Existence of radially symmetric stationary solutions for the compressible Navier-Stokes equation

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1 Introduction and main theorem

This is a short survey of our recent results announced in [5] on the existence of radially symmetric stationary solutions for exterior problems in $R^n (n \geq 2)$ to the compressible Navier-Stokes equation:

$$\begin{cases} \rho_t + \mathrm{div}(\rho U) = 0, \\ (\rho U)_t + \mathrm{div}(\rho U \otimes U) + \nabla p = \nu \triangle U + (\nu + \lambda)\nabla(\mathrm{div} U), \end{cases} \tag{1.1}$$

$t > 0,\ x \in \Omega$, where $\Omega = \{x \in \mathbb{R}^n\,(n \geq 2); |x| > r_0\}$ (r_0 is a positive constant), $\rho = \rho(t,x) > 0$ is the mass density, $U = (u_1(t,x), \cdots, u_n(t,x))$ is the fluid velocity, and $p = p(\rho)$ is the pressure given by a smooth function of ρ satisfying $p'(\rho) > 0$ $(\rho > 0)$. Furthermore, ν and λ are the shear and bulk viscosity coefficients respectively, which are assumed to be constants satisfying $\nu > 0, 2\nu + n\lambda > 0$. We focus our attention on the radially symmetric solutions, which have the form

$$\rho(t,x) = \rho(t,r), \quad U(t,x) = \frac{x}{r}\,u(t,r), \quad r = |x|, \tag{1.2}$$

where $u(t,r)$ is a scalar function. By plugging (1.2) to (1.1), we can rewrite (1.1) as in the form

$$\begin{cases} (r^{n-1}\rho)_t + (r^{n-1}\rho u)_r = 0, \\ (\rho u)_t + (\rho u^2 + p(\rho))_r + (n-1)\dfrac{\rho u^2}{r} = \mu\Big(\dfrac{(r^{n-1}u)_r}{r^{n-1}}\Big)_r, \quad t > 0,\ r > r_0, \end{cases} \tag{1.3}$$

where $\mu = 2\nu + \lambda$. Now, we consider the initial boundary value problems to (1.3) under the initial condition

$$(\rho, u)(0, r) = (\rho_0, u_0)(r), \quad r > r_0, \tag{1.4}$$

the far field condition

$$\lim_{r\to\infty} (\rho, u)(t, r) = (\rho_+, u_+), \quad t > 0, \tag{1.5}$$

and also the following two types of boundary conditions depending on the sign of the velocity on the boundary

$$\begin{cases} (\rho, u)(t, r_0) = (\rho_-, u_-), & t > 0, \quad (u_- > 0), \\ u(t, r_0) = u_-, \quad t > 0, & \qquad\qquad (u_- \leq 0), \end{cases} \tag{1.6}$$

where $\rho_\pm > 0, u_\pm$ are given constants. The case $u_- > 0$ is known as "inflow problem", the case $u_- = 0$ as "impermeable wall problem", and the case $u_- > 0$ as "outflow problem". For these initial boundary value problems, when the space dimension is one ($n = 1$), there have been many results on the existence of time-global solutions and their asymptotic behaviors toward various nonlinear waves depending on the far field and boundary conditions, for example, toward stationary waves, rarefaction waves, viscous shock waves, and even their composite waves (cf. [3],[6],[7],,, etc.). On the other hand, when the problems are multi-dimensional ($n \geq 2$), there seem no results except the case $u_\pm = 0$ studied by Jiang [2] and Nakamura-Nishibata-Yanagi [8]. They study more general compressible Navier-Stokes equation, describing the motion of viscous polytropic idea gas, Jiang [2] first showed the global asymptotic stability of the constant states, and later Nakamura-Nishibata-Yanagi [8] extended the results to the case with external potential forces. As the first one step to study the multi-dimensional problems in more general cases $u_- \neq 0$ or $u_+ \neq 0$, we showed in [5] the existence of the stationary solution in a suitably small neighborhood of the far field state, by using the similar arguments as in Germain-Iwabuchi [1]. In this survey we present the overview of our paper [5].

