Stochastic analysis on infinite-dimensional stochastic differential equations related to random matrices

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Abstract

This is an article for announcement of our papers [8, 10, 11]. We consider dynamics associated with random point fields with infinitely many particles. The dynamics is introduced through Dirichlet form theory, and it can be represented by infinite-dimensional stochastic differential equation (ISDE). In particular, we treat infinite particle systems with long-range interaction, which naturally appear in random matrix theory.

1 Introduction

1.1 Gaussian Ensembles and the Airy random point fields

We begin by recalling elemental facts in random matrix theory and introducing the Airy random point field (see [1, 6, 14, 22] etc. for random matrix theory). We consider Gaussian ensembles, that is, let μ_β^N be the random point field on $\mathbb{R}$ with N-particles such that its probability density function with respect to the Lebesgue measure $d\mathbf{x}_N = dx_1 \dots dx_N$ is given by

$$\mu_\beta^N(d\mathbf{x}_N) \propto \prod_{i<j}^N |x_i - x_j|^\beta \prod_{k=1}^N e^{-N\beta x_k^2} d\mathbf{x}_N.$$

Here, β is the inverse temperature, and we focus on the case $\beta = 1, 2, 4$ in what follows. Remark that rigorously speaking, a random point field is a probability measure on the configuration space, not on the product space of $\mathbb{R}$. In the former, particles are not distinguished each other, whilst in the latter each particle is labeled and distinguished, but we abuse the two concepts. The random point field μ_β^N gives an eigenvalue distribution of $N \times N$ Gaussian random matrices with symmetry.

To see local fluctuation around the top particle, we take the soft-edge scaling $x \mapsto \frac{s}{2N^{2/3}} + 1$, and let $\mu_{\mathrm{Ai},\beta}^N$ be the probability measure under this scaling, that is,

$$\mu_{\mathrm{Ai},\beta}^N(d\mathbf{s}_N) \propto \prod_{i<j}^N |s_i - s_j|^\beta \prod_{k=1}^N \exp\Big\{ -\beta N \Big(\frac{s_k}{2N^{\frac{2}{3}}} + 1 \Big)^2 \Big\} d\mathbf{s}_N.$$

Let $\mu_{\mathrm{Ai},\beta}$ be the Airy$_\beta$ random point field, which is a typical infinite particle system with long-range interaction potential. In fact, it is known as a classical result that

$$\lim_{N\to\infty} \mu^N_{\mathrm{Ai},\beta} = \mu_{\mathrm{Ai},\beta} \text{ weakly.}$$

For $\beta = 2$, n-correlation function $\rho^n_{\mathrm{Ai},2}$ for $\mu_{\mathrm{Ai},2}$ is given by

$$\rho^n_{\mathrm{Ai},2}(x_1\dots,x_n) = \det\Big[\frac{\mathrm{Ai}(x_i)\mathrm{Ai}'(x_j) - \mathrm{Ai}'(x_i)\mathrm{Ai}(x_j)}{x_i - x_j}\Big]_{1\le i,j\le n}.$$

For $\beta = 1, 4$, correlation functions for $\mu_{\mathrm{Ai},\beta}$ have similar expressions. The Airy$_\beta$ random point field has infinitely many particles on $\mathbb{R}$, but there are finite particles on $[0,\infty)$ almost surely. Note that the distribution of the right-most particle is the celebrated Tracy-Widom-β distribution.

1.2 Infinite-dimensional stochastic differential equations related to the Airy random point fields

We study a distorted Brownian motion associated with a random point field, which is a natural equilibrium dynamics with respect to the given probability measure. The distorted Brownian motion can be described as an infinite-dimensional stochastic differential equation (ISDE).

To find an ISDE related to the Airy random point field, consider N-dimensional stochastic differential equation (SDE) and take $N\to\infty$: then set the Dirichlet form on $L^2(\mathbb{R}^N, \mu^N_{\mathrm{Ai},\beta})$ given by

$$\mathcal{E}^N(f,g) = \frac{1}{2}\int_{\mathbb{R}^N}\sum_{i=1}^N \nabla_i f(\mathbf{s}_N)\cdot\nabla_i g(\mathbf{s}_N)\, d\mu^N_{\mathrm{Ai},\beta}(d\mathbf{s}_N).$$

This Dirichlet form gives a distorted Brownian motion associated with $\mu^N_{\mathrm{Ai},\beta}$.

