APPROXIMATION OF MINIMIZERS OF CONVEX FUNCTIONS IN HADAMARD SPACES

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Abstract. We introduce existence and convergence theorems on two modified proximal point algorithms for convex functions in Hadamard spaces.

1. Introduction

Let (X, d) be an Hadamard space and f a proper lower semicontinuous convex function of X into $(-\infty, \infty]$. Then we study the problem of finding a point $u \in X$ such that

$$f(u) = \inf f(X).$$

We denote by $\operatorname{argmin}_X f$ or $\operatorname{argmin}_{y\in X} f(y)$ the set of all solutions to this problem.

A well-known method for approximating a solution to this problem is the so-called proximal point algorithm first introduced by Martinet [10] in the case when X is a real Hilbert space. The proximal point algorithm generates a sequence $\{x_n\}$ by $x_1 \in X$ and

$$x_{n+1} = \operatorname*{argmin}_{y\in X} \left\{ f(y) + \frac{1}{2\lambda_n} d(y, x_n)^2 \right\} \quad (n = 1, 2, \dots),$$

where $\{\lambda_n\}$ is a sequence of positive real numbers.

We know the following results on the proximal point algorithm:

- If X is a real Hilbert space and $\inf_n \lambda_n > 0$, then $\{x_n\}$ is bounded if and only if $\operatorname{argmin}_X f$ is nonempty. In this case, the sequence $\{x_n\}$ is weakly convergent to an element of $\operatorname{argmin}_X f$; see Rockafellar [12];
- if X is a real Hilbert space, $\sum_{n=1}^{\infty} \lambda_n = \infty$, and $\operatorname{argmin}_X f$ is nonempty, then $\{x_n\}$ is weakly convergent to an element of $\operatorname{argmin}_X f$; see Brézis and Lions [3];
- if $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\operatorname{argmin}_X f$ is nonempty, then $\{x_n\}$ is Δ-convergent to an element of $\operatorname{argmin}_X f$; see Bačák [1].

Bačák [1] generalized the weak convergence theorem for $\{x_n\}$ by Brézis and Lions [3] to the case where X is an Hadamard space. However, the equivalence condition that $\{x_n\}$ is bounded if and only if $\operatorname{argmin}_X f$ is nonempty was not proved in the Hadamard space setting. This equivalence condition holds true as we see in Corollary 4.2.

In this paper, we introduce existence and convergence theorems for two modified proximal point algorithms in Hadamard spaces which was proved by Kimura and Kohsaka [8]. One of these algorithms is a generalization of the proximal point

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algorithm. These algorithms were studied by Kamimura and Takahashi [7] for maximal monotone operators in Hilbert spaces.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{R}$ and $\mathbb{N}$ the sets of real numbers and positive integers, respectively. We also denote by $\mathbb{R}^2$ the two dimensional Euclidean space with norm $|\cdot|_{\mathbb{R}^2}$.

A metric space (X,d) is said to be uniquely geodesic if for each $x,y \in X$, there exists a unique mapping $c\colon [0,l] \to X$ such that $c(0)=x$, $c(l)=y$, and

$$d\big(c(s),c(t)\big) = |s-t|$$

for all $s,t \in [0,l]$, where $l = d(x,y)$. In this case, we define the convex combination of x and y by

$$\alpha x \oplus (1-\alpha)y = c\big((1-\alpha)l\big)$$

for all $\alpha \in [0,1]$. A metric space (X,d) is called a CAT(0) space if it is uniquely geodesic and

$$d\big(\alpha x \oplus (1-\alpha)y, \beta x \oplus (1-\beta)z\big) \leq \big|\alpha\bar{x} + (1-\alpha)\bar{y} - \big(\beta\bar{x} + (1-\beta)\bar{z}\big)\big|_{\mathbb{R}^2}$$

whenever $\alpha,\beta \in [0,1]$, $x,y,z \in X$, $\bar{x},\bar{y},\bar{z} \in \mathbb{R}^2$,

$$d(x,y) = |\bar{x}-\bar{y}|_{\mathbb{R}^2}, \quad d(y,z) = |\bar{y}-\bar{z}|_{\mathbb{R}^2}, \quad \text{and} \quad d(z,x) = |\bar{z}-\bar{x}|_{\mathbb{R}^2}.$$

A complete CAT(0) space is called an Hadamard space. See [2,4] on geodesic spaces for more details.

