Representation theory of generalized quantum algebras using Weyl groupoids

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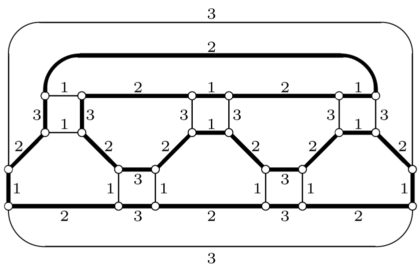
Hamiltonian circuit
Let $\Gamma = (V, E)$ be a graph, where V (resp. E) is the set of vertices (resp. edges).
Let $\varphi : \{1, 2, ..., |V|\} \to V$ be a bijection.
We call φ *a Hamiltonian circuit* of Γ if

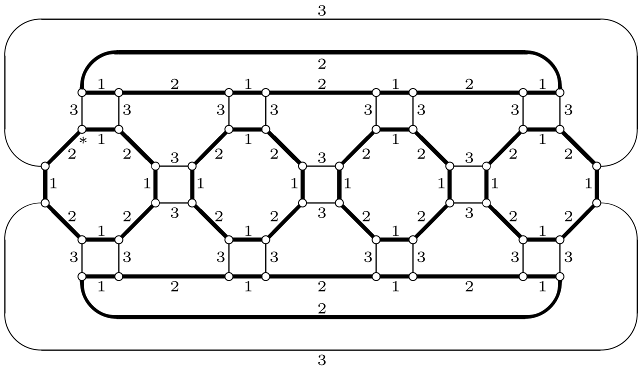
$$\{\varphi(i), \varphi(i+1)\} \in E \ (1 \leq i \leq |E| - 1)$$

and $\{\varphi(|E|), \varphi(1)\} \in E$.

In a very beautiful and easy way,
J.H. Conway, N.J.A. Sloane and Allan R. Wilks, (Gray codes for reflection groups, Graphs and Combinatorics 5 (1989) 315–325) showed an existence of a Hamiltonian circuit of the Caley graph of every finite Coxeter group $W = \langle s_i | i \in I \rangle$.
Example 1: Caley graph of Weyl group $W(A_3)$ of type A_3



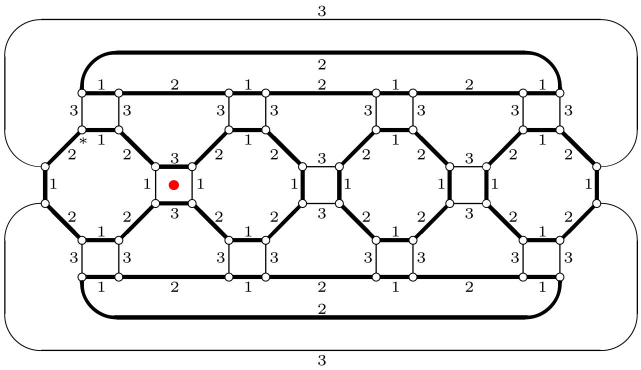
$W(A_3) = \langle s_1, s_2, s_3 \rangle$
$s_i^2 = e$, $s_1s_3 = s_3s_1$, $s_1s_2s_1 = s_2s_1s_2$, $s_2s_3s_2 = s_3s_2s_3$.
Example 2-process 0: Caley graph of Weyl group $W(B_3)$ of type B_3