This dynamics can be represented as a stochastic differential equation. By integration by parts for this Dirichlet integral, we obtain a generator. The generator gives the SDE representation as follows: for $1\le i\le N$,

$$dX^{N,i}_t = dB^i_t + \frac{\beta}{2}\Big\{\sum_{1\le j\ne i\le N}\frac{1}{X^{N,i}_t - X^{N,j}_t} - \frac{X^{N,i}_t}{2N^{\frac{1}{3}}} - N^{\frac{1}{3}}\Big\}dt, \tag{1}$$

where $\{B^i_t\}_{i=1,\dots,N}$ are independent Brownian motions. The SDE (1) shows that each Brownian particle repel each other by the logarithmic potential, which is a long-range potential.

Taking as $N\to\infty$, (1) must be become an ISDE related to the Airy$_\beta$ random point field. However, it is difficult to find limit formula, because (1) has divergent term and the informal limit is given by

$$dX^i_t = dB^i_t + \frac{\beta}{2}\Big\{\sum_{1\le j\ne i\le\infty}\frac{1}{X^{N,i}_t - X^{N,j}_t} - \infty\Big\}dt.$$

The limit ISDE is obtained by Osada-Tanemura [20]. For $\beta = 1, 2, 4$, an ISDE representation related to the Airy$_\beta$ random point field is given by the following Airy$_\beta$ interacting ISDE:

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{s\to\infty} \Big\{ \sum_{|X_t^j|<s, j\neq i} \frac{1}{X_t^i - X_t^j} - \int_{|x|<s} \frac{\hat{\rho}(x)}{-x} dx \Big\} dt,\ i \in \mathbb{N}, \tag{2}$$

where $\hat{\rho}(x) = \frac{\mathbf{1}_{(-\infty,0)}(x)}{\pi}\sqrt{-x}$.

To obtain (2), they showed exact cancellation of the divergent term and the interaction term with concrete computation using asymptotic formula of Hermite polynomials.

1.3 The universality of random matrices

Universality of random matrices (or log-gases), which is a central topic in random matrix theory, has been developed rapidly in the several decades (see [5, 4, 23] and references there in). The universality claims that finite particle systems such as eigenvalue distributions of random matrices of a quite wide class, or log-gases with general free potential, converge universal random point fields as the number of particles to infinity (up to suitable scaling). There are several universal limit, and the Airy$_\beta$ random point field is an example of these.

We focus on soft-edge universality for log-gases with general free potential. We generalise μ_β^N as follows. For $\beta = 1, 2, 4$ and even degree polynomial $V(x) = \sum_{i=0}^{2l} \kappa_i x^i$ ($\kappa_{2l} > 0$), set β-log-gas with free potential V

$$\mu_{\beta,V}^N(d\mathbf{x}_N) \propto \prod_{i<j}^N |x_i - x_j|^\beta \prod_{k=1}^N e^{-\beta N V(x_k)} d\mathbf{x}_N.$$

Take the soft-edge scaling $x \mapsto N^{-\frac{1}{2l}}\Big\{c_N\Big(1 + \frac{s}{\alpha_N N^{\frac{2}{3}}}\Big) + d_N\Big\}$, then the scaled probability measure $\mu_{\mathrm{Ai},\beta,V}^N$ is given by

$$\mu_{\mathrm{Ai},\beta,V}^N(d\mathbf{s}^N) \propto \prod_{i<j}^N |s_i - s_j|^\beta \prod_{k=1}^N e^{-\beta N V\left(N^{-\frac{1}{2l}}\left\{c_N\left(1+\frac{s_k}{\alpha_N N^{2/3}}\right)+d_N\right\}\right)} d\mathbf{s}^N.$$

Here, c_N, α_N, d_N are constants depending only on N and V explicitly given in [3].

Then the soft-edge universality holds:

$$\lim_{N\to\infty} \mu_{\mathrm{Ai},\beta,V}^N = \mu_{\mathrm{Ai},\beta} \text{ weakly.} \tag{3}$$

Note that the limit $\mu_{\mathrm{Ai},\beta}$ is independent of V, and in this sense, the Airy$_\beta$ random point field is universal. This is a typical result of universality of random matrices.

1.4 Dynamical Universality

The universality result (3) can be regarded as central limit theorem in random matrix theory. Then a natural question is what is invariance principle corresponding to (3). In other word, we consider what is a dynamical version of the soft-edge universality.

