CLASSICAL FIELD EQUATIONS AS EFFECTIVE THEORIES OF QUANTUM ELECTRODYNAMICS AND YUKAWA INTERACTIONS

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ABSTRACT. In this short note the so-called Bohr correspondence principle is extended to standard models of quantum electrodynamics and to the Yukawa theory of strong nuclear forces. The main motivation here is to provide firm mathematical foundations to the physics of interacting quantum fields at small energy scales and to provide a comprehensive parallel between quantum and classical field theories.

CONTENTS

1. INTRODUCTION

Quantum electrodynamics is the fundamental theory that describes the interactions between matter and radiation through phenomenological models of Quantum Field Theory (QFT). Since the fifties enormous progress has been made in the understanding of such theory with for example the breakthrough of perturbative renormalization and asymptotic freedom, see e.g. [23]. Nevertheless, several conceptual and analytical mathematical problems remain open as well as outstanding questions like the *Millennium Prize Problem of Yang-Mills*, see e.g. [17, 21]. On the other hand, some of the issues in the general topic of QFT, which have recently aroused the interest of the mathematical physics community, concern the relationship between classical and quantum field theories. The aim of this brief note, based on joint works with Marco Falconi and Fumio Hiroshima [4] and [5], is to explain the rigorous mathematical basis for such a relationship related to *Bohr's correspondence principle* and *canonical quantization*.

Specifically, we consider the two cases of electrodynamics and Yukawa interaction. Recall that quantum electrodynamics is usually described by it standard model, namely the Pauli-Fierz

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Hamiltonian, and is widely studied as an interesting example of QFT. Such model consists of non-relativistic spin zero particles interacting with the quantized electromagnetic fields in Coulomb's gauge. On the other hand, the Yukawa interaction describes the strong nuclear force as the nucleon-meson interaction of a Dirac field with a boson field of positive mass. If one fixes the number of nucleons and consider them to be non-relativistic then the Yukawa interaction reduces to the so-called Nelson Hamiltonian, see [20]. These two models will be recalled in Section 2 and 3 respectively and our main results will be stated in Section 4. Let us now give a brief sketch of the canonical quantization and Bohr's correspondence principles.

Canonical quantization: The general goal of constructive QFT is to provide solutions for nonlinear quantum field equations like for instance the ϕ^{2n} equation,

$$(\Box + m)\phi(t,x) + \lambda\phi^{2n+1}(t,x) = 0\,, \tag{1.1}$$

where $\phi(t;x)$ and $\pi(t;x) = \partial_t\phi(t;x)$ are the unknown quantum fields (distribution-valued operators) satisfying the canonical condition

$$[\phi(t,x),\phi(t,y)] = [\pi(t,x),\pi(t,y)] = 0, \;\; [\pi(t,x), i\phi(t,y)] = \hbar\delta(x-y). \tag{1.2}$$

Such evolution system is Hamiltonian and its energy $\hat{H}$ is a formally conserved quantity. Hence $\hat{H}$ can be expressed as a function of the zero-time canonical variables $\phi(0;x)$ and $\pi(0;x)$,

$$\hat{H} = \int \frac{1}{2}[\pi^2(0,x) + |\nabla\phi(0,x)|^2 + m\phi(0,x)^2] + \frac{\lambda}{2n+2}\phi(0,x)^{2n+2}\,dx\,, \tag{1.3}$$

and the time variation of quantum fields is given by the equation of motion

$$i\hbar\partial_t\phi(t;x) = [\phi(t;x);\hat{H}] \quad \text{and} \quad i\hbar\partial_t\pi(t;x) = [\pi(t;x);\hat{H}]\,. \tag{1.4}$$

In particular, if $\hat{H}$ is a self-adjoint operator over a Hilbert space on which a representation of the zero-time canonical commutation relation is realized, then one determines the quantum fields at any time

$$\phi(t;x) = e^{i\frac{t}{\hbar}\hat{H}}\phi(0;x)e^{-i\frac{t}{\hbar}\hat{H}} \quad \text{and } \pi(t;x) = e^{i\frac{t}{\hbar}\hat{H}}\pi(0;x)e^{-i\frac{t}{\hbar}\hat{H}}. \tag{1.5}$$

Hence, solving (1.1) requires on one hand the study of representations of canonical commutation relations (1.2) and the study of self-adjointness of the formal operator $\hat{H}$ on the other.

Consider the smeared fields

$$\phi(f) = \int_{\mathbb{R}^d}\phi(0,x)f(x)dx \quad \text{and} \quad \pi(f) = \int_{\mathbb{R}^d}\pi(0,x)f(x)dx, \quad \forall f\in L^2(\mathbb{R}^d,\mathbb{R}) \tag{1.6}$$

as self-adjoint operators on a given Hilbert space $\mathscr{H}$ then one can check that the *Heisenberg commutation relations* are satisfied:

$$[\phi(f),\phi(g)] = [\pi(f),\pi(g)] = 0, \qquad [\pi(g), i\phi(f)] = \hbar\langle f,g\rangle\, Id.$$

To avoid domain problems due to the unboundedness of ϕ and π, it is more convenient to deal with the following Weyl commutation relations:

$$(i) \qquad W(g_1,f_1)W(g_2,f_2) = e^{i\frac{\hbar}{2}\sigma[(g_1,f_1),(g_2,f_2)]}\, W(g_1+g_2, f_1+f_2).$$
$$(ii) \qquad W(g,f)^* = W(-g,-f),$$

where σ is the canonical symplectic form over the space $L^2(\mathbb{R}^d,\mathbb{R}) \oplus L^2(\mathbb{R}^d,\mathbb{R})$. In this topic it is known that the most relevant representation of Weyl commutation relations is the Fock representation acting on the symmetric Fock space,

$$\mathscr{H} = \Gamma_s(\mathscr{Z}) = \oplus_{n=0}^{\infty}\otimes_s^n \mathscr{Z}\,,$$

where $\mathscr{Z} = L^2(\mathbb{R}^d, \mathbb{C}) \simeq L^2(\mathbb{R}^d, \mathbb{R}) \oplus L^2(\mathbb{R}^d, \mathbb{R})$ and the subscript "s" stands for the symmetric tensor product. On such spaces one can rigorously defines the following operators for any $f \in \mathscr{Z}$:
Annihilation operator:

$$\hat{\alpha}(f) f_1 \otimes_s \cdots \otimes_s f_n = \sqrt{\hbar n}\ \frac{1}{n!} \sum_{\sigma \in S_n} \langle f, f_{\sigma_1} \rangle\, f_{\sigma_2} \otimes_s \cdots \otimes_s f_{\sigma_n}, \tag{1.7}$$

