On the locations and transcendency of the zeros of weakly holomorphic modular forms.

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1 Introduction

Let $k \geq 4$ be an even integer, $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ be the complex upper half plane and $\Gamma = SL_2(\mathbb{Z})$. The standard fundamental domain for Γ is given as follows.

$$\mathbb{F}(1) = \left\{ z \in \mathbb{H} \;\middle|\; |z| \geq 1,\ -\frac{1}{2} \leq \mathrm{Re}(z) \leq 0 \right\} \cup \left\{ z \in \mathbb{H} \;\middle|\; |z| > 1,\ 0 < \mathrm{Re}(z) < \frac{1}{2} \right\}.$$

The Eisenstein series of weight k for Γ is a function on $\mathbb{H}$ defined by

$$E_k(z) = \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} (cz+d)^{-k} = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \tag{1}$$

where $q = e^{2\pi i z}$, B_k is the kth Bernoulli number, and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$. Then E_k is a modular form of weight k for Γ

In 1970, Rankin and Swinnerton-Dyer proved that all of the zeros of E_k on $\mathbb{F}_1$ lie on the lower boundary arc[9]. Since then, the locations of the zeros of several types of holomorphic (or weakly holomorphic) modular forms have been studied by using the method introduced in [9](It is frequently called the RSD method). The RSD method is very straightforward, but it yields nontrivial results.

In 2008, Duke and Jenkins studied weakly holomorphic modular forms for Γ and constructed an integral formula of standard basis and studied their zeros[3]. The integral formula allows us to investigate the zeros of certain weakly holomorphic modular forms. Choi and Kim found a generalized integral formula for the Fricke groups of prime levels with genus zero[2].

In this paper, we introduce some results of the locations and transcendency of zeros of certain weakly holomorphic modular forms for the Fricke groups.

2 Fundamental domain of $\Gamma_0^*(p)$ for $p = 2, 3, 5, 7$

Let p be a prime number, $\Gamma_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \;\middle|\; c \equiv 0 \pmod{p} \right\}$ be the congruence subgroup of level p. We set the Fricke group of level p by

$$\Gamma_0^*(p) = N_{SL_2(\mathbb{R})}(\Gamma_0(p)) = \Gamma_0(p) \cup \begin{pmatrix} 0 & \frac{-1}{\sqrt{p}} \\ \sqrt{p} & 0 \end{pmatrix} \Gamma_0(p).$$

For $p = 2, 3, 5, 7$, The standard fundamental domain of $\Gamma_0^*(p)$ denoted by $\mathbb{F}^*(p)$ are given as follows.

(i) When $p = 2, 3$,

$$\mathbb{F}^*(p) = \left\{ z \in \mathbb{H} \;\middle|\; |z| \geq \frac{1}{\sqrt{p}},\ -\frac{1}{2} \leq \mathrm{Re}(z) \leq 0 \right\} \cup \left\{ z \in \mathbb{H} \;\middle|\; |z| > \frac{1}{\sqrt{p}},\ 0 < \mathrm{Re}(z) < \frac{1}{2} \right\}.$$

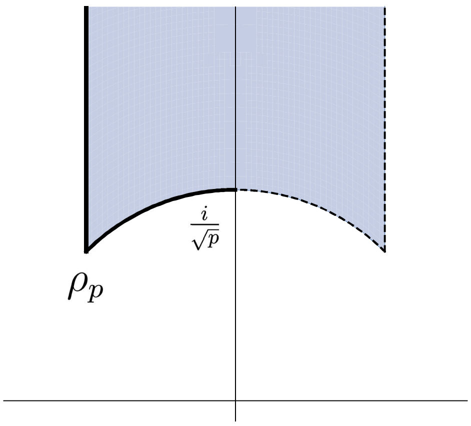

Figure 1: $\mathbb{F}^*(p)$ $(p = 2, 3)$

(ii) When $p = 5, 7$,

$$\mathbb{F}^*(p) = \left\{ z \in \mathbb{H} \;\middle|\; |z| \geq \frac{1}{\sqrt{p}},\ \left| z + \frac{1}{2} \right| \geq \frac{1}{2\sqrt{p}},\ -\frac{1}{2} \leq \mathrm{Re}(z) \leq 0 \right\}$$
$$\cup \left\{ z \in \mathbb{H} \;\middle|\; |z| > \frac{1}{\sqrt{p}},\ \left| z - \frac{1}{2} \right| > \frac{1}{2\sqrt{p}},\ 0 < \mathrm{Re}(z) < \frac{1}{2} \right\}.$$