By the same procedure as in the case $V(x) = x^2$, we deduce an N-dimensional SDE associated with soft-edge scaled log-gas $\mu^N_{\mathrm{Ai},\beta,V}$: for $1 \le i \le N$,

$$dX_t^{N,i} = dB_t^i + \frac{\beta}{2}\Bigg\{ \sum_{1\le j\ne i\le N} \frac{1}{X_t^{N,i} - X_t^{N,j}} \\ - \frac{N^{\frac{1}{3}-\frac{1}{2l}}c_N}{2\alpha_N} V'\Big(N^{-\frac{1}{2l}}\Big\{c_N\Big(1+\frac{X_t^{N,i}}{\alpha_N N^{\frac{2}{3}}}\Big)+d_N\Big\}\Big)\Bigg\}dt, \tag{4}$$

Keeping in mind the universality result (3), we expect that the limit ISDE of (4) as $N \to \infty$ is also the Airy$_\beta$ interacting ISDE (2). Here, the limit is independent of V, and (2) is supposed to be universal dynamics. In other words, we expect approximations of random point fields derive that of dynamics accordingly. Our problem is how to proof this dynamical transition rigorously.

One way to prove the limit transition is to calculate the drift term concretely, which is the same argument in [20], but it is easy to imagine such argument takes effort. It is because we have to show exact calculation of a divergent term and the logarithmic interaction term in (4), which is a sensitive problem, and even for the simplest case $V(x) = x^2$, it involves hard analysis. To avoid such hard calculation, we constructed a general framework such that approximation of random point fields implies that of dynamics automatically. Our general theorems show under some reasonable assumption of the limit dynamics, convergence of random point fields implies that of stochastic dynamics automatically without exact computation for drift terms. We shall see the general framework in the next section.

2 Settings and general convergence theorems

In this section, we see general theorems. In this paper, we specify two conditions which are crucial for convergence of dynamics. One is strong convergence of random point fields (see **(A1)** or **(A1')** below). The other one is uniqueness of Dirichlet forms associated with a limit random point field (**(A2)** below).

2.1 Approximation sequences of random point fields

Set $S = \mathbb{R}^d$, and let S be the configuration space over S given by

$$\mathsf{S} = \{\mathsf{s} = \sum_i \delta_{s_i}\ ;\ s_i \in S, \mathsf{s}(K) < \infty \text{ for any compact set } K \subset S\}.$$

A probability measure μ on S is called a random point field, also known as a point process.

Suppose that a sequence of random point fields $\{\mu^N\}_{N\in\mathbb{N}}$ with $\mu^N(\mathsf{s}(S)=N)=1$ satisfies

$$\lim_{N\to\infty}\mu^N=\mu \text{ in law.} \tag{5}$$

Hereafter, we suppose certain reasonable conditions for existence of dynamics associated with μ^N and μ without writing it. We will not pursue these conditions here (see [16, 19] for the existence of associated S-valued diffusion, [18] for ISDE representation).

For $S_r=\{x\in S\,;\,|x|\le r\}$, we define $m_{r,n}$ and $m_{r,n}^N$ as the n-point density of μ and μ^N with respect to the Lebesgue measure on S_r, respectively. These density are supposed to be continuous. Recall that weak convergence of random point fields is assumed in (5). Additionally, we assume strong convergence in the following sense:

(A1) $\lim_{N\to\infty}||\frac{m_{r,n}^N}{m_{r,n}}-1||_{S_r^n}=0$ for any $r,n\in\mathbb{N}$.

Here, $\|\cdot\|_A$ is the L^∞-norm on A. It is also enough to assume **(A1')** below instead of **(A1)**. Practically, **(A1)** is easier to confirm than **(A1')**.

(A1') For any $n,r\in\mathbb{N}$, the following two hold;

$$\mathrm{Cap}^\mu(\{\mathsf{s};m_{r,n}(\mathsf{s})=0\})=0, \tag{6}$$

$$\lim_{N\to\infty}||m_{r,n}^N-m_{r,n}||_{S_r^n}=0. \tag{7}$$

In (6), $m_{r,n}$ is regarded as a function on the configuration space. Furthermore, Cap^μ is the one capacity associated with Dirichlet form $(\mathcal{E},\mathcal{D})$, which will be introduced in the next subsection.