Creation operator:

$$\hat{\alpha}^*(f) f_1 \otimes_s \cdots \otimes_s f_n = \sqrt{\hbar(n+1)}\ f \otimes_s f_1 \cdots \otimes_s f_n. \tag{1.8}$$

Weyl operator:

$$W(f) = e^{\frac{i}{\sqrt{2}}(\hat{\alpha}^*(f) + \alpha(f))}. \tag{1.9}$$

Remark that the above Weyl operator depends in $\hbar \in (0,1)$. Hence, the above structures in the Fock space determine the zero-time quantum fields $\phi(0,\cdot)$ and $\pi(0,\cdot)$ using (1.6) and the relations

$$\phi(f) = \frac{\hat{\alpha}(f) + \hat{\alpha}^*(f)}{\sqrt{2}}, \quad \pi(f) = \frac{\hat{\alpha}(f) - \hat{\alpha}^*(f)}{i\sqrt{2}}.$$

Now a canonical or Wick quantization is a mapping for each polynomial functional $b(\bar{\alpha}, \alpha)$ of the complex classical fields $\bar{\alpha}, \alpha$ corresponds an operator in the Fock space

$$b(\bar{\alpha}, \alpha) \longrightarrow b(\hat{\alpha}^*, \hat{\alpha}) = b^{Wick}$$

according to the *Wick quantization rules* so that all the $\hat{\alpha}^*$'s are in the left and all the $\hat{\alpha}$'s are in the right. Such procedure can be implemented more systematically using Wick operators with homogenous polynomial functionals b defined as

$$b^{Wick}{}_{|\otimes_s^n \mathscr{Z}} = \frac{\sqrt{(n-p+q)!n!}}{(n-p)!}\, \hbar^{\frac{p+q}{2}}\ \tilde{b} \otimes_s Id^{\otimes(n-p)}$$

where $b(\bar{\alpha}, \alpha) = \langle \alpha^{\otimes q}, \tilde{b}\, \alpha^{\otimes p} \rangle$ for some $p, q \in \mathbb{N}$ and $\tilde{b} : \otimes_s^p \mathscr{Z} \to \otimes_s^q \mathscr{Z}$ is a given operator. For instance, the canonical quantization of the total mass and the Klein-Gordon classical energy yield

$$\int_{\mathbb{R}^d} \bar{\alpha}(k)\ \alpha(k) dk \quad \longrightarrow \quad \hat{N} = \langle \alpha,\ \alpha \rangle^{Wick} = \int_{\mathbb{R}^d} \hat{\alpha}^*(k)\ \ \hat{\alpha}(k) dk\,, \tag{1.10}$$

$$\int_{\mathbb{R}^d} \bar{\alpha}(k)\, \omega(k)\, \alpha(k) dk \quad \longrightarrow \quad \hat{H}_0 = \langle \alpha,\, \omega(k)\, \alpha \rangle^{Wick} = \int_{\mathbb{R}^d} \hat{\alpha}^*(k)\, \omega(k)\ \hat{\alpha}(k) dk\,, \tag{1.11}$$

where

$$\omega(k) = \sqrt{k^2 + m^2}\,,$$

and $\hat{\alpha}^*(\cdot), \hat{\alpha}(\cdot)$ are distributions valued operators defined similarly as in (1.6). Expressing the Hamiltonian $\hat{H}$ of the ϕ^{2n} theory in (1.1) according to the above procedure yields an operator or a quadratic form over the Fock space. The challenging question remaining is then to show that $\hat{H}$ corresponds to a well-defined self-adjoint operator. Such task for the ϕ^{2n} model is at least solved in $d = 1$ with spatial-cutoffs but still unsolved for $d = 3$ while for $d = 2$ is solved by a different euclidian approach. The above discussion describes briefly the purpose of canonical quantization and constructive quantum field theory from a Hamiltonian point of view.

Bohr's correspondence principle: A classical Hamiltonian system with an infinite number of degrees of freedom is described by pairs of momentum-position canonical variables $(p_1, q_1, \cdots, p_n, q_n, \cdots)$ and the equation of motion is derived from a classical Hamiltonian functional,

$$\mathcal{H}(p,q) = \mathcal{H}(p_1, q_1, \cdots, p_n, q_n, \cdots)\,,$$

as the following system of ODEs

$$\dot{q}_j = \frac{\delta \mathcal{H}}{\delta p_j}, \qquad \dot{p}_j = -\frac{\delta \mathcal{H}}{\delta q_j}, \qquad j = 1, \cdots \tag{1.12}$$

In such Hamiltonian systems there is usually a natural symplectic form and a compatible complex structure allowing to formulate the above equation of motion (1.12) in terms of classical complex fields,

$$\mathcal{H}(p,q) \equiv \mathcal{H}(\bar{\alpha}, \alpha), \qquad i\frac{d}{dt}\alpha(t) = \frac{\delta \mathcal{H}}{\delta \bar{\alpha}}(\bar{\alpha}(t), \alpha(t)). \tag{1.13}$$

There are mainly two approaches to describe the dynamics of such classical Hamiltonian systems:

(1) The evolution of *dynamical states*: One considers the equation (1.12) or (1.13) as a Cauchy problem with initial data given as phase-space points. Then one try to determine unique solutions or trajectories satisfying (1.12) or (1.13) for each initial datum. Within this point of view the main issues that are considered are more quantitative and they are related to PDE analysis, for instance well–posedness, Hadamard's stability, scattering and blow up.

