}{2}-\cos\frac{\tau}{2}}{\tau^2} &= \mathrm{sinc}\frac{\tau}{2}\Big(\cos\frac{\tau}{2}-\frac{1}{4}\Big)-\Big\{\mathrm{sinc}\frac{\tau}{4}\Big\}^2\Big(\frac{1}{4}\cos\frac{\tau}{2}+\frac{1}{8}\Big), \\
\frac{\tau\sin\tau+\cos\tau-1}{\tau^2} &= \mathrm{sinc}\tau-\frac{1}{2}\Big\{\mathrm{sinc}\frac{\tau}{2}\Big\}^2.
\end{aligned}
$$

A full-discretization of the inversion formula (3) which includes a sum with respect to coprime numbers, is known (see [11]). Proposition 2.2 gives a frame expansion of f directly from $\mathcal{R}(f)(r,\gamma)$ without the use of the Fourier-based inversion formula

$$f(x) = \frac{1}{8\pi^2}\int_0^{2\pi}\int_{\mathbf{R}_s} e^{isx\cdot\gamma}\mathcal{F}_r[\mathcal{R}(f)](s,\gamma)|s|dsd\theta. \tag{7}$$

Remark 2.4 *Let $\mathcal{H}f = \frac{i}{\pi}f * r^{-1} = \frac{i}{\pi}\int f(r-p)p^{-1}dp$. The solution of the Cauchy problem $\partial_t^2 u - (\partial_{x_1}^2 + \partial_{x_2}^2)u = 0$ with $u(0,x) = f(x)$ and $\partial_t u(0,x) = 0$ is given by*

$$u(t,x) = \frac{1}{4\pi i}\int_{S^1}\mathcal{H}\Big(\partial_r\mathcal{R}(f)(r,\gamma)\Big)\Big|_{r=t+x\cdot\gamma}d\sigma(\gamma) = \frac{1}{4\pi}(-\Delta)^{1/2}\mathcal{R}^*\Big(\mathcal{R}(f)(r+t,\gamma)\Big)(x).$$

This just corresponds to (3) with $\mathcal{R}(f)(r+t,\gamma)$ instead of $\mathcal{R}(f)(r,\gamma)$. In the same way, we arrive at the formula (6) with $\mathcal{R}(f)(r+t,\gamma)$. Finally, replacing $\mathcal{R}(f)$ of Proposition 2.2 with

$$\tilde{\mathcal{R}}_t(f) := \frac{1}{2}\Big\{\mathcal{R}(f)(r+t,\gamma) + \mathcal{R}(f)(r-t,\gamma)\Big\},$$

we can also represent the solution $u(t,x)$ as follows (see §2.4):

$$u(t,x) = \sum_{j\geq J+1}\sum_{1\leq\ell\leq 2^{N-1}}\sum_{k\in\mathbf{Z}^2}\alpha_{j,\ell,k}\Psi_{j,\ell}^{(N)}\Big(x-2^{-j}R_{-\ell}k'\Big) + \sum_{1\leq\ell\leq 2^{N-1}}\sum_{k\in\mathbf{Z}^2}\beta_{J,\ell,k}\Phi_{J,\ell}^{(N)}\Big(x-2^{-J}R_{-\ell}k'\Big).$$

2.2 Function of Pyramid Form

For the later simulation in §2.3 we derive the Fourier transform of the function of a pyramid form.

Proposition 2.5 *Let*

$$f(x_1,x_2) = \max\Big\{1-|x_1|-|x_2|, 0\Big\}.$$

Then, we have

$$\mathcal{R}(f)(r,\gamma) = \frac{|\gamma_1|(|r|+|\gamma_2|)^2 - \gamma_1\Big|\big|r|-|\gamma_2|\Big|(|r|-|\gamma_2|) - |\gamma_2|(|r|+|\gamma_1|)^2 + |\gamma_2|\Big|\big|r|-|\gamma_1|\Big|(|r|-|\gamma_1|)}{2|\gamma_1||\gamma_2|(|\gamma_2|^2-|\gamma_1|^2)}.$$

Proof. Define the triangular function $\mathrm{tri}(t)$ by

$$\mathrm{tri}(t) = \max\{1 - |t|, 0\}.$$

This function satisfies

$$a\ \mathrm{tri}\Big(\frac{t}{a}\Big) = a \max\Big\{1 - \Big|\frac{t}{a}\Big|, 0\Big\} = \max\{a - |t|, 0\} \quad \text{for} \quad a > 0.$$