The stationary problem corresponding to the problem (1.3)-(1.6) is written as

$$\begin{cases} (r^{n-1}\rho u)_r = 0, \\ \rho u u_r + p(\rho)_r = \mu\left(\dfrac{(r^{n-1}u)_r}{r^{n-1}}\right)_r, \qquad r > r_0, \\ \lim_{r\to\infty} (\rho, u)(r) = (\rho_+, u_+), \\ (\rho, u)(r_0) = (\rho_-, u_-) \quad (u_- > 0), \quad u(r_0) = u_- \quad (u_- \leq 0). \end{cases} \tag{1.7}$$

From the first equation in (1.7), we easily see it holds

$$r^{n-1}\rho(r)u(r) = \epsilon, \qquad r \geq r_0, \tag{1.8}$$

for some constant ϵ, and it also holds from the boundary conditions that

$$\epsilon = r_0^{n-1}\rho_- u_- \quad (u_- > 0), \qquad \epsilon = r_0^{n-1}\rho(r_0) u_- \quad (u_- \leq 0), \tag{1.9}$$

where note that in the case $u_- \leq 0$, ϵ includes the unknown $\rho(r_0)$ which should be determined later. The formula (1.8) implies that if $n \geq 2$,

$$u_+ = \lim_{r\to\infty} u(r) = \lim_{r\to\infty} \frac{\epsilon}{r^{n-1}\rho_+} = 0. \tag{1.10}$$

Hence, we need to assume $u_+ = 0$ for the existence of multi-dimensional stationary solutions. Now we are ready to state the main results.

Theorem 1.1. *Let $n \geq 2$ and $u_+ = 0$. Then, for any $\rho_+ > 0$, there exist positive constants ϵ_0 and C satisfying the following:*

(I) *Let $u_- > 0$. If $|u_-| + |\rho_- - \rho_+| \leq \epsilon_0$, there exists a unique smooth solution (ρ, u) of the problem (1.7) satisfying*

$$\begin{aligned} &|\rho(r) - \rho_+| \leq Cr^{-(n-1)}(|u_-|^2 + |\rho_- - \rho_+|), \\ &C^{-1}r^{-(n-1)}|u_-| \leq |u(r)| \leq Cr^{-(n-1)}|u_-|, \qquad r \geq r_0. \end{aligned} \tag{1.11}$$

Furthermore, for any positive constant h, there exists a positive constant C_h such that it holds

$$\sup_{r \geq r_0 + h} |\rho(r) - \rho_+| \leq C_h |u_-|^2. \tag{1.12}$$

(II) *Let $u_- \leq 0$. If $|u_-| \leq \epsilon_0$, there exists a unique smooth solution (ρ, u) of the problem (1.7) satisfying*

$$\begin{aligned} &|\rho(r) - \rho_+| \leq Cr^{-2(n-1)}|u_-|^2, \\ &C^{-1}r^{-(n-1)}|u_-| \leq |u(r)| \leq Cr^{-(n-1)}|u_-|, \qquad r \geq r_0. \end{aligned} \tag{1.13}$$

2 Preliminary

In this section, we reformulate the problem (1.7). First, we assume $u_- \neq 0$ in what follows. We also assume $r_0 = 1$ without of loss of generality. Next, introduce the specific volume v by $v = 1/\rho$ (accordingly, denote $v_\pm$ by $1/\rho_\pm$). Then, by (1.8) and (1.9), the velocity u is given in terms of v as

$$u(r) = \frac{\epsilon}{r^{n-1}} v(r), \quad r \geq 1, \tag{2.1}$$

where $\epsilon = u_-/v_-$ $(u_- > 0)$, and $\epsilon = u_-/v(1)$ $(u_- \leq 0)$. Substituting (2.1) into the second equation of (1.7), and introducing a new unknown function η, as the deviation of v from the far field state v_+, by

$$\eta(r) = v(r) - v_+, \qquad r \geq 1, \tag{2.2}$$

we obtain the following differential equation in terms of $\eta(r)$ in the following reformulated problem:

$$\begin{cases} \eta_r = \dfrac{r^{n-1}}{\epsilon\mu}\big(\tilde{p}(v_+ + \eta) - \tilde{p}(v_+)\big) \\ \qquad + \dfrac{\epsilon v_+}{2\mu}\dfrac{1}{r^{n-1}} + \dfrac{\epsilon\eta}{\mu r^{n-1}} - \dfrac{\epsilon(n-1)r^{n-1}}{\mu}\displaystyle\int_r^\infty \frac{\eta(s)}{s^{2n-1}}ds, \quad r > 1, \\ \begin{cases} \displaystyle\lim_{r\to\infty} \eta(r) = 0, \\ \eta(1) = \eta_- := v_- - v_+ \ (u_- > 0), \\ \textit{no boundary condition } (u_- < 0), \end{cases} \end{cases} \tag{2.3}$$

where $\epsilon = u_-/v_-$ $(u_- > 0)$, and $\epsilon = u_-/(v_+ + \eta(1))$ $(u_- < 0)$. We note $\tilde{p}(v) := p(1/v)$, and it holds $\tilde{p}'(v) < 0\,(v > 0)$ by the assumption on $p(v)$.