(2) The evolution of *statistical states*: Instead one considers an initial ensemble of data given by a probability distribution on the phase-space and then attempts to characterize the time evolution of such distribution. This leads to the Liouville equation and thus to a more qualitative study concerned with ergodic, chaotic and asymptotic statistical behaviors of such dynamical systems.

This general picture is complemented by few exceptional Hamiltonian systems that have some form of integrability or solvability on which specific techniques like the KAM theory are used.

On the other hand, a quantum mechanical system with *infinite degrees of freedom* is formally described by a Hamiltonian

$$\mathcal{H}(\hat{p}, \hat{q}) = \mathcal{H}(\hat{p}_1, \hat{q}_1, \cdots, \hat{p}_n, \hat{q}_n, \cdots),$$

where the pairs $(\hat{p}_j, \hat{q}_j)$ are conjugate canonical variables satisfying the canonical commutation relations (CCR's):

$$[\hat{q}_j, \hat{p}_k] = i\hbar\delta_{j,k}, \quad [\hat{q}_j, \hat{q}_k] = [\hat{p}_j, \hat{p}_k] = 0.$$

or equivalently using quantum complex fields,

$$\mathcal{H}(\hat{p}, \hat{q}) \equiv \mathcal{H}(\hat{\alpha}^*, \hat{\alpha}), \qquad [\hat{\alpha}_j, \hat{\alpha}_k^*] = \hbar\delta_{j,k}.$$

The equation of motion in this case is given by the Schrödinger equation

$$i\hbar\partial_t \Psi_\hbar = \mathcal{H}(\hat{\alpha}^*, \hat{\alpha})\,\Psi_\hbar, \tag{1.14}$$

and the dynamics of such quantum Hamiltonian system can be described either by:

(1) *Schrödinger picture*: Wave functions over Hilbert spaces.
(2) *Heisenberg picture*: States over C^* or W^*-algebras of observables.

The key issues in this topic are energy levels, resonances, spectral analysis, scattering, KMS states and statistical behaviors. Again this formal general landscape is embellished by a few exceptional solvable models like Spin chains, Hubbard and Lieb-Liniger models.

The relationship between classical and quantum fields is given by the following diagram reflecting the so-called canonical quantization and classical limit:

$$\text{Classical field theories} \underset{\text{Classical limit}}{\overset{\text{Quantization}}{\rightleftarrows}} \text{Quantum field theories} \tag{1.15}$$

and stating that canonical quantization of classical fields yields QFT as explained in the previous paragraph while the classical limit recovers the original classical fields. The latter link is the so-called Bohr's correspondence principle. Taking again the formal example of ϕ^{2n} quantum field theory then insightful parallel between these quantum and classical field theories is summarized in the following table:

Classical system	Quantum system
$\mathscr{Z} = L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$ Classical phase space (infinite dim.)	$\Gamma_s(\mathscr{Z})$ Symmetric Fock space
$\bar{\alpha}(k), \alpha(k)$ Classical variables (complex fields)	$\hat{\alpha}^*(k), \hat{\alpha}(k)$ Quantum variables (Dist.valued operators)
$b(\bar{\alpha}, \alpha) : D(b) \to \mathbb{R}$ Classical observables (functionals)	$b(\bar{\alpha}, \alpha)^{Wick}$ Quantum observables (operators on Fock sp.)
μ Classical states (Prob. on phase sp.)	ϱ Quantum states (density matrices on Fock sp.)
$\mathcal{H}(\bar{\alpha}, \alpha)$ Classical energy (functional on phase sp.)	$\hat{H} = \mathcal{H}(\bar{\alpha}, \alpha)^{Wick}$ Quantum Hamiltonian (self-adjoint op. on Fock sp.)
$(\Phi_{\mathcal{H}})_{t_0}^t$ Classical evolution (flow on phase sp.)	$e^{-\frac{i}{\hbar}(t-t_0)\hat{H}}$ Quantum evolution (unitary group on Fock sp.)

In this respect one naturally expects that complex quantum dynamics out of reach by numerical simulations can be approximated by more tractable flow of related nonlinear PDEs governing the classical fields. From this duality (1.15) one could in principle deduce accurate expansion of quantum correlation functions, eigenvalues asymptotics and scattering amplitudes of quantum field theories in the classical regime. To address such questions there are few mathematical tools that are non perturbative with respect to the coupling constant; namely the coherent state method and the more recent Wigner measure approach. The latter is the main topic of this note and is useful for the interesting regimes:

- Mean field limit $N \to \infty$ in quantum many-body theory where N is the number of particles,
- Classical limit $\hbar \to 0$ in QED or QFT where $\hbar$ is the rescaled Planck constant.

And it can be applied to study the following examples of physical systems and related phenomena:

- Many-Body theory and Bose-Einstein condensate.
- Relativistic Quantum field theory ($(\varphi)_2^4, P(\varphi)_2$ models).
- Yukawa interaction theory and renormalization.
- Quantum electrodynamics and Lamb shift.

Wigner or semiclassical measures: At zero temperature the states describing quantum systems are trace class normalized non-negative operators (density matrices) while for classical systems states are probability measures over the phase-space. Thus the classical limit implementing the Bohr's correspondence principle can be interpreted as a scaling limit that corresponds to an arbitrary sequence of scaled quantum states one or several probability measures over the classical phase-space called Wigner or semiclassical measures. Such type of convergence is rather weak and the non uniqueness of the limit is a natural feature that can be circumvented by selecting subsequences. Note that Wigner or semiclassical measures have been extensively studied in finite dimension (see e.g. [18] and references therein). This concept has been extended to infinite dimensional phase spaces in a series of articles by Ammari and Nier (see e.g. [1, 2, 6, 7]).