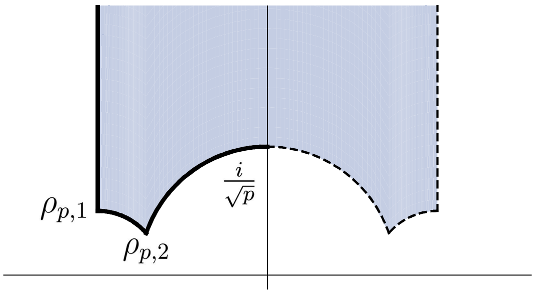

Figure 2: $\mathbb{F}^*(p)$ $(p = 5, 7)$

Here, we put $\rho_2 = \frac{1}{2} + \frac{i}{2}$, $\rho_3 = -\frac{1}{2} + \frac{i}{2\sqrt{3}}$, $\rho_{5,1} = -\frac{1}{2} + \frac{i}{2\sqrt{5}}$, $\rho_{5,2} = -\frac{2}{5} + \frac{i}{5}$, $\rho_{7,1} = -\frac{1}{2} + \frac{i}{2\sqrt{7}}$, and $\rho_{7,2} = -\frac{5}{14} + \frac{\sqrt{3}}{14} i$.

3 The locations of the zeros

3.1 The Eisenstein series

The Eisenstein series of weight $k \geq 4$ for $\Gamma_0^*(p)$ is defined by

$$E_{p,k}^*(z) = \frac{1}{1 + p^{\frac{k}{2}}} (E_k(z) + p^{\frac{k}{2}} E_k(pz)).$$

At first, we briefly recall the RSD method introduced in [9]. The RSD method is based on considering the following function

$$F_k(\theta) = e^{\frac{ik\theta}{2}} E_k(e^{i\theta}),\ \theta \in (0, \pi) \tag{2}$$

and the valence formula for Γ given by

$$v_\infty(f) + \frac{1}{2} v_i(f) + \frac{1}{3} v_{\frac{-1+\sqrt{3}i}{2}}(f) + \sum_{\substack{\rho \neq i, \frac{-1+\sqrt{3}i}{2} \\ \rho \in \Gamma \backslash \mathbb{H}}} v_\rho(f) = \frac{k}{12}$$

where f is a holomorphic modular form of weight k and $v_\rho(f)$ is the order of f at ρ. Rankin and Swinnerton-Dyer proved that F_k is real valued function. Picking out the four terms of the right hand side of (1) with $c^2+d^2=1$, they showed that

$$\left| F_k(\theta) - 2\cos\frac{k\theta}{2} \right| < 2 \tag{3}$$

for all $\theta \in [\frac{\pi}{2}, \frac{2\pi}{3}]$. By using intermediate value theorem, the valence formula, and careful estimates of E_k at i and $\frac{-1+\sqrt{3}i}{2}$, we can obtain the distribution of the zeros of E_k on $\mathbb{F}^*(1)$.

Second, we shall applicate their method in the case of $\Gamma_0^*(p)$ for $p = 2, 3, 5, 7$. By applying the above method, Miezaki, Nozaki, and Shigezumi constructed the RSD method for $p = 2, 3$ and proved the following theorem.

Theorem 3.1. [8] *Let $p = 2, 3$ and $k \geq 4$ be an even integer. Then all of the zeros of $E_{k,p}^*$ on $\mathbb{F}^*(p)$ lie on the lower boundary arc.*

In [12], Shigezumi proved similar results for $p = 5, 7$ under some assumptions. His results are incomplete because they allowed infinitely many exceptions about k. Our first main result is giving the solution of this problem.