Let $\{x_n\}$ be a sequence in a metric space (X,d). The asymptotic center $\mathcal{A}\big(\{x_n\}\big)$ of $\{x_n\}$ is defined by

$$\mathcal{A}\big(\{x_n\}\big) = \left\{ z \in X : \limsup_{n\to\infty} d(z,x_n) = \inf_{y\in X} \limsup_{n\to\infty} d(y,x_n) \right\}.$$

The sequence $\{x_n\}$ is said to be Δ-convergent to $p \in X$ if

$$\mathcal{A}\big(\{x_{n_i}\}\big) = \{p\}$$

for each subsequence $\{x_{n_i}\}$ of $\{x_n\}$. In this case, the sequence $\{x_n\}$ is bounded and every subsequence of $\{x_n\}$ is Δ-convergent to p. If X is a real Hilbert space, then $\{x_n\}$ is Δ-convergent to p if and only if $\{x_n\}$ is weakly convergent to p. If X is an Hadamard space and $\{x_n\}$ is a bounded sequence in X, then $\mathcal{A}\big(\{x_n\}\big)$ is a singleton and there exists a subsequence of $\{x_n\}$ which is Δ-convergent to some point in X; see [2,5,9] for more details.

Let (X,d) be a CAT(0) space and f a function of X into $(-\infty,\infty]$. The function f is said to be

- proper if $f(a) \in \mathbb{R}$ for some $a \in X$;
- lower semicontinuous if $\{x \in X : f(x) \leq \lambda\}$ is closed for each $\lambda \in \mathbb{R}$;
- convex if

  $$f\big(\alpha x \oplus (1-\alpha)y\big) \leq \alpha f(x) + (1-\alpha)f(y)$$

  whenever $x,y \in X$ and $\alpha \in (0,1)$.

We also denote by $\operatorname{argmin}_X f$ or $\operatorname{argmin}_{y\in X} f(y)$ the set

$$\{u \in X : f(u) = \inf f(X)\}$$

of all minimizers of f. In the case when $\operatorname{argmin}_X f$ is a singleton $\{p\}$ for some $p \in X$, we sometimes identify $\operatorname{argmin}_X f$ with p.

If f is a proper lower semicontinuous convex function of an Hadamard space (X, d) into $(-\infty, \infty]$ and $x \in X$, then there exists a unique point $\hat{x} \in X$ such that

$$f(\hat{x}) + \frac{1}{2} d(\hat{x}, x)^2 = \inf_{y \in X} \left\{ f(y) + \frac{1}{2} d(y, x)^2 \right\}.$$

The resolvent J_f of f is defined by $J_f(x) = \hat{x}$ for all $x \in X$. We know that $\mathcal{F}(J_f) = \operatorname{argmin}_X f$, where $\mathcal{F}(J_f)$ denotes the set of all fixed points of J_f. For each $\lambda > 0$, the function λf is proper, lower semicontinuous, and convex. In this case, we have

$$J_{\lambda f}(x) = \operatorname*{argmin}_{y \in X} \left\{ \lambda f(y) + \frac{1}{2} d(y, x)^2 \right\} = \operatorname*{argmin}_{y \in X} \left\{ f(y) + \frac{1}{2\lambda} d(y, x)^2 \right\}$$

for all $x \in X$. See [2, 6, 11] on resolvents of convex functions for more details.

A subset C of a CAT(0) space is said to be convex if

$$\alpha x \oplus (1 - \alpha) y \in C$$

whenever $x, y \in C$ and $\alpha \in [0, 1]$. If C is a nonempty closed convex subset of an Hadamard space (X, d) and $x \in X$, then there exists a unique point $\hat{x} \in C$ such that

$$d(\hat{x}, x) = \inf_{y \in C} d(y, x).$$

The metric projection of X onto C is defined by $P_C(x) = \hat{x}$ for all $x \in X$. The indicator function i_C of C is defined by

$$i_C(x) = \begin{cases} 0 & (x \in C); \\ \infty & (x \in X \setminus C). \end{cases}$$

This is a proper lower semicontinuous convex function of X into $(-\infty, \infty]$ satisfying $J_{i_C} = P_C$.