Definition 1.1 (Wigner measures)**.** A probability μ over the phase-space $\mathscr{Z}$ is a Wigner measure of a family of quantum states $(\varrho_\hbar)_{\hbar\in(0,1)}$ on the Fock space $\Gamma_s(\mathscr{Z})$ if there exists a countable subset $\mathscr{E} \subset (0,1)$ with $0 \in \overline{\mathscr{E}}$ such that for any $\xi \in \mathscr{Z}$:

$$\lim_{\hbar\to 0, \hbar\in\mathscr{E}} \mathrm{Tr}[\varrho_\hbar\, W(\xi)] = \int_{\mathscr{Z}} e^{\sqrt{2}i\,\Re\mathrm{e}\langle\xi,z\rangle} d\mu(z)\,.$$

Recall that the Weyl operator $W(\cdot)$ depends on $\hbar$ and is given in (1.9). The set of all Wigner measures of a given family of quantum states $\big(\varrho_\hbar\big)_{\hbar\in(0,1)}$ is denoted by

$$\mathscr{M}(\varrho_\hbar, \hbar \in (0,1))\,. \tag{1.16}$$

Note that we know that the set of Wigner measures $\mathscr{M}(\varrho_\hbar, \hbar \in (0,\bar{\hbar}))$ is not empty under suitable assumptions of the quantum states. Moreover, usually one can only consider families of quantum states with a unique Wigner measure without loss of generality.

The above concept allows to establish the Bohr's correspondence principle in an effective and flexible way. Indeed, the convergence of quantum dynamics towards classical dynamics while taking $\hbar \to 0$ can be reformulated as the following question. Consider for instance the above example of ϕ^{2n} theory, hence if the quantum system is in the state $\varrho_\hbar$ at time $t_0 = 0$ then its time evolution satisfies the Heisenberg equation and yields

$$\varrho_\hbar(t) = e^{-\frac{i}{\hbar}t\,\hat{H}}\varrho_\hbar e^{\frac{i}{\hbar}t\,\hat{H}}\,.$$

Suppose now that the family of states $(\varrho_\hbar)_{\hbar\in(0,1)}$ admits a unique Wigner measure μ i.e.,

$$\mathscr{M}(\varrho_\hbar, \hbar \in (0,1)) = \{\mu\}\,,$$

then can we determine the Wigner measures of $\big(\varrho_\hbar(t)\big)_{\hbar\in(0,1)}$ for any time?
The formal or expected answer is yes and says that the family $\big(\varrho_\hbar(t)\big)_{\hbar\in(0,1)}$ admits a unique Wigner measure at any time given by the push-forward measure

$$\mu_t = (\Phi^t_{\mathcal{H}})_\#\,\mu_0 \tag{1.17}$$

where $\Phi^t_{\mathcal{H}} = (\Phi_{\mathcal{H}})^t_0$ is in our case the flow of the classical field equation related to the ϕ^{2n} theory admitting the following classical energy

$$\mathcal{H}(\bar{\alpha},\alpha) = \int \bar{\alpha}(k)\,\sqrt{k^2+m^2}\,\alpha(k)\;dk + \int \frac{\lambda}{2n+2}\varphi(x)^{2n+2}\,dx. \tag{1.18}$$

Remark that our discussion here is formal and in particular the self-adjointenss of $\hat{H}$ and the well posedness of the nonlinear classical field equation are presumed without proof. Such result can also take the following form for all $\xi \in \mathscr{Z}$,

$$\lim_{\hbar\to 0}\mathrm{Tr}\left[\varrho_\hbar\; e^{i\frac{t}{\hbar}\hat{H}}\;W(\xi)\;e^{-i\frac{t}{\hbar}\hat{H}}\right] = \int_{\mathscr{Z}} e^{i\sqrt{2}\Re\mathrm{e}\langle\xi,z\rangle}\;d\mu_t(z)\,.$$

The above Bohr's principle is proved rigorously for the Yukawa interaction with and without ultraviolet cutoffs in [3, 4] respectively; and more recently it is considered for non-relativistic quantum electro dynamics in [5]. Such results are briefly given in Section 4.

2. Non-Relativistic ElectroDynamics

We briefly review below the standard model of non-relativistic quantum electrodynamics.

2.1. Pauli-Fierz Hamiltonian. The dynamics of a quantum extended charge interacting with the quantized electromagnetic field in Coulomb's gauge is usually described by the Pauli-Fierz Hamiltonian, see [22]. Such operator is proved to be essentially self-adjoint under some regularity assumptions on the charge distribution, see for instance [11, 14, 15, 16, 19].

The particle-field Hilbert space is:

$$\mathcal{Q} = L^2(\mathbb{R}^3_x,\mathbb{C}) \otimes \Gamma_s\big(L^2(\mathbb{R}^3_k,\mathbb{C}^2)\big)\,, \tag{2.1}$$

where $\Gamma_s(\cdot)$ stands for the symmetric Fock space. The particle's momentum and position operators are respectively:

$$\hat{p} = -i\hbar\nabla_x\,, \quad \hat{q} = x\,,$$

while the creation-annihilation operators for $f \in L^2(\mathbb{R}^3, \mathbb{C}^2)$ are given by:

$$\hat{\alpha}(f) = \sum_{j=1}^{2} \int_{\mathbb{R}^3} \bar{f}(k,j)\; \hat{\alpha}(k,j)\, dk\;, \quad \hat{\alpha}^*(f) = \sum_{j=1}^{2} \int_{\mathbb{R}^3} f(k,j)\; \hat{\alpha}^*(k,j)\, dk\;,$$

with $\hat{\alpha}(k,j)$ and $\hat{\alpha}^*(k,j)$ are the annihilation-creation fields satisfying the canonical commutation relations:

$$[\hat{\alpha}(k,j), \hat{\alpha}^*(k',j')] = \hbar\, \delta(k-k')\, \delta_{jj'}\;.$$