$W(B_3) = \langle s_1, s_2, s_3 \rangle$

$s_i^2 = e,\ s_1s_3 = s_3s_1,\ (s_1s_2)^2 = (s_2s_1)^2,\ s_2s_3s_2 = s_3s_2s_3.$

Example 2-process 1: Caley graph of Weyl group $W(B_3)$ of type B_3



$W(B_3) = \langle s_1, s_2, s_3 \rangle$

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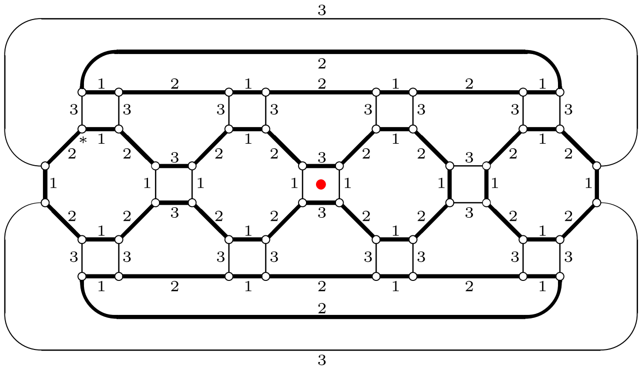
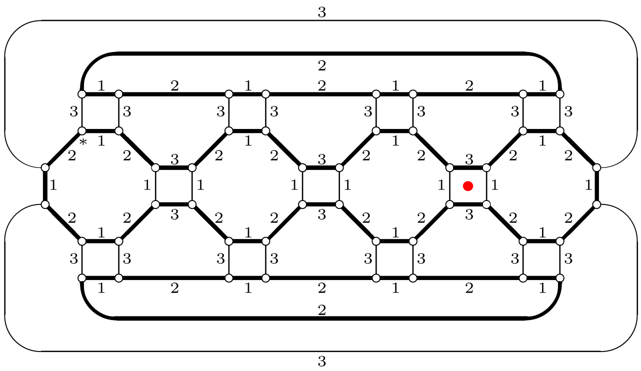
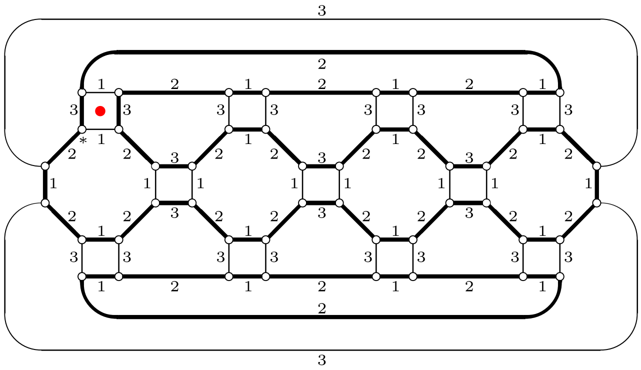
Example 2-process 2: Caley graph of Weyl group $W(B_3)$ of type B_3



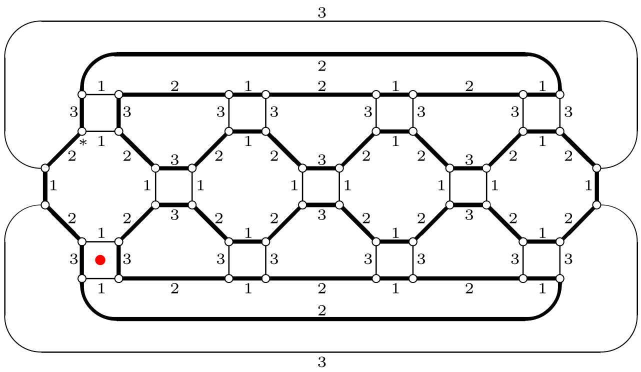
$W(B_3) = \langle s_1, s_2, s_3 \rangle$
$s_i^2 = e$, $s_1 s_3 = s_3 s_1$, $(s_1 s_2)^2 = (s_2 s_1)^2$, $s_2 s_3 s_2 = s_3 s_2 s_3$.
Example 2-process 3: Caley graph of Weyl group $W(B_3)$ of type B_3



$W(B_3) = \langle s_1, s_2, s_3 \rangle$
$s_i^2 = e$, $s_1 s_3 = s_3 s_1$, $(s_1 s_2)^2 = (s_2 s_1)^2$, $s_2 s_3 s_2 = s_3 s_2 s_3$.
Example 2-process 4: Caley graph of Weyl group $W(B_3)$ of type B_3



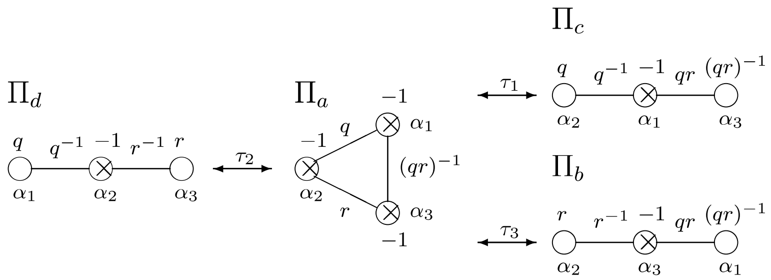
$W(B_3) = \langle s_1, s_2, s_3 \rangle$
$s_i^2 = e$, $s_1 s_3 = s_3 s_1$, $(s_1 s_2)^2 = (s_2 s_1)^2$, $s_2 s_3 s_2 = s_3 s_2 s_3$.
Example 2-process 5-completed: Caley graph of Weyl group $W(B_3)$ of type B_3