Once the desired solution η of (2.3) is obtained, the velocity u is immediately obtained by (2.1) as

$$u(r) = \frac{u_-(v_+ + \eta(r))}{v_- r^{n-1}}, \ (u_- > 0), \quad u(r) = \frac{u_-(v_+ + \eta(r))}{(v_+ + \eta(1))r^{n-1}}, \ (u_- < 0). \tag{2.4}$$

The theorem for the reformulated problem (2.3) which we need to prove is

Theorem 2.1. *Let $n \geq 2$. Then, for any $v_+ > 0$, there exist positive constants ϵ_0 and C satisfying the following:*

(I) *Let $u_- > 0$. If $|u_-| + |\eta_-| \leq \epsilon_0$, there exists a unique smooth solution η of the problem* (2.3) *satisfying*

$$|\eta(r)| \leq Cr^{-(n-1)}(|u_-|^2 + |\eta_-|), \qquad r \geq 1. \tag{2.5}$$

Furthermore, for any positive constant h, there exists a positive constant C_h satisfying

$$\sup_{r \geq r_0+h} |\eta(r)| \leq C_h |u_-|^2. \tag{2.6}$$

(II) *Let $u_- < 0$. If $|u_-| \leq \epsilon_0$, there exists a unique smooth solution η of the problem* (2.3) *satisfying*

$$|\eta(r)| \leq C r^{-2(n-1)} |u_-|^2, \qquad r \geq 1. \tag{2.7}$$

We can easily see that the main Theorem 1.1 is a direct consequence by Theorem 2.1, the formula (2.4).

3 Proof of theorem

3.1 Inflow problem

In this subsection, we consider the case $u_- > 0$, that is, inflow problem, and show the result (I) in the Theorem 1.2. In this case, recalling $\epsilon = u_-/v_- > 0$, we further rewrite the equation of η in (2.3) as in the form

$$\eta_r - \frac{\tilde{p}'(v_+)}{\mu\epsilon} r^{n-1}\eta = F[\eta], \quad r > 1, \tag{3.1}$$

where

$$F[\eta](r) := \frac{\epsilon v_+}{2\mu}\frac{1}{r^{n-1}} + \frac{\epsilon\eta(r)}{\mu r^{n-1}} - \frac{\epsilon(n-1)r^{n-1}}{\mu}\int_r^\infty \frac{\eta(s)}{s^{2n-1}}\,ds + \frac{r^{n-1}}{\mu\epsilon}N(\eta(r)),$$

$$N(\eta) := \tilde{p}(v_+ + \eta) - \tilde{p}(v_+) - \tilde{p}'(v_+)\eta.$$

Solving the linear differential equation (3.1) in terms of η with the initial data $\eta(1) = \eta_-$ and the inhomogeneous term F, we have, by the Duhamel's principle,

$$\eta(r) = \eta_- e^{-\frac{\kappa}{\epsilon}(r^n-1)} + \int_1^r e^{-\frac{\kappa}{\epsilon}(r^n-s^n)} F[\eta](s)\,ds, \quad r \geq 1, \tag{3.2}$$

where $\kappa := -\tilde{p}'(v_+)/(\mu n) > 0$. Thus, to prove the existence of the solution of (2.3) with the decay rate estimate (2.5), we look for a solution of (3.2) in the Banach space X, with its norm $\|\cdot\|_X$, defined by

$$X = \{\eta \in C([1,\infty));\ \sup_{r\geq 1} |r^{n-1}\eta(r)| < \infty\}, \quad \|\eta\|_X = \sup_{r\geq 1} |r^{n-1}\eta(r)|. \tag{3.3}$$

To do that, we construct the approximate sequence $\{\eta^{(m)}\}_{m\geq 0}$ by

$$\begin{cases} \eta^{(0)}(r) = \eta_- e^{-\frac{\kappa}{\epsilon}(r^n-1)}, \\ \eta^{(m+1)}(r) = \eta^{(0)}(r) + \displaystyle\int_1^r e^{-\frac{\kappa}{\epsilon}(r^n-s^n)} F[\eta^{(m)}](s)\,ds \quad (m \geq 0). \end{cases} \tag{3.4}$$

To show $\{\eta^{(m)}\}_{m\geq 0}$ is a Cauchy sequence in X for suitably small $|u_-| + |\eta_-|$, we prepare the following lemma.