In this framework the Hamiltonian of free fields is

$$\hat{H}_f = \sum_{j=1}^{2} \int_{\mathbb{R}^3} |k|\;\; \hat{\alpha}^*(k,j)\; \hat{\alpha}(k,j)\, dk\;.$$

Consider now a smooth function $\varphi : \mathbb{R}^3 \to \mathbb{C}$ representing the Fourier transform of the particle's charge distribution and satisfying the assumption:

$$|\cdot|^{-1}\varphi(\cdot), |\cdot|^{\frac{1}{2}}\varphi(\cdot) \in L^2(\mathbb{R}^3, \mathbb{C})\,. \tag{A0}$$

The quantum electromagnetic vector potential $\hat{A}_\varphi = \big(\hat{A}_{\varphi,1}, \hat{A}_{\varphi,2}, \hat{A}_{\varphi,3}\big)$ is defined by

$$\hat{A}_{\varphi,\ell}(\hat{q}) = \sum_{j=1}^{2} \int_{\mathbb{R}^3} |k|^{-\frac{1}{2}} e_\ell(k,j) \Big(\varphi(k)\, e^{-ik\cdot\hat{q}}\;\hat{\alpha}^*(k,j) + \bar{\varphi}(k)\, e^{ik\cdot\hat{q}}\;\hat{\alpha}(k,j)\Big) dk\;,$$

where $\{e(\cdot,j)\}_{j=1,2}$ are the polarization vectors satisfying for a.e. $k \in \mathbb{R}^3$,

$$e(k,j)\cdot k = 0, \qquad e(k,j)\cdot e(k,j) = \delta_{j,j'} \qquad \forall j \neq j'. \tag{2.2}$$

The Pauli-Fierz Hamiltonian is then given by

$$\hat{H}_{PF} = \frac{1}{2}\big(\hat{p} - \hat{A}_\varphi\big)^2 + \hat{H}_f\;.$$

The Pauli-Fierz Hamiltonian is self-adjoint according to the following result of F. Hiroshima.

Proposition 2.1 ([16]). *Assume* (A0)*, then $\hat{H}_{PF}$ is self-adjoint on $\mathscr{D}(\hat{p}^2) \cap \mathscr{D}(\hat{H}_f)$.*

This in particular gives the existence and uniqueness of quantum dynamics related to the Pauli-Fierz model of non-relativistic quantum electrodynamics.

2.2. **Newton-Maxwell equation.** The dynamics of an extended classical non-relativistic charge coupled to the classical electromagnetic field is described by the Newton-Maxwell equation. The latter is a coupled system of an ODE and a PDE consisting in Newton's equation for the particle and Maxwell's equation for the field. Consider the same function φ as in (A0) and assume for simplicity that it satisfies

$$\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^3 \setminus \{0\})\,, \tag{A0'}$$

to avoid ultraviolet and infrared problems. More general conditions will be considered in [5]. Denote by $(q,p) \in \mathbb{R}^3 \times \mathbb{R}^3 \cong \mathbb{R}^6$ the phase-space coordinates of the particle and by $(E,B) : \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$ the electromagnetic field. Define the smeared fields

$$E_\varphi = \check{\varphi} * E, \quad B_\varphi = \check{\varphi} * B,$$

and the particle's current $J_{q,p}(\cdot) = p\check{\varphi}(q-\cdot)$, then the Newton-Maxwell equation takes the form:

$$\begin{cases} \dot{q} = p\;, & \dot{p} = E_\varphi(q) + p \times B_\varphi(q) \\ \dot{E} = \nabla \times B - J_{q,p}\;, & \dot{B} = -\nabla \times E \end{cases}$$
$$\begin{cases} \nabla \cdot E = \check{\varphi}(q-\cdot) \\ \nabla \cdot B = 0 \end{cases} \tag{N-M}$$

One can introduce the vector potential A for the electromagnetic field and chose the Coulomb gauge to reformulate (N-M) in terms of A, $\dot{A}$ and (q,p) only. Moreover, one can introduce new complex canonical variables $(\bar{\alpha},\alpha)$ instead of A, $\dot{A}$. Indeed, define for $j\in\{1,2\}$,

$$\alpha(k,j)=\frac{1}{\sqrt{2}}\,e(k,j)\cdot\big(|k|^{\frac12}\mathscr{F}(A)(k)-i\,|k|^{-\frac12}\mathscr{F}(\dot{A})\big)\,, \tag{2.3}$$

where $\{e(\cdot,j)\}_{j=1,2}$ are the polarization vectors satisfying (2.2) and $\mathscr{F}$ denotes the Fourier transform. Then one can write

$$A(x)=\sum_{j=1}^{2}\sqrt{2}\,\Re\mathrm{e}\,\mathscr{F}^{-1}\big(e(\cdot,j)\,|\cdot|^{-\frac12}\alpha(\cdot,j)\big)(x) \tag{2.4}$$

$$\dot{A}(x)=\sum_{j=1}^{2}\sqrt{2}\,\Im\mathrm{m}\,\mathscr{F}^{-1}\big(e(\cdot,j)\,|\cdot|^{\frac12}\alpha(\cdot,j)\big)(x) \tag{2.5}$$

such that

$$A_\varphi=\check{\varphi}*A\,.$$

Using such new canonical variables $(q,p,\bar{\alpha},\alpha)$ one can reformulate the Newton-Maxwell equation as a classical Hamiltonian system with an energy functional:

$$\mathcal{H}_{NM}(q,p,\bar{\alpha},\alpha)=\frac12\Big(p-A_\varphi(q)\Big)^2+\sum_{j=1}^{2}\int_{\mathbb{R}^3}\overline{\alpha}(k,j)\,|\,k\,|\,\alpha(k,j)\,dk\,.$$

and a classical equation of motion:

$$\dot{q}(t)=\partial_p\mathcal{H}_{NM}(q,p,\bar{\alpha},\alpha)\,,\qquad \dot{p}(t)=-\partial_q\mathcal{H}_{NM}(q,p,\bar{\alpha},\alpha)\,,$$

$$i\partial_t\alpha(\cdot,j)=\partial_{\overline{\alpha}}\mathcal{H}_{NM}(q,p,\bar{\alpha},\alpha)\,.$$