$W(B_3) = \langle s_1, s_2, s_3 \rangle$
$s_i^2 = e$, $s_1 s_3 = s_3 s_1$, $(s_1 s_2)^2 = (s_2 s_1)^2$, $s_2 s_3 s_2 = s_3 s_2 s_3$. In a similar but non-easy way, the speaker showed an existence of a Hamiltonian circuit of the Caley graph of the Weyl groupiods associated to the generalized quantum groups by [Hiroyuki Yamane, Hamilton circuits of Cayley graphs of Weyl groupoids of generalized quantum groups, arXiv.2103.16126].

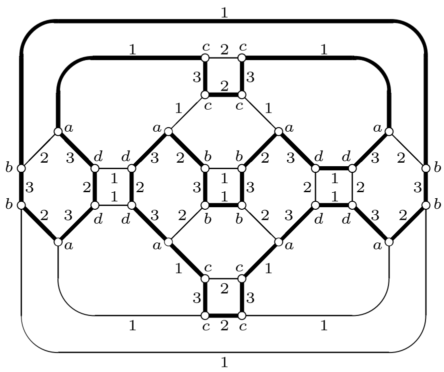
In rank 3 and 4 cases, he exactly obtained Hamilton circuits.

Weyl groupoid of the Lie superalgebra $D(2, 1; \alpha)$



$R_+^{\Pi_d} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1+\alpha_2, \alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3, \alpha_1+2\alpha_2+\alpha_3\}$. $R_+^{\Pi_a} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1+\alpha_2, \alpha_2+\alpha_3, \alpha_1+\alpha_3, \alpha_1+\alpha_2+\alpha_3\}$.

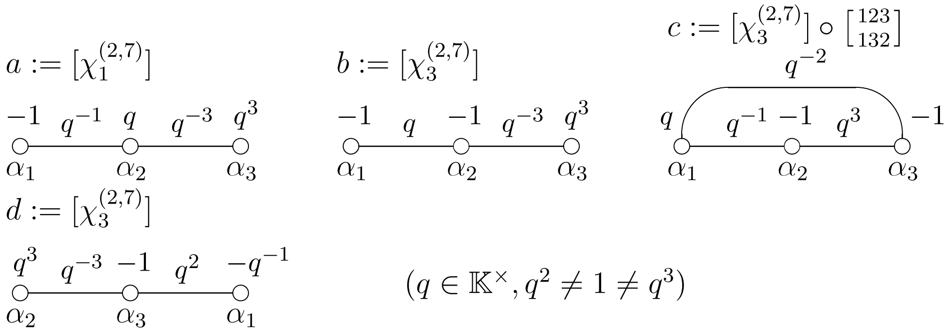
Example 3: Caley graph of Weyl groupoid $W(D(2,1;\alpha))$ of the Lie superalgebra $D(2,1;\alpha)$