Let us rewrite $f(x_1, x_2) = \max\Big\{(1-|x_2|) - |x_1|, 0\Big\}$. Suppose that x_2 is given first and fixed in $|x_2| \le 1$. Then, x_1 is forced to satisfy $|x_1| \le 1 - |x_2| =: a(> 0)$. Define $\mathrm{sinc}t = \frac{\sin t}{t}$. By the formula $\mathrm{tri}\Big(\frac{x_1}{a}\Big) = |a|\mathrm{sinc}^2\Big(\frac{a\xi_1}{2}\Big)$, it holds that

$$\mathcal{F}_{x_1}[f](\xi_1, x_2) = \mathcal{F}_x\Big[a\mathrm{tri}\Big(\frac{x_1}{a}\Big)\Big](\xi_1, x_2) = a^2\mathrm{sinc}^2\Big(\frac{a\xi_1}{2}\Big) = \frac{4}{\xi_1^2}\sin^2\frac{(1-|x_2|)\xi_1}{2}$$

for $|x_2| \le 1$, and otherwise equals 0. Therefore, we also obtain

$$\mathcal{F}_{x_1x_2}[f](\xi_1, \xi_2) = \frac{8}{\xi_1^2}\int_0^1 \cos(x_2\xi_2)\sin^2\frac{(1-|x_2|)\xi_1}{2}dx_2,$$

since

$$\int_{-1}^0 e^{-ix_2\xi_2}\sin^2\frac{(1-|x_2|)\xi_1}{2}dx_2 = \int_0^1 e^{ix_2\xi_2}\sin^2\frac{(1-|x_2|)\xi_1}{2}dx_2,$$

and $e^{-ix_2\xi_2} + e^{ix_2\xi_2} = 2\cos(x_2\xi_2)$. Moreover, we can compute the following:

$$\begin{aligned}
&\mathcal{F}_{x_1x_2}[f](\xi_1, \xi_2) \\
=\ & \frac{8}{\xi_1^2}\int_0^1 \cos(x_2\xi_2)\sin^2\frac{(1-x_2)\xi_1}{2}dx_2 = \frac{4}{\xi_1^2}\int_0^1 \cos(x_2\xi_2)\Big\{1 - \cos\Big((1-x_2)\xi_1\Big)\Big\}dx_2 \\
=\ & \frac{4}{\xi_1^2}\int_0^1 \cos(x_2\xi_2)dx_2 - \frac{2}{\xi_1^2}\int_0^1 \cos\Big(x_2\xi_2 + (1-x_2)\xi_1\Big)dx_2 \\
&\quad -\frac{2}{\xi_1^2}\int_0^1 \cos\Big(x_2\xi_2 - (1-x_2)\xi_1\Big)dx_2 \\
=\ & \frac{4}{\xi_1^2}\Big[\frac{\sin(x_2\xi_2)}{\xi_2}\Big]_0^1 - \frac{2}{\xi_1^2}\Big[\frac{\sin\Big(x_2\xi_2 + (1-x_2)\xi_1\Big)}{\xi_2 - \xi_1}\Big]_0^1 - \frac{2}{\xi_1^2}\Big[\frac{\sin\Big(x_2\xi_2 - (1-x_2)\xi_1\Big)}{\xi_2 + \xi_1}\Big]_0^1 \\
=\ & \frac{4}{\xi_1^2}\mathrm{sinc}\xi_2 - \frac{2}{\xi_1^2}\Big\{\frac{\sin\xi_2}{\xi_2 - \xi_1} - \frac{\sin\xi_1}{\xi_2 - \xi_1}\Big\} - \frac{2}{\xi_1^2}\Big\{\frac{\sin\xi_2}{\xi_2 + \xi_1} + \frac{\sin\xi_1}{\xi_2 + \xi_1}\Big\} \\
=\ & 4\frac{\xi_1\sin\xi_2 - \xi_2\sin\xi_1}{\xi_1\xi_2(\xi_1^2 - \xi_2^2)}.
\end{aligned}$$