3. Fundamental properties of resolvents

In this section, we state some fundamental results on resolvents of convex functions in Hadamard spaces.

Lemma 3.1 ([8, Lemma 3.1]). *Let X be an Hadamard space and f a proper lower semicontinuous convex function of X into $(-\infty, \infty]$. If $\lambda, \mu > 0$ and $x, y \in X$, then the inequalities*

$$d(J_{\lambda f} x, J_{\mu f} y)^2 + d(J_{\lambda f} x, x)^2 + 2\lambda \big(f(J_{\lambda f} x) - f(J_{\mu f} y) \big) \le d(J_{\mu f} y, x)^2$$

and

$$\begin{aligned} &(\lambda + \mu) d(J_{\lambda f} x, J_{\mu f} y)^2 + \mu d(J_{\lambda f} x, x)^2 + \lambda d(J_{\mu f} y, y)^2 \\ &\quad \le \lambda d(J_{\lambda f} x, y)^2 + \mu d(J_{\mu f} y, x)^2 \end{aligned}$$

hold.

Corollary 3.2 ([8, Corollary 3.2]). *Let X be an Hadamard space and f a proper lower semicontinuous convex function of X into $(-\infty, \infty]$. Then*

$$2d(J_{\lambda f} x, J_{\lambda f} y)^2 + d(J_{\lambda f} x, x)^2 + d(J_{\lambda f} y, y)^2 \le d(J_{\lambda f} x, y)^2 + d(J_{\lambda f} y, x)^2$$

and

$$d(J_{\lambda f} x, J_{\lambda f} y) \le d(x, y)$$

for all $\lambda > 0$ *and* $x, y \in X$.

Using Lemma 3.1, we can prove the following lemma.

Lemma 3.3 ([8, Lemma 3.3]). *Let* X *be an Hadamard space,* f *a proper lower semicontinuous convex function of* X *into* $(-\infty, \infty]$, $\{\lambda_n\}$ *a sequence of positive real numbers, and* p *an element of* X. *Then the following hold.*

(i) *If* $\inf_n \lambda_n > 0$ *and* $\mathcal{A}(\{z_n\}) = \{p\}$ *for some sequence* $\{z_n\}$ *in* X *satisfying* $d(J_{\lambda_n f} z_n, z_n) \to 0$, *then* p *is an element of* $\operatorname{argmin}_X f$;
(ii) *if* $\lim_n \lambda_n = \infty$ *and* $\mathcal{A}(\{J_{\lambda_n f} z_n\}) = \{p\}$ *for some bounded sequence* $\{z_n\}$ *in* X, *then* p *is an element of* $\operatorname{argmin}_X f$;

We need the following minimization theorem.

Theorem 3.4 ([8, Theorem 4.1]). *Let* X *be an Hadamard space,* f *a proper lower semicontinuous convex function of* X *into* $(-\infty, \infty]$, $\{z_n\}$ *a bounded sequence in* X, $\{\beta_n\}$ *a sequence of positive real numbers such that* $\sum_{n=1}^{\infty} \beta_n = \infty$, *and* g *the real function defined by*

$$g(y) = \limsup_{n\to\infty} \frac{1}{\sum_{l=1}^{n} \beta_l} \sum_{k=1}^{n} \beta_k d(y, z_k)^2$$

for all $y \in X$. *Then* g *is a continuous and convex function on* X *such that* $\operatorname{argmin}_X g$ *is a singleton.*

4. Two modified proximal point algorithms

The following is one of our two main results in this paper.

Theorem 4.1 ([8, Theorem 4.2]). *Let* X *be an Hadamard space,* f *a proper lower semicontinuous convex function, and* $\{x_n\}$ *a sequence in* X *defined by* $x_1 \in X$ *and*

$$x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n) J_{\lambda_n f} x_n \quad (n = 1, 2, \dots),$$

where $\{\alpha_n\}$ *is a sequence in* $[0, 1)$ *and* $\{\lambda_n\}$ *is a sequence of positive real numbers satisfying*

$$\sum_{n=1}^{\infty} (1 - \alpha_n)\lambda_n = \infty.$$

Then the following hold.