Theorem 3.2 (K)**.** *Let $p = 5, 7$ and $k \geq 4$ be an even integer. Then all of the zeros of $E_{k,p}^*$ on $\mathbb{F}^*(p)$ lie on the lower boundary arcs.*

The following is a short proof of Theorem 3.2 for $p = 5$. Let A_5 be the lower boundary arcs of $\mathbb{F}^*(5)$. Then A_5 consists of two arcs of radiuses $\frac{1}{\sqrt{5}}$ and $\frac{1}{2\sqrt{5}}$ centered at 0 and $-\frac{1}{2}$ respectively(See Figure 2). More precisely

$$A_5^* = A_{5,1}^* \cup A_{5,2}^* \cup \left\{ \frac{i}{\sqrt{5}}, \rho_{5,1}, \rho_{5,2} \right\}$$

where

$$A_{5,1}^* = \left\{ \frac{1}{\sqrt{5}} e^{i\theta} \;\middle|\; \frac{\pi}{2} < \theta < \frac{\pi}{2} + \alpha_5 \right\}$$
$$A_{5,2}^* = \left\{ -\frac{1}{2} + \frac{1}{2\sqrt{5}} e^{i\theta} \;\middle|\; \alpha_5 < \theta < \frac{\pi}{2} \right\}$$

and α_5 is the angle such that $\tan\alpha_5 = 2$.
As analogies of F_k, we define

$$F_{k,5,1}^*(\theta) = e^{\frac{ik\theta}{2}} E_{k,5}^* \left(\frac{1}{\sqrt{5}} e^{i\theta} \right), \ \theta \in \left[\frac{\pi}{2}, \frac{\pi}{2} + \alpha_5 \right],$$
$$F_{k,5,2}^*(\theta) = e^{\frac{ik\theta}{2}} E_{k,5}^* \left(-\frac{1}{2} + \frac{1}{2\sqrt{5}} e^{i\theta} \right), \ \theta \in \left[\alpha_5, \frac{\pi}{2} \right]$$

Then we can write

$$F^*_{k,5,1}(\theta) = \frac{1}{2} \sum_{\substack{(c,d)=1 \\ 5\nmid c}} \left\{ (ce^{\frac{i\theta}{2}} + \sqrt{5}de^{-\frac{i\theta}{2}})^{-k} + (ce^{-\frac{i\theta}{2}} + \sqrt{5}de^{\frac{i\theta}{2}})^{-k} \right\}, \tag{4}$$

$$\begin{aligned} F^*_{k,5,2}(\theta) = &\frac{1}{2} \sum_{\substack{(c,d)=1 \\ 5\nmid c,\ 2\mid cd}} \left\{ (ce^{\frac{i\theta}{2}} + \sqrt{5}de^{-\frac{i\theta}{2}})^{-k} + (ce^{-\frac{i\theta}{2}} + \sqrt{5}de^{\frac{i\theta}{2}})^{-k} \right\} \\ &+ \frac{2^k}{2} \sum_{\substack{(c,d)=1 \\ 5\nmid c,\ 2\nmid cd}} \left\{ (ce^{\frac{i\theta}{2}} + \sqrt{5}de^{-\frac{i\theta}{2}})^{-k} + (ce^{-\frac{i\theta}{2}} + \sqrt{5}de^{\frac{i\theta}{2}})^{-k} \right\}. \end{aligned} \tag{5}$$

It is obvious that (4) and (5) are invariant under the complex conjugate, and hence $F^*_{k,5,j}$ $(j = 1, 2)$ are real valued functions. Unfortunately, $F^*_{k,5,j}$ does not satisfy an inequality like (3) on whole interval. To resolve this problem, we consider the first few terms of (4) and (5). We define

$$\begin{aligned} f^*_{k,5,1}(\theta) &= \frac{1}{2} \sum_{\substack{(c,d)=\pm(1,0), \\ \pm(2,1)}} \left\{ (ce^{\frac{i\theta}{2}} + \sqrt{5}de^{-\frac{i\theta}{2}})^{-k} + (ce^{-\frac{i\theta}{2}} + \sqrt{5}de^{\frac{i\theta}{2}})^{-k} \right\} \\ &= 2\cos\frac{k\theta}{2} + (2e^{\frac{i\theta}{2}} + \sqrt{5}e^{-\frac{i\theta}{2}})^{-k} + (2e^{-\frac{i\theta}{2}} + \sqrt{5}e^{\frac{i\theta}{2}})^{-k}, \\ f^*_{k,5,2}(\theta) &= \frac{1}{2} \sum_{(c,d)=\pm(1,0)} \left\{ (ce^{\frac{i\theta}{2}} + \sqrt{5}de^{-\frac{i\theta}{2}})^{-k} + (ce^{-\frac{i\theta}{2}} + \sqrt{5}de^{\frac{i\theta}{2}})^{-k} \right\} \\ &\quad + \frac{2^k}{2} \sum_{(c,d)=\pm(1,-1)} \left\{ (ce^{\frac{i\theta}{2}} + \sqrt{5}de^{-\frac{i\theta}{2}})^{-k} + (ce^{-\frac{i\theta}{2}} + \sqrt{5}de^{\frac{i\theta}{2}})^{-k} \right\} \\ &= 2\cos\frac{k\theta}{2} + \left(\frac{e^{\frac{i\theta}{2}} - \sqrt{5}e^{-\frac{i\theta}{2}}}{2} \right)^{-k} + \left(\frac{e^{-\frac{i\theta}{2}} - \sqrt{5}e^{\frac{i\theta}{2}}}{2} \right)^{-k}, \end{aligned}$$