Note that $4\frac{\xi_1 \sin\xi_2 - \xi_2 \sin\xi_1}{\xi_1\xi_2(\xi_1^2-\xi_2^2)}$ is an analytic function by the Maclaurin expansion. By the Fourier slice theorem, the formulas $\mathcal{F}_\rho\Big[\mathrm{sinc}(a\rho)\Big] = \frac{\pi}{|a|}\chi_{(-1/2,1/2)}\Big(\frac{r}{2a}\Big) = \frac{\pi}{|a|}\chi_{(-|a|,|a|)}(r)$ and $\mathcal{F}_\rho[\rho^{-2}] = -\pi|r|$ yield

$$
\begin{aligned}
& \mathcal{R}(f)(r,\gamma) \\
= & \frac{1}{2\pi}\int_{\mathbf{R}_\rho} e^{i\rho r}\mathcal{F}_{x_1x_2}[f](\rho\gamma_1,\rho\gamma_2)d\rho = \frac{2}{\pi}\int_{\mathbf{R}_\rho} e^{i\rho r}\frac{\gamma_1\sin(\rho\gamma_2)-\gamma_2\sin(\rho\gamma_1)}{\rho^3\gamma_1\gamma_2(\gamma_1^2-\gamma_2^2)}d\rho \\
= & \frac{2}{\pi(\gamma_1^2-\gamma_2^2)}\int_{\mathbf{R}_\rho}\frac{e^{i\rho r}}{\rho^2}\Big\{\mathrm{sinc}(\rho\gamma_2)-\mathrm{sinc}(\rho\gamma_1)\Big\}d\rho \\
= & \frac{1}{\pi^2(\gamma_1^2-\gamma_2^2)}\int_{\mathbf{R}_{r'}}\mathcal{F}_\rho\Big[\frac{e^{i\rho r}}{\rho^2}\Big](r')\mathcal{F}_\rho\Big[\mathrm{sinc}(\rho\gamma_2)-\mathrm{sinc}(\rho\gamma_1)\Big](r')dr' \\
= & \frac{1}{\gamma_2^2-\gamma_1^2}\int_{\mathbf{R}_{r'}}|r'-r|\Big\{\frac{1}{|\gamma_2|}\chi_{(-|\gamma_2|,|\gamma_2|)}(r')-\frac{1}{|\gamma_1|}\chi_{(-|\gamma_1|,|\gamma_1|)}(r')\Big\}dr' \\
= & \frac{1}{|\gamma_2|^2-|\gamma_1|^2}\int_{\mathbf{R}_{r''}}|r''|\Big\{\frac{1}{|\gamma_2|}\chi_{(-|\gamma_2|-r,|\gamma_2|-r)}(r'')-\frac{1}{|\gamma_1|}\chi_{(-|\gamma_1|-r,|\gamma_1|-r)}(r'')\Big\}dr''.
\end{aligned}
$$

We also see that for $p = 1, 2$

$$
\begin{aligned}
& \int_{\mathbf{R}_{r''}}|r''|\chi_{(-|\gamma_p|-r,|\gamma_p|-r)}(r'')dr'' \\
= & \int_0^\infty r''\chi_{(-|\gamma_p|-r,|\gamma_p|-r)}(r'')dr'' - \int_{-\infty}^0 r''\chi_{(-|\gamma_p|-r,|\gamma_p|-r)}(r'')dr'' \\
= & \int_{\max\{0,-|\gamma_p|-r\}}^{\max\{0,|\gamma_p|-r\}} r''dr'' - \int_{\min\{0,-|\gamma_p|-r\}}^{\min\{0,|\gamma_p|-r\}} r''dr'' \\
= & \frac{(\max\{0,|\gamma_p|-r\})^2-(\max\{0,-|\gamma_p|-r\})^2-(\min\{0,|\gamma_p|-r\})^2+(\min\{0,-|\gamma_p|-r\})^2}{2} \\
= & \begin{cases} 2r|\gamma_p| & \text{for} \quad r > |\gamma_p| \\ |\gamma_p|^2 + r^2 & \text{for} \quad -|\gamma_p| \le r \le |\gamma_p| \\ -2r|\gamma_p| & \text{for} \quad r < -|\gamma_p| \end{cases} = \begin{cases} 2|r||\gamma_p| & \text{for} \quad |r| > |\gamma_p| \\ |\gamma_p|^2 + r^2 & \text{for} \quad |r| \le |\gamma_p| \end{cases} \\
= & \frac{(|r|+|\gamma_p|)^2 - \Big||r|-|\gamma_p|\Big|(|r|-|\gamma_p|)}{2}.
\end{aligned}
$$