Specifically, if we consider (q,p) and $\alpha(\cdot)$ as the unknowns, the explicit equation of motion takes the form:

$$\begin{cases}\dot{q}=p-A_\varphi(q)\\[4pt] \dot{p}=\displaystyle\sum_{\ell=1}^{3}\Big(p_\ell-A_{\varphi,\ell}(q)\Big)\nabla A_{\varphi,\ell}(q)\\[4pt] i\partial_t\alpha(\cdot,j)=|\cdot|\alpha(\cdot,j)-\displaystyle\sum_{\ell=1}^{3}\frac{1}{\sqrt{2}}|\cdot|^{-\frac12}\varphi(\cdot)\,e_\ell(\cdot,j)\Big(p_\ell-A_{\varphi,\ell}(q)\Big)\end{cases}. \tag{N-M*}$$

Global well-posedness of (N-M) on various spaces has been studied in the literature, see for instance [8, 9, 10]. The natural functional space for the field $\alpha(\cdot)$ is $L^2(\mathbb{R}^3,\mathbb{C}^2)$. Therefore, we recall existence and uniqueness of (N-M*) solutions on the space

$$\mathscr{Z}_{NM}=\underbrace{\mathbb{R}^3\times\mathbb{R}^3}_{(q,p)}\oplus\underbrace{L^2(\mathbb{R}^3,\mathbb{C}^2)}_{\alpha(\cdot)}\,.$$

Proposition 2.2. *For each initial datum $(q_0,p_0,\alpha_0)\in\mathscr{Z}_{NM}$ there exists a unique global solution $\big(q(t),p(t),\alpha(t,\cdot)\big)\in\mathscr{C}\big(\mathbb{R},\mathbb{R}^6\oplus L^2(\mathbb{R}^3,\mathbb{C}^2)\big)$ of (N-M*). The map*

$$\Phi^t_{NM}(q_0,p_0,\alpha_0)=\big(q(t),p(t),\alpha(t,\cdot)\big)$$

defines the Hamiltonian flow of the Newton-Maxwell equation.

3. Yukawa Interaction model

In this section we consider the quantum non- relativistic Yukawa field theory and the Schrödinger-Klein-Gordon classical system.

3.1. **Reduced Yukawa Hamiltonian.** The formal Hamiltonian of the reduced Yukawa model is understood as a non-relativistic boson field interacting with a meson field according to an Yukawa type interaction given by

$$\begin{aligned} \hat{H}_{Yu} \equiv & \int_{\mathbb{R}^3} \hat{u}^*(x)\Big(-\Delta_x\Big)\hat{u}(x)dx + \int_{\mathbb{R}^3} \hat{\alpha}^*(k)\omega(k)\hat{\alpha}(k)dk \\ & + \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^6} \hat{u}^*(x) \frac{1}{\sqrt{2\omega(k)}} \Big(\hat{\alpha}^*(k)e^{-ik\cdot x} + \hat{\alpha}(k)e^{ik\cdot x}\Big)\hat{u}(x)dxdk\,, \end{aligned} \tag{3.1}$$

where $\omega(k) = \sqrt{k^2+m^2}$, $m > 0$ is the meson mass and $\hat{u}^\sharp, \hat{\alpha}^\sharp$ are respectively the creation-annihilation operators of the particle and meson fields satisfying the CCR's:

$$[\hat{u}(x), \hat{u}^*(y)] = \hbar\, \delta(x-y)\,; \quad [\hat{\alpha}(k), \hat{\alpha}^*(k')] = \hbar\, \delta(k-k')\,.$$

The number of non-relativistic particles is a conserved quantity for the formal Hamiltonian (3.1) and hence one can decompose $\hat{H}_{Yu}$ as direct sum over fixed number of particles. In an influential article E. Nelson constructed in 1964 a renormalization procedure allowing to define quantum dynamics from the formal expression (3.1) by means of dressing transformations and cancellation of the infinite self-energy, see [20]. In particular, there exists a bounded from below self-adjoint operator $\hat{H}_{Yu}$ implementing the Yukawa dynamics and related to the quadratic form (3.1), see [4] for details. The dressing transformation involved in this renormalization procedure consists of a unitary transformation

$$\hat{U}_\infty = e^{i\hbar^{-1}\hat{T}_\infty}\,, \tag{3.2}$$

where T_∞ is a self-adjoint operator given as

$$\hat{T}_\infty = \int_{\mathbb{R}^3} \hat{u}^*(x)\Big(\hat{\alpha}^*(k)g_\infty(k)e^{-ik\cdot x} + \hat{\alpha}(k)\bar{g}_\infty(k)e^{ik\cdot x}\Big)\hat{u}(x)dx\,. \tag{3.3}$$

and

$$g_\infty(k) = -\frac{i}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega(k)}} \frac{1-\chi_{\sigma_0}(k)}{\frac{k^2}{2M}+\omega(k)}\,, \tag{3.4}$$

for some fixed value $0 < \sigma_0$ and a cutoff function $\chi_{\sigma_0}(\cdot) = \chi(\frac{\cdot}{\sigma_0})$ with $\chi \in \mathscr{C}_0^\infty(\mathbb{R}^3)$ such that $\chi \equiv 1$ around the origin.