and

$$R^*_{k,5,j}(\theta) = F^*_{k,5,j}(\theta) - f^*_{k,5,j}(\theta).$$

Then $R^*_{k,5,j}$ contributes little to the behavior of $F^*_{k,5,j}$ by the following lemma.

Lemma 3.1. *For $k \geq 4$, we have*

$$|R^*_{k,5,1}(\theta)| \leq 4\left(\frac{1}{2}\right)^{\frac{k}{2}} + \frac{160}{k-3}\left(\frac{1}{4}\right)^{\frac{k}{2}},$$

$$|R^*_{k,5,2}(\theta)| \leq \frac{160}{k-3}\left(\frac{1}{4}\right)^{\frac{k}{2}} + 2\left(\frac{2}{3}\right)^{\frac{k}{2}} + 2\left(\frac{1}{2}\right)^{\frac{k}{2}} + \frac{260\sqrt{13}}{k-3}\left(\frac{4}{13}\right)^{\frac{k}{2}},$$

$$\left|(R^*_{k,5,1})'\left(\frac{\pi}{2}+\alpha_5\right)\right| \leq 6k\left(\frac{1}{2}\right)^{\frac{k}{2}} + \frac{720k}{k-3}\left(\frac{1}{4}\right)^{\frac{k}{2}},$$

$$\left|(R^*_{k,5,2})'(\alpha_5)\right| \leq \sqrt{2}k\left(\frac{1}{2}\right)^{\frac{k}{2}} + \frac{240k}{k-3}\left(\frac{1}{4}\right)^{\frac{k}{2}} + \frac{570\sqrt{19}k}{2(k-3)}\left(\frac{4}{19}\right)^{\frac{k}{2}},$$

By Lemma 3.1 and some careful estimates of $f^*_{k,5,j}$ around at the end points of interval, we can prove the following lemmas.

Lemma 3.2. *Let $k \geq 40$ be an even integer. Then we have*

(i) $|F^*_{k,5,1}(\theta) - 2\cos\frac{k\theta}{2}| < 1 \quad (\theta \in [\frac{\pi}{2}, \frac{\pi}{2}+\alpha_5-\frac{2\pi}{3k}])$,

(ii) $|F^*_{k,5,2}(\theta) - 2\cos\frac{k\theta}{2}| < 1 \quad (\theta \in [\alpha_5+\frac{2\pi}{3k}, \frac{\pi}{2}])$.

Lemma 3.3. *Let $k \geq 40$ be an even integer.*

(i) *When $k \equiv 0 \pmod 4$, we have*

$$\operatorname{sgn}(F^*_{k,1}(\tfrac{\pi}{2}+\alpha_5)) = \operatorname{sgn}(f^*_{k,1}(\tfrac{\pi}{2}+\alpha_5)),$$
$$\operatorname{sgn}(F^*_{k,2}(\alpha_5)) = \operatorname{sgn}(f^*_{k,2}(\alpha_5)).$$

(ii) *When $k \equiv 2 \pmod 4$, we have $F^*_{k,1}(\frac{\pi}{2}+\alpha_5) = F^*_{k,2}(\alpha_5) = 0$ and*

$$\operatorname{sgn}((F^*_{k,1})'(\tfrac{\pi}{2}+\alpha_5)) = \operatorname{sgn}((f^*_{k,1})'(\tfrac{\pi}{2}+\alpha_5)),$$
$$\operatorname{sgn}((F^*_{k,2})'(\alpha_5)) = \operatorname{sgn}((f^*_{k,2})'(\alpha_5)).$$

Lemmas 3.2, 3.3, and the intermediate value theorem tell us that $E^*_{k,5}$ has at least $\begin{cases} [\frac{k}{4}] & (k \equiv 0 \pmod 4) \\ [\frac{k}{4}]-1 & (k \equiv 2 \pmod 4) \end{cases}$ distinct zeros on the arcs. Theorem 3.2 follows from the valence formula for $\Gamma_0^*(5)$ when $k \geq 40$.