(i) *The sequence* $\{J_{\lambda_n f} x_n\}$ *is bounded if and only if* $\operatorname{argmin}_X f$ *is nonempty;*
(ii) *if* $\sup_n \alpha_n < 1$ *and* $\operatorname{argmin}_X f$ *is nonempty, then* $\{x_n\}$ *and* $\{J_{\lambda_n f} x_n\}$ *are* Δ*-convergent to an element* x_∞ *of* $\operatorname{argmin}_X f$.

Corollary 4.2 ([8, Corollary 4.3]). *Let* X *be an Hadamard space,* f *a proper lower semicontinuous convex function, and* $\{x_n\}$ *a sequence in* X *defined by* $x_1 \in X$ *and*

$$x_{n+1} = J_{\lambda_n f} x_n \quad (n = 1, 2, \dots),$$

where $\{\lambda_n\}$ *is a sequence of positive real numbers satisfying*

$$\sum_{n=1}^{\infty} \lambda_n = \infty.$$

Then the following hold.

(i) *The sequence* $\{x_n\}$ *is bounded if and only if* $\operatorname{argmin}_X f$ *is nonempty;*

(ii) *if* $\operatorname{argmin}_X f$ *is nonempty, then* $\{x_n\}$ *is* Δ*-convergent to an element of* $\operatorname{argmin}_X f$.

Remark 4.3. The result (ii) was obtained by Bačák [1, Theorem 1.4].

Corollary 4.4 ([8, Corollary 4.5]). *Let* X *be a real Hilbert space,* f *a proper lower semicontinuous convex function, and* $\{x_n\}$ *a sequence in* X *defined by* $x_1 \in X$ *and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) J_{\lambda_n f} x_n \quad (n = 1, 2, \dots),$$

where $\{\alpha_n\}$ *is a sequence in* $[0, 1)$ *and* $\{\lambda_n\}$ *is a sequence of positive real numbers satisfying*

$$\sum_{n=1}^{\infty} (1 - \alpha_n)\lambda_n = \infty.$$

Then the following hold.

(i) *The sequence* $\{J_{\lambda_n f} x_n\}$ *is bounded if and only if* $\operatorname{argmin}_X f$ *is nonempty;*
(ii) *if* $\sup_n \alpha_n < 1$ *and* $\operatorname{argmin}_X f$ *is nonempty, then* $\{x_n\}$ *and* $\{J_{\lambda_n f} x_n\}$ *are weakly convergent to an element* x_∞ *of* $\operatorname{argmin}_X f$.

Remark 4.5. The result (ii) for the special case when $\lim_n \lambda_n = \infty$ was obtained by Kamimura and Takahashi [7, Theorem 3].

The following is the other of our two main results in this paper.

Theorem 4.6 ([8, Theorem 5.1]). *Let* X *be an Hadamard space,* f *a proper lower semicontinuous convex function,* v *an element of* X*, and* $\{y_n\}$ *a sequence in* X *defined by* $y_1 \in X$ *and*

$$y_{n+1} = \alpha_n v \oplus (1 - \alpha_n) J_{\lambda_n f} y_n \quad (n = 1, 2, \dots),$$

where $\{\alpha_n\}$ *is a sequence in* $[0, 1]$ *and* $\{\lambda_n\}$ *is a sequence of positive real numbers satisfying* $\lim_n \lambda_n = \infty$. *Then the following hold.*

(i) *The sequence* $\{J_{\lambda_n f} y_n\}$ *is bounded if and only if* $\operatorname{argmin}_X f$ *is nonempty;*
(ii) *if* $\lim_n \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $\operatorname{argmin}_X f$ *is nonempty, then* $\{y_n\}$ *and* $\{J_{\lambda_n f} y_n\}$ *are convergent to* Pv*, where* P *denotes the metric projection of* X *onto* $\operatorname{argmin}_X f$.