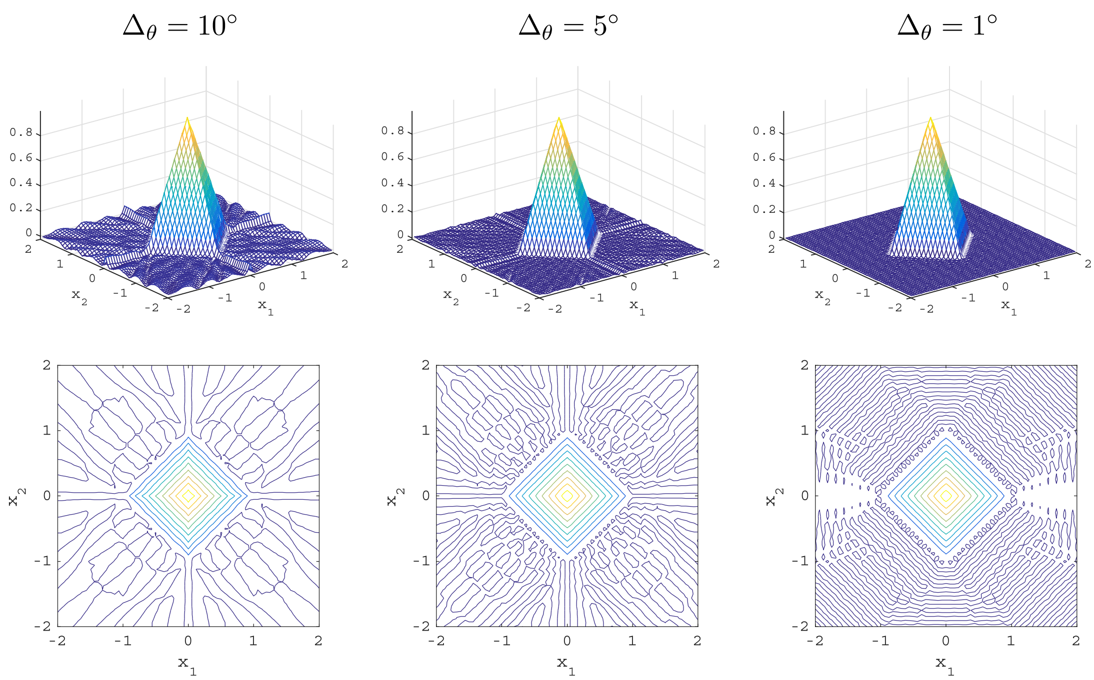
Thus, we have

$$
\begin{aligned}
& \mathcal{R}(f)(r,\gamma) \\
= & \frac{|\gamma_1|(|r|+|\gamma_2|)^2-\gamma_1|\Big||r|-|\gamma_2|\Big|(|r|-|\gamma_2|)-|\gamma_2|(|r|+|\gamma_1|)^2+|\gamma_2|\Big||r|-|\gamma_1|\Big|(|r|-|\gamma_1|)}{2|\gamma_1||\gamma_2|(|\gamma_2|^2-|\gamma_1|^2)}.
\end{aligned}
$$