Lemma 3.1. (I) *If $\epsilon \leq \frac{n\kappa}{n-1}$, then it holds that*

$$r^{n-1} e^{-\frac{\kappa}{\epsilon}(r^n-1)} \leq 1, \qquad r \geq 1. \tag{3.5}$$

(II) *If $\epsilon \leq \frac{n\kappa}{4(n-1)}$, then there exists a positive constant C which is independent of ϵ such that*

$$r^{n-1} \left| \int_1^r e^{-\frac{\kappa}{\epsilon}(r^n-s^n)} f(s)\,ds \right| \leq C\epsilon \, \|f\|_X \qquad r \geq 1, \quad f \in X. \tag{3.6}$$

The proof is given by elementary calculations. Refer to [5] for the details.

Now, by using Lemma 3.1, we show the uniform boundedness of $\eta^{(m)}$ $(m \geq 0)$ in X for suitably small $|u_-| + |\eta_-|$. That is, we show that for any fixed v_+, there exist positive constants ϵ_0 and C which are independent of u_- and η_- such that if $|u_-| + |\eta_-| \leq \epsilon_0$, then there exists a positive constant M satisfying

$$\|\eta^{(m)}\|_X \leq M \leq C(|u_-|^2 + |\eta_-|), \quad m \geq 0. \tag{3.7}$$

Here and in what follows, we use the letter C and ϵ_0 to denote generic positive constants which are independent of u_- and η_-, but may depend on v_+ and other fixed constants like $\mu, n, \ldots, etc.$ For the proof, in particular, to use Lemma 3.1, we first assume

$$|\eta_-| = |v_- - v_+| \leq \frac{v_+}{2}, \quad u_- \leq \frac{n\kappa v_+}{8(n-1)}, \tag{3.8}$$

which assures

$$\frac{v_+}{2} \leq v_- \leq \frac{3v_+}{2}, \quad \epsilon = \frac{u_-}{v_-} \leq \frac{n\kappa}{4(n-1)}. \tag{3.9}$$

Let us show (3.7) by mathematical induction:

<u>Case $m = 0$</u>. Due to (3.9), we have from Lemma 3.1 that

$$|r^{n-1}\eta^{(0)}(r)| = |\eta_-||r^{n-1}e^{-\frac{\kappa}{\epsilon}(r^n-1)}| \leq |\eta_-|, \quad r \geq 1. \tag{3.10}$$

Hence, we ask the constant M satisfy

$$|\eta_-| \le M, \tag{3.11}$$

so that it holds $\|\eta^{(0)}\|_X \le M$.
Case $m = k+1$ $(k \ge 0)$. Suppose (3.7) with $m = k$ hold. Here we ask the constant M satisfy another assumption

$$M \le \frac{v_+}{2}, \tag{3.12}$$

which, in particular, implies

$$|\eta^{(k)}(r)| \le \frac{v_+}{2}, \quad r \ge 1. \tag{3.13}$$

Then, by using Lemma 3.1, (3.12),(3.13), and also the Taylor's theorem, we estimate η^{k+1} as follows:

$$\begin{aligned} r^{n-1}|\eta^{(k+1)}(r)| &\le r^{n-1}|\eta^{(0)}(r)| + r^{n-1}\int_1^r e^{-\frac{\kappa}{\epsilon}(r^n-s^n)}|F[\eta^{(k)}](s)|\,ds \\ &\le |\eta_-| + r^{n-1}\Big|\int_1^r e^{-\frac{\kappa}{\epsilon}(r^n-s^n)}\Big(\frac{\epsilon v_+}{2\mu}\frac{1}{s^{n-1}}\Big)\,ds\Big| \\ &\quad + r^{n-1}\Big|\int_1^r e^{-\frac{\kappa}{\epsilon}(r^n-s^n)}\Big(\frac{\epsilon\eta^{(k)}(s)}{\mu s^{n-1}}\Big)\,ds\Big| \\ &\quad + r^{n-1}\Big|\int_1^r e^{-\frac{\kappa}{\epsilon}(r^n-s^n)}\Big(\frac{\epsilon(n-1)s^{n-1}}{\mu}\int_s^\infty \frac{\eta^{(k)}(\tau)}{\tau^{2n-1}}\,d\tau\Big)\,ds\Big| \\ &\quad + r^{n-1}\Big|\int_1^r e^{-\frac{\kappa}{\epsilon}(r^n-s^n)}\Big(\frac{r^{n-1}}{\mu\epsilon}N(\eta^{(k)}(s))\Big)\,ds\Big| \\ &=: |\eta_-| + I_1 + I_2 + I_3 + I_4; \end{aligned} \tag{3.14}$$