3.2. **Schrödinger-Klein-Gordon system.** The Schrödinger-Klein-Gordon (S-KG) equation with Yukawa coupling is a well studied system of non-linear PDEs, see e.g. [12, 13], given as:

$$\begin{cases} i\partial_t u = -\dfrac{\Delta}{2M}u + Au \\ (\Box + m_0^2)A = -|u|^2 \end{cases}; \tag{3.5}$$

where $m_0, M > 0$ are positive masses. Introducing the complex fields $(\bar{\alpha}, \alpha)$ according to the relations

$$A(x) = \tfrac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \tfrac{1}{\sqrt{2\omega(k)}}\big(\bar{\alpha}(k)e^{-ik\cdot x} + \alpha(k)e^{ik\cdot x}\big)dk\,, \tag{3.6}$$

$$\dot{A}(x) = -\tfrac{i}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \sqrt{\tfrac{\omega(k)}{2}}\big(\alpha(k)e^{ik\cdot x} - \bar{\alpha}(k)e^{-ik\cdot x}\big)dk\,, \tag{3.7}$$

one can reformulate the S-KG equation (3.5) with these complex fields $(\bar{\alpha}, \alpha)$ instead of the vector potentials $(A, \dot{A})$ as for the Newton-Maxwell equation. In particular, the S-KG equation is a Hamiltonian system with the following energy functional:

$$\mathcal{H}_{SKG}(u,\alpha) = \Big\langle u, \Big(-\tfrac{\Delta}{2M}\Big)u\Big\rangle_{L^2} + \langle \alpha, \omega\alpha\rangle_{L^2} + \tfrac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^6} \tfrac{1}{\sqrt{2\omega(k)}}\Big(\bar{\alpha}(k)e^{-ik\cdot x} + \alpha(k)e^{ik\cdot x}\Big)|u(x)|^2 dxdk\,.$$

And hence the equation of motion takes the form:

$$\begin{cases} i\partial_t u = -\dfrac{\Delta}{2M}u + Au \\ i\partial_t \alpha = \omega\alpha + \dfrac{1}{\sqrt{2\omega}}\mathcal{F}\big(|u|^2\big) \end{cases} . \tag{S-KG}$$

Moreover, it is well known that such system is well posed on its energy space, see [4] for more details.

Proposition 3.1. *For any initial condition* $(u_0, \alpha_0) \in \mathscr{D}(\sqrt{-\Delta}) \oplus \mathscr{D}(\sqrt{\omega})$ *there exists a unique global solution* $(u, \alpha) \in \mathscr{C}\big(\mathbb{R}, \mathscr{D}(\sqrt{-\Delta})\oplus\mathscr{D}(\sqrt{\omega})\big)$ *of the Schrödinger-Klein-Gordon system* (S-KG) *satisfying* $(u_0, \alpha_0) = (u(0), \alpha(0))$*. Moreover, the map*

$$\Phi^t_{SKG} : (u_0, \alpha_0) \in \mathscr{D}(\sqrt{-\Delta}) \oplus \mathscr{D}(\sqrt{\omega}) \longrightarrow (u, \alpha) \in \mathscr{C}\big(\mathbb{R}, \mathscr{D}(\sqrt{-\Delta}) \oplus \mathscr{D}(\sqrt{\omega})\big) .$$

defines a global flow of the (S-KG) *system.*

4. Main results

The three results below reflect the Bohr's correspondence principle in electrodynamics and Yukawa theory.

4.1. **First contribution.** Recall that the space $\mathcal{Q}$ is given in (2.1) and the scaled number operator $\hat{N}$ is defined similarly as in (1.10). The following formal statement will appear in [5].

Theorem 4.1 (Dynamical Pauli-Fierz model)**.** *Assume* (A0') *and let* $\big(\psi_\hbar\big)_{\hbar\in(0,1)}$ *be a family of normalized vectors on the Hilbert space* $\mathcal{Q}$ *satisfying :*

$$\exists C_1 > 0, \forall \hbar \in (0,1), \quad \big\|(\frac{\hat{p}^2}{2} + \hat{H}_f)\, \psi_\hbar\big\|_{\mathcal{Q}} \leq C_1, \tag{A1}$$

$$\exists C_2 > 0, \forall \hbar \in (0,1), \quad \big\langle \psi_\hbar,\ (\hat{q}^2 + \hat{N}^2)\, \psi_\hbar \big\rangle_{\mathcal{Q}} \leq C_2\,, \tag{A2}$$

and admitting a single Wigner measure μ*. Then the evolved quantum states*

$$\varrho_\hbar(t) = e^{-\frac{i}{\hbar}t\, \hat{H}_{PF}}\, \big|\psi_\hbar\big\rangle\big\langle\psi_\hbar\big|\, e^{\frac{i}{\hbar}t\, \hat{H}_{PF}}$$

has a unique Wigner measure given by:

$$\mu_t = (\Phi^t_{\mathcal{H}_{NM}})\,\#\, \mu\,,$$

with $\Phi^t_{\mathcal{H}_{NM}}$ *is the flow of the Newton-Maxwell equation.*

4.2. **Second contribution.** The relationship between the Yukawa Hamiltonian $\hat{H}_{Yu}$ and the classical (S-KG) system is altered by the renormalization procedure and it is not obvious even formally how these quantum and classical theories are still related. Nevertheless, we are able in [4] to prove the following Bohr's correspondence principle. Recall that the operators $\hat{N}, \hat{H}_0$ defined on the Fock space $\Gamma_s(L^2(\mathbb{R}^3))$ are given in (1.10)-(1.11). We denote by $L^2_s(\mathbb{R}^{3n})$ the space of symmetric square integrable functions.

Theorem 4.2 (Dynamical Yukawa theory)**.** *Let* $(\hbar_n)_{n\in\mathbb{N}}$ *be a sequence such that* $\hbar_n \in (0,1)$*,* $\lim \hbar_n = 0$ *and* $(n\hbar_n)_{n\in\mathbb{N}}$ *is bounded. Let* $(\varrho_n)_{n\in\mathbb{N}}$ *be a family of density matrices on* $L^2_s(\mathbb{R}^{3n}) \otimes \Gamma_s\big(L^2(\mathbb{R}^3)\big)$ *satisfying:*

$$\exists C > 0, \quad \forall n \in \mathbb{N}, \quad \mathrm{Tr}[\varrho_n\, (\hat{N} + \hat{U}_\infty \hat{H}_0 \hat{U}_\infty)] \leq C\,.$$

Then:

(i) $\mathcal{M}\big(\varrho_n, n \in \mathbb{N}\big) \neq \emptyset$.