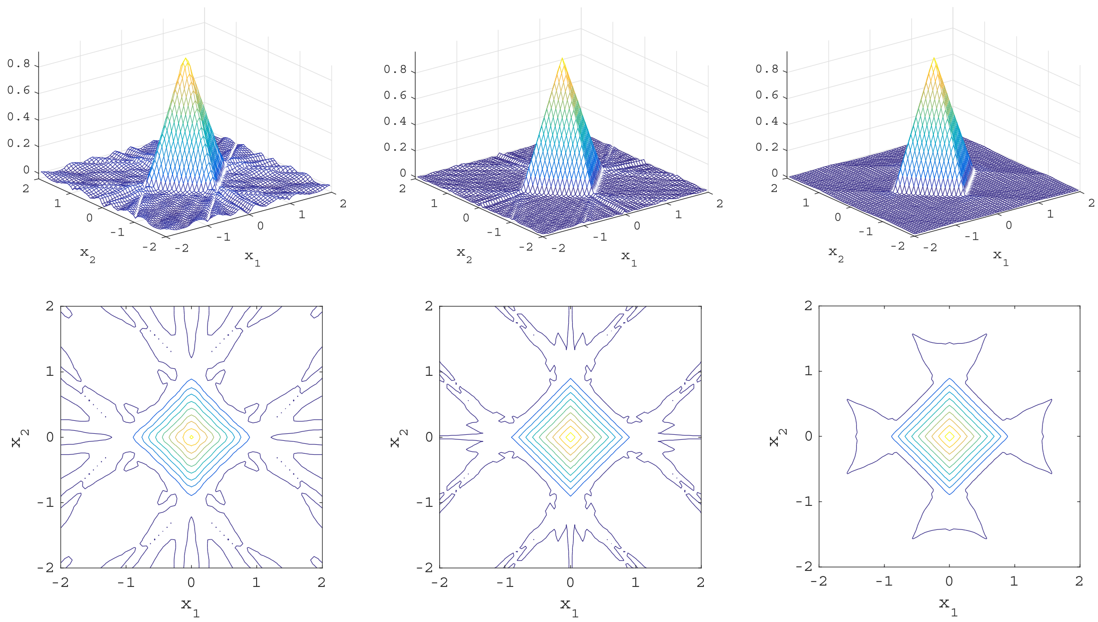
2.3 Inversion Formula of the Radon Transform

In Proposition 2.2, the coefficients $\alpha_{j,\ell,k}$ are computed using a kernel and the Radon transform $\mathcal{R}(f)(r,\gamma)$ instead of taking the inner product. This yields a reconstruction of f directly from its Radon transform without the use of the Fourier-based inversion formula (7). In order to demonstrate the performance for the proposed Radon inversion formula, we compared its reconstruction quality with that of (7). The results for both normal and contour plots are shown in Figure 1. Here, we denote the step width of the projection angles of the Radon transform by Δ_θ [deg]. We considered three projection angles of the Radon transform $\mathcal{R}(f)(r,\gamma)$, which are $\Delta_\theta = 10°, 5°, 1°$ because the integration with respect to angle θ has an influence on the reconstruction quality. We set $N=2$ for our frame expansion to make the comparison simple. $f(x_1,x_2)=\max\{1-|x_1|-|x_2|,0\}$ does not satisfy the assumption of $f\in\mathcal{S}(\mathbf{R}_x^2)$ in Proposition 2.2. But, the formula of $\mathcal{R}(\max\{1-|x_1|-|x_2|,0\})(r,\gamma)$ in Proposition 2.5 can be used for the simulation. Figure 1(a) shows the results of the reconstructions using the Fourier-based inversion formula. Obviously, the reconstruction quality depends on Δ_θ, and the reconstructed f seems to be well represented, especially when $\Delta_\theta = 1°$. These results are also true for our PTF-based inversion formula shown in Figure 1(b).

Let us now compare edge components of the graphs shown in their contour plots. For the case of $\Delta_\theta = 10°$, serious distortions appear at some edges for both cases. However, when we have finer projection steps, such as $\Delta_\theta = 5°$ or $\Delta_\theta = 1°$, the sharp edge components of a pyramid shape are well represented for both cases. To describe the pyramid (inside of the support f), there are slight differences between the two cases, but no definite statement can be made as to which is better. On the other hand, when we focus outside the support of f in the contour plots, the both results are completely different. The Fourier-based method has some particular noise-like patterns outside the support of f. Remarkably, this phenomenon is dramatically reduced in our cases. The Fourier-based inversion formula is sensitive to noise due to the use of the ramp filter $|s|$ in (7), which amplifies high-frequency components. The inversion formula considered herein uses the multidirectional frame expansion and computes the reconstruction of f directly from $\mathcal{R}(f)(r,\gamma)$. Furthermore, we can control the precision of the linear combination of the PTFs with respect to the scale, shift, and rotation. These results clearly show that the proposed PTF-based inversion formula of the Radon transform has some advantages over the conventional Fourier-based method.