Remark (1) The condition (6) can be checked easily. In fact, non-colliding property of particles of the limit diffusion implies (6).
(2) A crucial assumption in **(A1')** is strong convergence of densities (7). Remark that weak convergence is not enough to show convergence of corresponding dynamics even in finite-dimension. Hence, stronger condition than weak convergence is needed.
(3) With a marginal assumption, compact uniform convergence of correlation functions implies (7).

2.2 Two Dirichlet forms associated with μ

To state the second essential assumption, we introduce two schemes of Dirichlet forms associated with μ. They are natural infinite volume limit of Dirichlet forms on compact subsets.

Let $\mathcal{D}_\circ$ be all of local smooth functions $f:\mathsf{S}\to\mathbb{R}$. For $f,g\in\mathcal{D}_\circ$ we set

$$\mathbb{D}[f,g](\mathsf{s})=\frac{1}{2}\sum_i\nabla_{s_i}\check{f}(\mathbf{s})\cdot\nabla_{s_i}\check{g}(\mathbf{s}),$$

$$\mathbb{D}_r^m[f,g](\mathsf{s})=\begin{cases}\frac{1}{2}\sum_{i;s_i\in S_r}\nabla_{s_i}\check{f}(\mathbf{s})\cdot\nabla_{s_i}\check{g}(\mathbf{s}) & (\mathsf{s}\in\mathsf{S}_r^m),\\ 0 & (\mathsf{s}\notin\mathsf{S}_r^m).\end{cases}$$

Here for $\mathsf{s} = \sum_i \delta_{s_i}$ and $\mathbf{s} = (s_i)$, $\check{f}$ is a symmetric function satisfying $\check{f}(\mathbf{s}) = f(\mathsf{s})$. Furthermore, let $\mathsf{S}_r^m = \{\mathsf{s} \in \mathsf{S}\,;\, \mathsf{s}(S_r) = m\}$. We set

$$\mathcal{E}(f,g) = \int_{\mathsf{S}} \mathbb{D}[f,g](\mathsf{s}) d\mu(\mathsf{s}), \quad \mathcal{E}_r^m(f,g) = \int_{\mathsf{S}} \mathbb{D}_r^m[f,g](\mathsf{s}) d\mu(\mathsf{s}),$$
$$\mathcal{D}_\circ^\mu = \{f \in D_\circ \cap L^2(\mu)\,;\, E(f,f) < \infty\}.$$

Under suitable condition for μ, $(\mathcal{E}_r^m, \mathcal{D}_\circ^\mu)$ is closable on $L^2(\mu)$ for $m, r \in \mathbb{N}$. Hence, all operations taking closure below are justified.

Set $\mathcal{E}_r^\infty = \sum_{m=1}^\infty \mathcal{E}_r^m$. Then let $(\underline{\mathcal{E}}_r, \underline{\mathcal{D}}_r)$ be the closure of $(\mathcal{E}_r^\infty, \mathcal{D}_\circ^\mu)$ on $L^2(\mu)$. Moreover, $(\mathcal{E}_r, \mathcal{D}_r)$ denotes the closure of $(\mathcal{E}, \mathcal{D}_\circ^\mu \cap \mathcal{B}_r^b)$ on $L^2(\mu)$, where $\mathcal{B}_r^b$ is all of bounded functions f such that f depends only on particles on S_r.

We write $(\mathcal{E}_1, \mathcal{D}_1) \le (\mathcal{E}_2, \mathcal{D}_2)$ if $\mathcal{D}_1 \supset \mathcal{D}_2$ and $\mathcal{E}_1(f,f) \le \mathcal{E}_2(f,f)$ for $f \in \mathcal{D}_2$. Then we see $\{(\underline{\mathcal{E}}_r, \underline{\mathcal{D}}_r)\}_{r\in\mathbb{N}}$ is increasing and $\{(\mathcal{E}_r, \mathcal{D}_r)\}_{r\in\mathbb{N}}$ is decreasing with respect to this order. Using the monotonicity, we define $(\underline{\mathcal{E}}, \underline{\mathcal{D}})$ and $(\mathcal{E}, \mathcal{D})$ as the limit of $\{(\underline{\mathcal{E}}_r, \underline{\mathcal{D}}_r)\}_{r\in\mathbb{N}}$, $\{(\mathcal{E}_r, \mathcal{D}_r)\}_{r\in\mathbb{N}}$, respectively. We see $(\underline{\mathcal{E}}_r, \underline{\mathcal{D}}_r) \le (\mathcal{E}_r, \mathcal{D}_r)$ for each $r \in \mathbb{N}$, therefore we obtain $(\underline{\mathcal{E}}, \underline{\mathcal{D}}) \le (\mathcal{E}, \mathcal{D})$. Then we assume the uniqueness of Dirichlet forms:

(A2) $(\underline{\mathcal{E}}, \underline{\mathcal{D}}) = (\mathcal{E}, \mathcal{D})$.