$$I_1 \le C\epsilon \sup_{r\ge1}|r^{n-1}(\frac{\epsilon v_+}{2\mu}\frac{1}{r^{n-1}})| \le C\epsilon^2\frac{v_+}{2\mu} \le C|u_-|^2; \tag{3.15}$$

$$I_2 \le C\epsilon \sup_{r\ge1}|r^{n-1}(\frac{\epsilon\eta^{(k)}(r)}{\mu r^{n-1}})| \le C\frac{\epsilon^2}{\mu}M \le C|u_-|^2; \tag{3.16}$$

$$I_3 \le \frac{C\epsilon^2(n-1)}{\mu}\sup_{r\ge1}|r^{2(n-1)}\int_r^\infty \frac{s^{n-1}\eta^{(k)}(s)}{s^{3n-2}}\,ds| \le \frac{C\epsilon^2 M}{3\mu} \le C|u_-|^2 M; \tag{3.17}$$

$$
\begin{aligned}
I_4 &\le C\epsilon \sup_{r\ge 1} |r^{n-1}(\frac{r^{n-1}}{\mu\epsilon} N(\eta^{(k)}(r)))| \\
&\le \frac{C}{2\mu} \cdot \sup_{v_+/2\le t\le 3v_+/2} |\tilde{p}''(t)| \cdot \sup_{r\ge 1}(r^{2(n-1)}|\eta^{(k)}(r)|^2) \le CM^2.
\end{aligned}
\tag{3.18}
$$

Substituting (3.15)-(3.18) into (3.14), we obtain

$$
\|\eta^{(k+1)}\|_X \le |\eta_-| + C|u_-|^2 + CM^2. \tag{3.19}
$$

Therefore, we further assume

$$
|\eta_-| + C|u_-|^2 \le \frac{M}{2}, \qquad CM \le \frac{1}{2}, \tag{3.20}
$$

so that (3.19) gives the desired estimate $\|\eta^{(k+1)}\|_X \le M$. By elementary calculations, it is easy to see there exists a positive constant ϵ_0 such that if $|\eta_-| + |u_-| \le \epsilon_0$, all the assumptions (3.8),(3.11),(3.12), and (3.20) hold, and in particular, M can be chosen by

$$
M = 2(|\eta_-| + C|u_-|^2) \quad (\le C(|\eta_-| + |u_-|^2)\,). \tag{3.21}
$$

Thus, the uniform boundedness of $\eta^{(m)}$ $(m \ge 0)$ in X with the estimate (3.7) is proved. Once (3.7) is proved, the proof to show $\{\eta^{(m)}\}_{m\ge 0}$ is a Cauchy sequence in X is very standard. In fact, by using Lemma 3.1, we may estimate

$$
\begin{aligned}
\|\eta^{(m+1)} - \eta^{(m)}\|_X &= \sup_{r\ge 1} |r^{n-1} \int_1^r e^{-\frac{\kappa}{\epsilon}(r^n - s^n)} (F[\eta^{(m)}] - F[\eta^{(m-1)}])(s)\, ds| \\
&\le C\epsilon \|F[\eta^{(m)}] - F[\eta^{(m-1)}]\|_X, \quad m \ge 1,
\end{aligned}
\tag{3.22}
$$

in the same way as in (3.15)-(3.18), and taking ϵ_0 suitably small again if needed, we can show

$$
\|\eta^{(m+1)} - \eta^{(m)}\|_X \le \frac{1}{2} \|\eta^{(m)} - \eta^{(m-1)}\|_X, \quad m \ge 1, \tag{3.23}
$$

which proves that $\{\eta^{(m)}\}_{m\ge 0}$ is a Cauchy sequence in X. Thus, as the limit, the solution η of (2.3) with the desired estimate (2.5) is obtained. The arguments on the regularity and uniqueness of the solution are also very standard, so we omit them.