Corollary 4.7 ([8, Corollary 5.2]). *Let* X *be a real Hilbert space,* f *a proper lower semicontinuous convex function,* v *an element of* X*, and* $\{y_n\}$ *a sequence in* X *defined by* $y_1 \in X$ *and*

$$y_{n+1} = \alpha_n v + (1 - \alpha_n) J_{\lambda_n f} y_n \quad (n = 1, 2, \dots),$$

where $\{\alpha_n\}$ *is a sequence in* $[0, 1]$ *and* $\{\lambda_n\}$ *is a sequence of positive real numbers satisfying* $\lim_n \lambda_n = \infty$. *Then the following hold.*

(i) *The sequence* $\{J_{\lambda_n f} y_n\}$ *is bounded if and only if* $\operatorname{argmin}_X f$ *is nonempty;*
(ii) *if* $\lim_n \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $\operatorname{argmin}_X f$ *is nonempty, then* $\{y_n\}$ *and* $\{J_{\lambda_n f} y_n\}$ *are strongly convergent to* Pv*, where* P *denotes the metric projection of* X *onto* $\operatorname{argmin}_X f$.

Remark 4.8. The result (ii) is also a corollary of a strong convergence theorem for maximal monotone operators in Hilbert spaces obtained by Kamimura and Takahashi [7, Theorem 1].

We can also obtain the following convergence theorem to the case when $\{\alpha_n\}$ and $\{\lambda_n\}$ satisfy conditions which are different from those in Theorem 4.6.

Theorem 4.9 ([8, Theorem 5.4]). *Let X be an Hadamard space, f a proper lower semicontinuous convex function, v an element of X, and $\{y_n\}$ a sequence in X defined by $y_1 \in X$ and*

$$y_{n+1} = \alpha_n v \oplus (1 - \alpha_n) J_{\lambda_n f} y_n \quad (n = 1, 2, \dots),$$

where $\{\alpha_n\}$ is a sequence in $(0, 1]$ and $\{\lambda_n\}$ is a sequence of positive real numbers satisfying

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \text{and} \quad \inf_n \lambda_n > 0.$$

Then $\{y_n\}$ and $\{J_{\lambda_n f} y_n\}$ are convergent to Pv, where P denotes the metric projection of X onto $\operatorname{argmin}_X f$.

References

[1] M. Bačák, *The proximal point algorithm in metric spaces*, Israel J. Math. **194** (2013), 689–701.

[2] ———, *Convex Analysis and Optimization in Hadamard Spaces*, De Gruyter, Berlin, 2014.

[3] H. Brézis and P.-L. Lions, *Produits infinis de résolvantes*, Israel J. Math. **29** (1978), 329–345.

[4] M. R. Bridson and A. Haefliger, *Metric Spaces of Non-positive Curvature*, Springer-Verlag, Berlin, 1999.

[5] S. Dhompongsa, W. A. Kirk, and B. Sims, *Fixed points of uniformly Lipschitzian mappings*, Nonlinear Anal. **65** (2006), 762–772.

[6] J. Jost, *Convex functionals and generalized harmonic maps into spaces of nonpositive curvature*, Comment. Math. Helv. **70** (1995), 659–673.

[7] S. Kamimura and W. Takahashi, *Approximating solutions of maximal monotone operators in Hilbert spaces*, J. Approx. Theory **106** (2000), 226–240.

[8] Y. Kimura and F. Kohsaka, *Two modified proximal point algorithms for convex functions in Hadamard spaces*, Linear and Nonlinear Analysis **2** (2016), 69–86.

[9] W. A. Kirk and B. Panyanak, *A concept of convergence in geodesic spaces*, Nonlinear Anal. **68** (2008), 3689–3696.

[10] B. Martinet, *Régularisation d'inéquations variationnelles par approximations successives*, Rev. Française Informat. Recherche Opérationnelle **4** (1970), 154–158 (French).

[11] U. F. Mayer, *Gradient flows on nonpositively curved metric spaces and harmonic maps*, Comm. Anal. Geom. **6** (1998), 199–253.

[12] R. T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control Optimi. **14** (1976), 877–898.

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