Remark (1) Under reasonable assumptions, $(\mathcal{E}, \mathcal{D})$ is a local and quasi-regular Dirichlet form [16]. Therefore, there exists an S-valued diffusion associated with $(\mathcal{E}, \mathcal{D})$. See [7, 13] for the general theory of Dirichlet forms.
(2) If uniqueness (in distribution) of solutions to an associated ISDE holds, then **(A2)** holds [11]. The uniqueness of solution was obtained under the tail triviality of μ [21]. If μ is not tail trivial, uniqueness of solution is not known; however ISDE representation also yields **(A2)** for non-tail trivial μ [8].

2.3 General theorems

Let $(\mathcal{E}^N, \mathcal{D}^N)$ be the closure of $(\mathcal{E}^N, \mathcal{D}_\circ^{\mu^N})$ on $L^2(\mu^N)$. Let X^N and X be S-valued diffusions associated with $(\mathcal{E}^N, \mathcal{D}^N, L^2(\mu^N))$ and $(\mathcal{E}, \mathcal{D}, L^2(\mu))$, respectively. Then we obtain Mosco convergence, and then convergence of solutions to SDEs. We note that Mosco convergence is equivalent to strong convergence of corresponding semi-groups.

Theorem 2.1 ([10])**.** Under reasonable conditions, assume **(A1)** (or **(A1')**) and **(A2)**. Then we have

$$\lim_{N\to\infty} (\mathcal{E}^N, \mathcal{D}^N, L^2(\mu^N)) = (\mathcal{E}, \mathcal{D}, L^2(\mu)) \text{ in Mosco in the sense of Kuwae-Shioya [12].}$$

In particular, if initial distributions are equilibrium ($\mathsf{X}_0^N \overset{d}{=} \mu^N$ and $\mathsf{X}_0 \overset{d}{=} \mu$), then

$$\lim_{N\to\infty} \mathsf{X}^N = \mathsf{X} \text{ weakly in } C([0,\infty), \mathsf{S}).$$

Theorem 2.1 is convergence theorem of S-valued diffusion (unlabeled diffusion). By putting suitable labels to particles of the unlabeled diffusion, we get a labeled diffusion. Then Theorem 2.1 implies convergence of labeled diffusion, especially, that of solutions to SDEs.

Theorem 2.2 ([10]). Suppose the same conditions as in Theorem 2.1. Furthermore, for labelings which are assumed to be suitable, let $(X^{N,i})_{i=1}^N$ and $(X^i)_{i=1}^\infty$ be labeled diffusions of X^N and X, respectively. Then for any $m \in \mathbb{N}$,

$$\lim_{N\to\infty} (X^{N,i})_{i=1}^m = (X^i)_{i=1}^m \text{ weakly in } C([0,\infty); S^m).$$

Remark We constructed another general theorem to conclude similar results of convergence of dynamics [9]. This result requires exact estimates of drift terms. On the other hand, Theorem 2.1 and Theorem 2.2 does not require such concrete estimates.

3 Examples

In this section, we exhibit two examples of Theorem 2.2.

3.1 Dynamical soft-edge universality for log-gases

The dynamical version of soft-edge universality mentioned in Section 1 is justified by Theorem 2.2. In fact, we see **(A1')** and **(A2)**.

Non-colliding property for the Airy$_\beta$ interacting ISDE is known [17], which implies **(A1')** (6). We see **(A1')** (7) from the following strong convergence result:

Lemma 3.1. [3] For $\beta = 1, 2, 4$ and $V(x) = \sum_{i=0}^{2l} \kappa_i x^i$ ($\kappa_{2l} > 0$), we have (3). Furthermore, compact uniform convergence of correlation functions also holds.

We also check the uniqueness of Dirichlet forms **(A2)**.

Theorem 3.2. [[11] for $\beta = 2$, [8] for $\beta = 1, 2, 4$] For $\beta = 1, 2, 4$, **(A2)** holds for Airy$_\beta$ random point fields. .

Hence, summarising the above, we obtain the next dynamical universality.