Finally, we show the estimate (2.6), that is, the existence of the boundary layer for the density. We turn back to the equation in (2.3), and again rewrite it as

$$
\eta_r - \frac{a(\eta) r^{n-1}}{\epsilon} \eta = q[\eta], \quad r > 1, \tag{3.24}
$$

where

$$
\begin{aligned}
a(\eta) &= \frac{\tilde{p}(v_+ + \eta) - \tilde{p}(v_+)}{\eta\mu}, \\
q[\eta](r) &= \frac{\epsilon v_+}{2\mu}\frac{1}{r^{n-1}} + \frac{\epsilon\eta(r)}{\mu r^{n-1}} - \frac{r^{n-1}\epsilon(n-1)}{\mu}\int_r^\infty \frac{\eta(s)}{s^{2n-1}}\,ds.
\end{aligned} \tag{3.25}
$$

Here we note that we already know the existence of the solution η of (3.24) with the estimate (2.5). Solving the equation (3.24) in terms of η with the initial data $\eta(1) = \eta_-$, we have, by the Duhamel's principle,

$$
\eta(r) = e^{\int_1^r \frac{a(\eta(s))}{\epsilon}s^{n-1}ds}\eta_- + \int_1^r e^{\int_\tau^r \frac{a(\eta(s))}{\epsilon}s^{n-1}ds} q[\eta](\tau)\,d\tau. \tag{3.26}
$$

By the estimate (2.5) and the assumption $\tilde{p}(v) < 0$ $(v > 0)$, it is easy to see that there exist positive constants δ and C satisfying

$$
-a(\eta(r)) \geq \delta, \quad |q[\eta](r)| \leq C\epsilon, \qquad r \geq 1. \tag{3.27}
$$

Therefore, for any positive constant h, if $r \geq 1 + h$, it follows from (3.26) and (3.27) that

$$
\begin{aligned}
|\eta(r)| &\leq e^{-\frac{\delta}{\epsilon}\int_1^r s^{n-1}ds}|\eta_-| + \int_1^r e^{-\frac{\delta}{\epsilon}\int_\tau^r s^{n-1}ds}|q[\eta](\tau)|\,d\tau \\
&\leq e^{-\frac{\delta}{\epsilon}(r-1)}|\eta_-| + C\epsilon\int_1^r e^{-\frac{\delta}{\epsilon}(r-\tau)}\,d\tau \\
&\leq e^{-\frac{\delta}{\epsilon}h}|\eta_-| + \frac{C}{\delta}\epsilon^2 \leq C_h|u_-|^2,
\end{aligned} \tag{3.28}
$$

which proves the desired estimate (2.6). Thus, the proof for the case $u_- > 0$ in Theorem 2.1 is completed.

3.2 Outflow problem

In this section, we consider the case $u_- < 0$, that is, outflow problem, and show the result (II) in the Theorem 1.2. In this case, recalling $\epsilon = u_-/(v_+ + \eta(1)) < 0$, we again rewrite the equation of η in (2.3) as in the form

$$
\eta_r - \frac{v_+\tilde{p}'(v_+)}{\mu u_-}r^{n-1}\eta = G[\eta], \quad r > 1, \tag{3.29}
$$

where

$$G[\eta](r) := \frac{v_+\tilde{p}'(v_+)\eta(r)\eta(1)}{\mu u_-}r^{n-1} + \frac{u_-v_+}{2\mu(v_+ + \eta(1))}\frac{1}{r^{n-1}}$$
$$+ \frac{u_-\eta(r)}{\mu(v_+ + \eta(1))r^{n-1}} - \frac{(n-1)u_-r^{n-1}}{\mu}\int_r^\infty \frac{\eta(s)}{(v_+ + \eta(1))s^{2n-1}}\,ds$$
$$+ \frac{(v_+ + \eta(1))r^{n-1}}{\mu u_-}N(\eta(r)).$$