(ii) For any $t \in \mathbb{R}$,

$$\mathcal{M}\Big(e^{-i\frac{t}{\hbar_n}\hat{H}_{Yu}}\,\varrho_n\,e^{i\frac{t}{\hbar_n}\hat{H}_{Yu}}, n \in \mathbb{N}\Big) = \Big\{(\Phi^t_{SKG})_{\#}\mu\,,\,\mu \in \mathcal{M}\big(\varrho_n, n \in \mathbb{N}\big)\Big\}\,. \tag{4.1}$$

Furthermore, suppose that $\mathcal{M}\big(\varrho_n, n \in \mathbb{N}\big) = \{\mu\}$ then for any $\xi \in L^2 \oplus L^2$ and any $t \in \mathbb{R}$,

$$\lim_{k\to\infty} \mathrm{Tr}\Big[e^{-i\frac{t}{\hbar_n}\hat{H}_{Yu}}\,\varrho_n\,e^{i\frac{t}{\hbar_n}\hat{H}_{Yu}}\,W(\xi)\Big] = \int_{L^2\oplus L^2} e^{i\sqrt{2}\Re\mathrm{e}\langle\xi,z\rangle} d(\Phi^t_{SKG})_{\#}\mu(z)\,,$$

where $W(\cdot)$ is the Weyl operator rescaled with $\hbar_n$ similarly as in (1.9).

4.3. **Third contribution.** In the case where an ultraviolet cutoff and a particle confinement is imposed in the Yukawa interaction model of Section 3, we are able to prove a further result concerning the convergence of the ground state energy towards the infimum of the Schrödinger-Klein-Gordon energy functional when $\hbar \to 0$. Consider for instance a cutoff function $\chi \in \mathscr{C}_0^\infty(\mathbb{R}^3)$ such that $\chi \equiv 1$ in a neighborhood of the origin. Then the corresponding Schrödinger-Klein-Gordon equation with ultraviolet cutoff takes the form:

$$\begin{cases} i\partial_t u = -\Delta u + Vu + A_\chi u \\ i\partial_t \alpha = \omega\alpha + \dfrac{\chi}{\sqrt{2\omega}}\widehat{(\bar{u}u)} \end{cases}$$

with $\omega(k) = \sqrt{k^2+m^2}$, $m > 0$, and the real field A_χ given by:

$$A_\chi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3} \frac{\chi(k)}{\sqrt{2\omega(k)}}\big(\bar{\alpha}(k)e^{-ik\cdot x} + \alpha(k)e^{ik\cdot x}\big)dk\,.$$

Here V is a confining potential satisfying

$$V \in L^2_{loc}(\mathbb{R}^3, \mathbb{R}^+), \qquad \lim_{|x|\to\infty} V(x) = +\infty\,. \tag{A3}$$

Using the canonical quantization explained in Section 1, one can define the ultraviolet-cutoff quantum Nelson-Yukawa Hamiltonian:

$$\begin{aligned} \hat{H}_{NY} = \mathcal{H}_{SKG_\chi}(\hat{u},\hat{\alpha}) &= \int_{\mathbb{R}^3} \hat{u}^*(x)\Big(-\Delta_x + V(x)\Big)\hat{u}(x)dx + \int_{\mathbb{R}^3}\hat{\alpha}^*(k)\omega(k)\hat{\alpha}(k)dk \\ &+ \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^6} \hat{u}^*(x)\frac{\chi(k)}{\sqrt{2\omega(k)}}\Big(\hat{\alpha}^*(k)e^{-ik\cdot x} + \hat{\alpha}(k)e^{ik\cdot x}\Big)\hat{u}(x)dxdk\,, \end{aligned} \tag{4.2}$$

related to the classical energy of Schrödinger-Klein-Gordon equation with cutoff Yukawa type interaction,

$$\begin{aligned} \mathcal{H}_{SKG_\chi}(u,\alpha) &= \int_{\mathbb{R}^3} \bar{u}(x)\Big(-\Delta_x + V(x)\Big)u(x)dx + \int_{\mathbb{R}^3}\bar{\alpha}(k)\omega(k)\alpha(k)dk \\ &+ \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^6} \bar{u}(x)\frac{\chi(k)}{\sqrt{2\omega(k)}}\Big(\bar{\alpha}(k)e^{-ik\cdot x} + \alpha(k)e^{ik\cdot x}\Big)u(x)dxdk\,. \end{aligned} \tag{4.3}$$

The number of particle is conserved and taking $(\hat{H}_{NY})_{|L^2_s(\mathbb{R}^{3n})}$ one retrieves the Nelson Hamiltonian. In particular, it is known that such expression defines a bounded from below self-adjoint operator. In this framework the following result is proved in [3].

Theorem 4.3 (Ground state energy limit)**.** *Assume that $m > 0$, $\chi \in \mathscr{C}_0^\infty(\mathbb{R}^3)$ and V satisfying (A3). Then the ground state energy of the Nelson-Yukawa Hamiltonian $\hat{H}_{NY}$ has the following limit, for any $\lambda > 0$,*

$$\lim_{\hbar\to 0, n\hbar=\lambda^2} \inf\sigma\big((\hat{H}_{NY})_{|L^2_s(\mathbb{R}^{3n})\otimes\Gamma_s(L^2)}\big) = \inf_{||u||_{L^2(\mathbb{R}^3)}=\lambda} \mathcal{H}_{SKG_\chi}(u,\alpha)\,, \tag{4.4}$$

where the infimum on the right hand side is taken over all $u \in \mathscr{D}(\sqrt{-\Delta + V})$ *and* $\alpha \in \mathscr{D}(\sqrt{\omega})$ *with the constraint* $||u||_{L^2(\mathbb{R}^3)} = \lambda$.

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