Theorem 3.3. [[10] for $\beta = 2$, [8] for $\beta = 1, 4$] For $\beta = 1, 2, 4$ and $V(x) = \sum_{i=0}^{2l} \kappa_i x^i$, the conclusion of Theorem 2.1 holds for SDEs (4) and (2). In particular, the limit ISDE does not depend on V.

Remark Note that the soft-edge universality of log-gases has been generalised to wider class of V [2], but this result shows only weak convergence. If we strengthen their result to convergence in compact uniform sense, then our framework immediately yields corresponding dynamical universality accordingly.

3.2 Dynamical soft-edge universality for biorthogonal ensemble

Next we see other example. For $\theta > 0$ and $\alpha > 1$, biorthogonal Laguerre ensemble on $[0, \infty)$ is defined as

$$\mu_{\alpha,\theta}^{N}(d\mathbf{x}_N) \propto \prod_{i<j}^{N} |x_i - x_j||x_i^{\theta} - x_j^{\theta}| \prod_{k=1}^{N} x_k^{\alpha} e^{-Nx_k} d\mathbf{x}_N.$$

This ensembles were introduced by Muttalib [15] as a generalisation of usual Laguerre ensembles.

Soft-edge scaling limit of the biorthogonal ensembles also yields the Airy$_2$ random point field. In this case the soft-edge scaling is $x \mapsto \theta\big(x_* + \frac{c_* s}{N^{2/3}}\big)^{\frac{1}{\theta}}$, where x_* and c_* are certain constants, and the scaled probability measure is

$$\begin{aligned}\mu_{\mathrm{Ai},\alpha,\theta}^{N}(d\mathbf{s}_N) \propto & \prod_{i<j}^{N} |s_i - s_j| \Big| \big(x_* + \frac{c_* s_i}{N^{\frac{2}{3}}}\big)^{\frac{1}{\theta}} - \big(x_* + \frac{c_* s_j}{N^{\frac{2}{3}}}\big)^{\frac{1}{\theta}} \Big| \\ & \times \prod_{k=1}^{N} \big(x_* + \frac{c_* s_k}{N^{\frac{2}{3}}}\big)^{\frac{\alpha}{\theta}} e^{-N\theta\big(x_* + \frac{c_* s_k}{N^{2/3}}\big)^{\frac{1}{\theta}}} d\mathbf{s}_N.\end{aligned}$$

Then we have the following soft-edge scaling limit.

Lemma 3.4. [24] We have

$$\lim_{N\to\infty} \mu_{\mathrm{Ai},\alpha,\theta}^{N} = \mu_{\mathrm{Ai},2}.$$

Furthermore, we have compact uniform convergence of correlation functions.

From this lemma we have strong convergence holds in the sense of (7). Then we can immediately apply Theorem 2.1 and Theorem 2.2.

Theorem 3.5. [10] The conclusion of Theorem 2.2 holds for the following SDEs:

$$\begin{aligned} dX_t^{N,i} = dB_t^i + \frac{1}{2}\Big\{ & \sum_{j\neq i}^{N} \Big(\frac{1}{X_t^{N,i} - X_t^{N,j}} + \frac{\frac{c_*}{\theta N^{\frac{2}{3}}}\big(x_* + \frac{c_* X_t^{N,i}}{N^{\frac{2}{3}}}\big)^{\frac{1-\theta}{\theta}}}{\big(x_* + \frac{c_* X_t^{N,i}}{N^{\frac{2}{3}}}\big)^{\frac{1}{\theta}} - \big(x_* + \frac{c_* X_t^{N,j}}{N^{\frac{2}{3}}}\big)^{\frac{1}{\theta}}} \Big) \\ & + \frac{\alpha c_*}{\theta(x_* N^{\frac{2}{3}} + c_* X_t^{N,i})} - c_* N^{\frac{1}{3}} \Big(x_* + \frac{c_* X_t^{N,i}}{N^{\frac{2}{3}}} \Big)^{\frac{1-\theta}{\theta}} \Big\} dt, \qquad 1 \le i \le N, \\ dX_t^i = dB_t^i + \lim_{s\to\infty} \Big\{ & \sum_{|X_t^j|<s, j\neq i} \frac{1}{X_t^i - X_t^j} - \int_{|x|<s} \frac{\hat{\rho}(x)}{-x} dx \Big\} dt, \qquad i \in \mathbb{N}. \end{aligned}$$

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