Noting $u_- < 0$, and solving the equation (3.29) in terms of η with the inhomogeneous term G and the far field condition $\eta(\infty) = 0$, we obtain

$$\eta(r) = -\int_r^\infty e^{-\frac{\kappa v_+}{|u_-|}(s^n - r^n)}G[\eta](s)\,ds, \quad r \geq 1, \tag{3.30}$$

where we recall $\kappa = -\tilde{p}'(v_+)/(\mu n) > 0$. This time, to prove the existence of the solution of (3.30) with the decay rate estimate (2.7), we look for a solution of (3.30) in the Banach space Y, with its norm $\|\cdot\|_Y$, defined by

$$\begin{aligned} Y = \{\eta \in & C([1,\infty));\ \sup_{r\geq 1}|r^{2(n-1)}\eta(r)| < \infty\}, \\ & \|\eta\|_Y = \sup_{r\geq 1}|r^{2(n-1)}\eta(r)|. \end{aligned} \tag{3.31}$$

By the same way as in the last section, we construct the approximate sequence $\{\eta^{(m)}\}_{m\geq 0}$ by

$$\begin{cases} \eta^{(0)}(r) = -\int_r^\infty e^{-\frac{\kappa v_+}{|u_-|}(s^n - r^n)}G[0](s)\,ds, \\ \eta^{(m+1)}(r) = -\int_r^\infty e^{-\frac{\bar{\kappa} v_+}{|u_-|}(s^n - r^n)}G[\eta^{(m)}](s))\,ds \quad (m \geq 0). \end{cases} \tag{3.32}$$

To show the uniform boundedness, we prepare the following lemma.

Lemma 3.2. *For $g \in X$, it holds that*

$$r^{2(n-1)}|\int_r^\infty e^{-\frac{\kappa v_+}{|u_-|}(s^n - r^n)}g(s)\,ds| \leq \frac{|u_-|}{n\kappa v_+}\sup_{s\geq r}|s^{n-1}g(s)|, \quad r \geq 1. \tag{3.33}$$

Proof. It holds that

$$|\int_r^\infty e^{-\frac{\kappa v_+}{|u_-|}(s^n - r^n)}g(s)\,ds| \leq \int_r^\infty e^{-\frac{n\kappa v_+}{|u_-|}r^{n-1}(s-r)}s^{-(n-1)}s^{(n-1)}|g(s)|\,ds$$
$$\leq r^{-2(n-1)}\frac{|u_-|}{n\kappa v_+}\sup_{s\geq r}|s^{n-1}g(s)|, \quad r \geq 1.$$

Thus, the proof of Lemma 3.2 is completed.

By using Lemma 3.2, we can show that for any fixed $v_+ > 0$, there exist positive constants ϵ_0 and C such that if $|u_-| \leq \epsilon_0$, then there exists a positive constant M satisfying

$$\|\eta^{(m)}\|_Y \leq M \leq C|u_-|^2, \quad m \geq 0, \tag{3.34}$$

as in the same manner of proof for the inequality (3.7). So we omit the details of the proofs of (3.34) and that $\{\eta^{(m)}\}_{m>0}$ forms a Cauchy sequence in Y.

References

[1] P. Germain, T. Iwabuchi, Self-similar solutions for compressible Navier-Stokes equations, to appear, arXiv:1903.09958.

[2] S. Jiang, Global spherically symmetric solutions to the equations of a viscous polytropic ideal gas in an exterior domain, Comm. Math. Phys. 178 (1996), 339-374.

[3] S. Kawashima, S. Nishibata, and P. Zhu, Asymptotic stability of the stationary solution to the compressible Navier-Stokes equations in the half space, Comm. Math. Phys. 240 (2003), 483-500.

[4] I. Hashimoto, A. Matsumura, Asymptotic behavior toward nonlinear waves for radially symmetric solutions of the multi-dimensional Burgers equation, J. Differential Equations, 266 (2019), 2805-2829.

[5] I. Hashimoto, A. Matsumura, Existence of radially symmetric stationary solutions for the compressible Navier-Stokes equation, to appear in Methods and Applications of Analysis.

[6] A. Matsumura, Inflow and outflow problems in the half space for a one-dimensional isentropic model system of compressible viscous gas, Methods and Applications of Analysis 8 (2001), 645-666.

[7] A. Matsumura and K. Nishihara, Large-time behaviors of solutions to an inflow problem in the half space for a one-dimensional system of compressible viscous gas, Commun. Math. Phys. 222 (2001), 449-474.

[8] T. Nakamura, S. Nishibata, and S. Yanagi, Large-time behavior of spherically symmetric solutions to an isentropic model of compressible viscous fluid in a field of potential forces, Math. Models Methods Appl. Sci. 14 (2004), 1849-1879.

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