Proposition 3.1. *Let f be a holomorphic modular form for $\Gamma_0^*(5)$ of weight $k \geq 4$, which is not identically zero. We have*

$$v_\infty(f) + \frac{1}{2}v_{\frac{i}{\sqrt{5}}}(f) + \frac{1}{2}v_{\rho_{5,1}}(f) + \frac{1}{2}v_{\rho_{5,2}}(f) + \sum_{\substack{\rho \neq \frac{i}{\sqrt{5}}, \rho_{5,1}, \rho_{5,2} \\ \rho \in \mathbb{F}^*(5)}} v_\rho(f) = \frac{k}{4}.$$

The proof of Proposition 3.1 is very similar to that of the valence formula for Γ (see [11]). When $4 \leq k \leq 38$, we can check directly that Theorem 3.2 is true in each case.

3.2 The natural basis

Let $p = 1, 2, 3, 5, 7$. A holomorphic function f on $\mathbb{H}$ is a weakly holomorphic modular form of weight $k \in 2\mathbb{Z}$ for $\Gamma_0^*(p)$ if f satisfies

- $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ for any $z \in \mathbb{H}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^*(p)$.
- f has a $q-$expansion of the form $f(z) = \sum_{n \in \mathbb{Z}} a_f(n) q^n$

  such that $a_f(n) = 0$ for almost all $n < 0$.

We denote the space of weakly holomorphic modular forms of weight k for $\Gamma_0^*(p)$ by $M_k^!(\Gamma_0^*(p))$.

Put $\delta = \begin{cases} 12 & \text{if } p = 1,3,7 \\ 8 & \text{if } p = 2 \\ 4 & \text{if } p = 5 \end{cases}$ and $m' = m_{p,k} = \frac{p+1}{24}\delta\ell_k + \dim S_{r_k}(\Gamma_0^+(p))$. Theorem 2.4 of [2] says that there exists a unique weakly holomorphic modular form $f_{k,m} \in M_k^!(\Gamma_0^+(p))$ such that

$$f_{k,m}(z) = q^{-m} + O(q^{m'+1})$$

for each integer $m \geq -m'$. Then $\{f_{k,m}\}_{m \geq -m'}$ forms a natural basis for $M_k^!(\Gamma_0^*(p))$. We introduce some results of the locations of the zeros of $f_{k,m}$ without proofs.

Theorem 3.3. [3, Theorem 1] *Let $\{f_{k,m}\}_{m \geq -m'}$ be the natural basis for $M_k^!(\Gamma)$. If $m \geq |\ell_k| - \ell_k$, then all of the zeros of $f_{k,m}$ in $\mathbb{F}^*(1)$ lie on the arc.*

Theorem 3.4. [1, Theorem 1.2] *Let $\{f_{k,m}\}_{m \geq -m'}$ be the natural basis for $M_k^!(\Gamma_0^*(2))$. If $m \geq 2|\ell_k| - \ell_k + 8$, then all of the zeros of $f_{k,m}$ in $\mathbb{F}^*(2)$ lie on the arc.*

Theorem 3.5. [5, Theorem 1.1] *Let $\{f_{k,m}\}_{m\geq -m'}$ be the natural basis for $M_k^!(\Gamma_0^*(3))$. If $m \geq 18|\ell_k| + 23$, then all of the zeros of $f_{k,m}$ in $\mathbb{F}^*(3)$ lie on the arc.*

Theorem 3.6. [7] *Let $p = 5, 7$ and $\{f_{k,m}\}_{m\geq -m'}$ be the natural basis for $M_k^!(\Gamma_0^*(p))$. If m is sufficiently large, then all of the zeros of $f_{k,m}$ in $\mathbb{F}^*(p)$ lie on the arcs.*

4 The transcendency of the zeros

In [6], Kohnen proved that all of the zeros of E_k except for the points equivalent to i or $\frac{-1+\sqrt{3}i}{2}$ under the action of Γ are transcendental. The proof is based on the theory of complex multiplication and the result of [9]. Gun and Saha generalize his method and obtain many result about the transcendency of the zeros of modular forms for several groups[4]. For example, they proved similar results of [6] for the natural basis for $M_k^!(\Gamma)$, $E_{k,2}^*$, and $E_{k,3}^*$. The author considered the cases of $\Gamma_0^*(5)$ and $\Gamma_0^*(7)$. Our second result is the following.