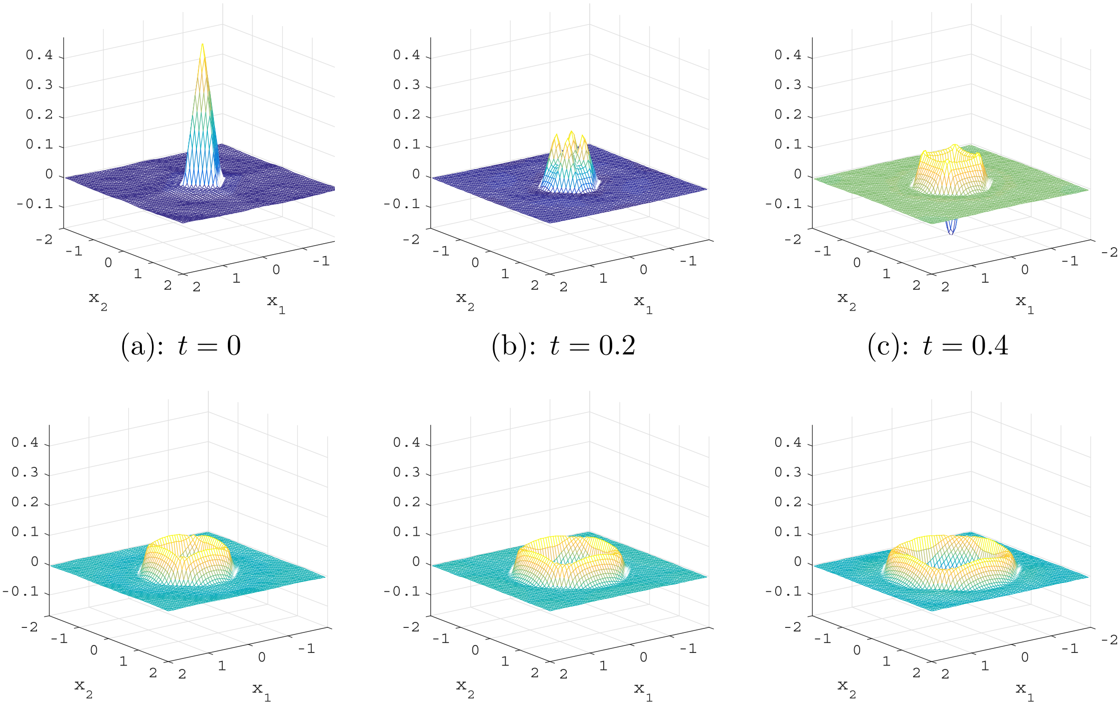


(a): Fourier



(b): PTF ($N = 2$)

Figure 1: Reconstructions of f from $\mathcal{R}(f)(r, \gamma)$. (a): Fourier-based inversion formula by (7); (b): PTF-based inversion formula by Proposition 2.2.



(a): $t = 0$ (b): $t = 0.2$ (c): $t = 0.4$

(d): $t = 0.6$ (e): $t = 0.8$ (f): $t = 1$

Figure 2: $u(t, x)$ of the Cauchy problem for function of pyramid form.

2.4 Wave Equation

We show an application of our PTF-based Radon inversion formula for the solution of the Cauchy problem of the wave equation, as mentioned in Remark 2.4. The solution $u(t, x)$ can be represented by using our frame expansion whose $\alpha_{j,\ell,k}$ and $\beta_{j,\ell,k}$ are computed with $\tilde{\mathcal{R}}_t(f)$. Here we slightly modify the function of a pyramid form as $f(x_1, x_2) = 2^{-1} \max\{1 - |2x_1| - |2x_2|, 0\}$ to make the wave propagation easy to see. We set $N = 4$ for the frame expansion. Figure 2 shows $u(t, x)$ with $t = 0, 0.2, 0.4, 0.6, 0.8$, and 1, which demonstrates how the wave propagates to the function that has sharp edge components. We observe sharp edges at four corners of the wave with $t = 0.2$. After that, the largest wave component propagates in the form of a rectangle. This implies that our frame expansion with the Radon inversion formula can capture multidirectional information of a function well.

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Addresses of the authors

Kensuke Fujinoki, Department of Mathematical Sciences, Tokai University, Kitakaname, Hiratsuka, Kanagawa 259-1292, Japan
e-mail: fujinoki@tokai-u.jp

Hirofumi Hashimoto, Institute of Mathematics, Tsukuba University, Tsukuba Ibaraki 305-8571, Japan
e-mail: h.hashimoto@math.tsukuba.ac.jp

Tamotu Kinoshita, Institute of Mathematics, Tsukuba University, Tsukuba Ibaraki 305-8571, Japan
e-mail: kinosita@math.tsukuba.ac.jp