Theorem 4.1. *Let $p = 5, 7$, $k \in 2\mathbb{Z}$, $f = \sum_{n\geq n_f} a_n q^n \in M_k^!(\Gamma_0^*(p))$ such that*

- *$a_n \in \mathbb{Q}$ for any n.*
- *All of the zeros of f on $\mathbb{F}^*(p)$ lie on the lower boundary arcs.*

If $z_0 \in \mathbb{H}$ is a zero of f which is not equivalent to the following points, then z_0 is transcendental.

(i) $p = 5$

$$\frac{i}{\sqrt{5}},\ \frac{-1+\sqrt{19}i}{10},\ \frac{-1+2i}{5},\ \frac{-3+\sqrt{11}i}{10},\ \frac{-2+i}{5},\ \frac{-5+\sqrt{5}i}{10}.$$

(ii) $p = 7$

$$\frac{i}{\sqrt{7}},\ \frac{-1+3\sqrt{3}i}{14},\ \frac{-1+\sqrt{6}i}{7},\ \frac{-3+\sqrt{19}i}{14},$$
$$\frac{-2+\sqrt{3}i}{7},\ \frac{-5+\sqrt{3}i}{14},\ \frac{-6+\sqrt{6}i}{14},\ \frac{-7+\sqrt{7}i}{14}.$$

Corollary 4.1. *Let $p = 5, 7$. The same is true for $E_{k,p}^*$, $f_{k,m} \in M_k^!(\Gamma_0^*(p))$ for sufficiently large m.*

Sketch of proof of Theorem 4.1.
We put

$$g := \prod_{\gamma \in \Gamma_0(p)\backslash\Gamma} f|_k\gamma \ \in M^!_{k(p+1)}(\Gamma),$$

$$h := \frac{g^{12}}{\Delta^{k(p+1)}} \in M^!_0(\Gamma)$$

where $f|_k\gamma(z) := (cz+d)^{-k} f(\frac{az+b}{cz+d})$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, and

$$\Delta = \frac{1}{1728}(E_4^3 - E_6^2) \in S_{12}(\Gamma).$$

Since the Fourier coefficients of h are rational, we have

$$h = P(j) \text{ for some } P \in \mathbb{Q}[x]$$

where

$$j = \frac{E_4^3}{\Delta} \in M^!_0(\Gamma)$$

is the j-function.
Suppose that $z_0 \in \mathbb{H}$ is algebraic with $f(z_0) = 0$. Then

$$h(z_0) = P(j(z_0)) = 0.$$

Hence $j(z_0)$ is algebraic. By the Schneider's theorem[10], z_0 is imaginary quadratic. Therefore, z_0 satisfies

$$az_0^2 + bz_0 + c = 0 \quad (a, b, c \in \mathbb{Z},\ a > 0,\ \gcd(a, b, c) = 1)$$

Put $D := b^2 - 4ac$ and

$$z_1 := \begin{cases} \frac{i\sqrt{|D|}}{2} & (D \equiv 0 \pmod 4) \\ \frac{-1+i\sqrt{|D|}}{2} & (D \equiv 1 \pmod 4) \end{cases}.$$

By the theory of complex multiplication, there exists $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{D}))$ such that $\sigma(j(z_0)) = j(z_1)$. Therefore

$$\begin{aligned} P(j(z_0)) = 0 \Longleftrightarrow \sigma(P(j(z_0))) = P(\sigma(j(z_0))) = P(j(z_1)) = h(z_1) = 0 \\ \Longleftrightarrow f|\gamma(z_1) = 0 \text{ for some } \gamma \in \Gamma_0(p)\backslash\Gamma. \end{aligned}$$

By the assumption of the zeros on $\mathbb{F}^*(p)$, the only possibility for D is the following.

$$D = \begin{cases} -4, -11, -16, -19, -20 & (p = 5) \\ -3, -7, -12, -19, -24, -27, -28 & (p = 7) \end{cases}.$$

Therefore, we can find exceptions stated in Theorem 4.1. □

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