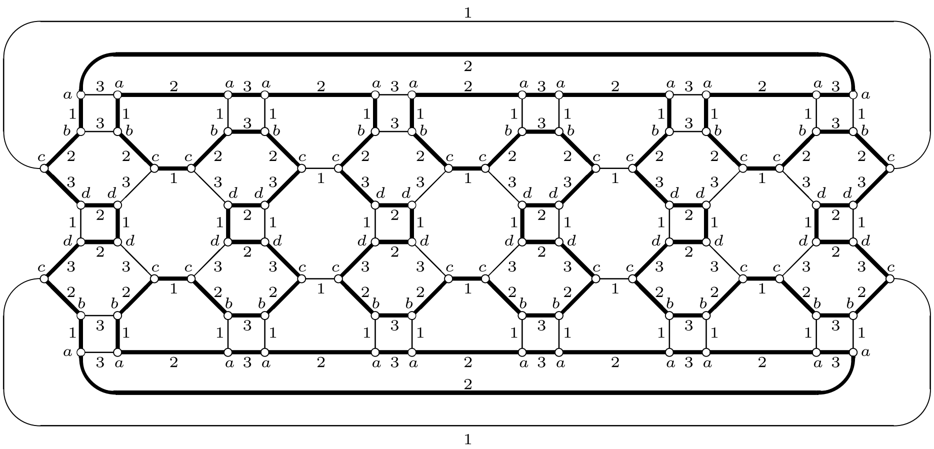
$W = W(D(2,1;\alpha)) = \langle 0, e^x, s_1^x, s_2^x, s_3^x (x \in \{a,b,c,d\})\rangle$ is the semigroup with $0w = w0 = 0$ $(w \in W)$, $(e^x)^2 = e^x$, $e^x e^y = 0$ $(x \neq y)$, $s_1^c s_1^a = e^a$, $s_2^b s_2^a = e^a$, $s_3^d s_3^a = e^a$, $s_1^b s_1^b = e^b$, $s_2^a s_2^b = e^b$, $s_3^b s_3^b = e^b$, $s_1^a s_1^c = e^c$, $s_2^c s_2^c = e^c$, $s_3^c s_3^c = e^c$, $s_1^d s_1^d = e^d$, $s_2^d s_2^d = e^d$, $s_3^a s_3^d = e^d$,

$s_1^c s_2^c s_1^a = s_2^b s_1^b s_2^a$, $s_2^b s_3^b s_2^a = s_3^d s_2^d s_3^a$, $s_1^c s_3^c s_1^a = s_3^d s_1^d s_3^a$,
$s_1^a s_2^b s_1^b = s_2^c s_1^a s_2^b$, $s_2^d s_3^a s_2^b = s_3^a s_2^b s_3^b$, $s_3^b s_1^b = s_1^b s_3^b$,
$s_1^b s_2^a s_1^c = s_2^a s_1^c s_2^c$, $s_3^c s_2^c = s_2^c s_3^c$, $s_1^d s_3^a s_1^c = s_3^a s_1^c s_3^c$,
$s_2^d s_1^d = s_1^d s_2^d$, $s_2^a s_3^d s_2^d = s_3^b s_2^a s_3^d$, $s_1^a s_3^d s_1^d = s_3^c s_1^a s_3^d$

Generalized quantum group $U_q(G(3))$ of Lie superalgebra $G(3)$



$(q \in \mathbb{K}^\times, q^2 \neq 1 \neq q^3)$



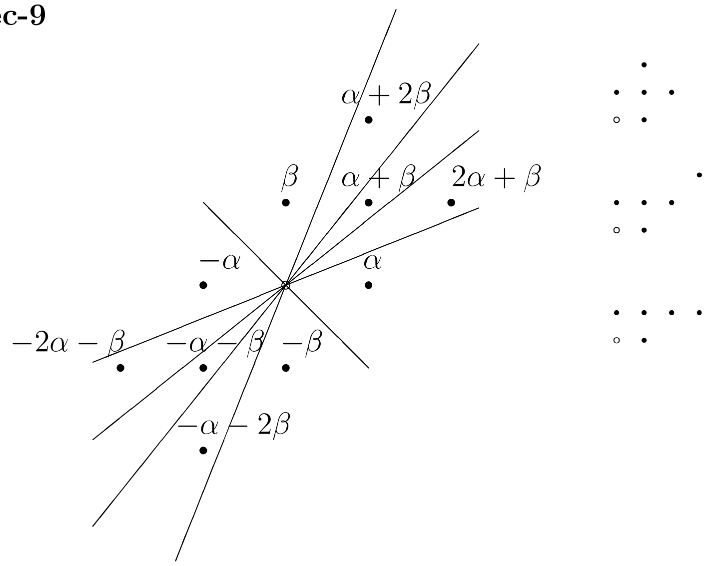
Generalized root system

Let $n \in \mathbb{N}$. Let $\widetilde{\mathfrak{B}}$ be the set of all the $\mathbb{Z}$-bases of $\mathbb{Z}^n$.
Let R be a nonempty subset of $\mathbb{Z}^n$.
Let $\mathfrak{B}$ be the subset of $\widetilde{\mathfrak{B}}$ formed by all $\Pi \in \widetilde{\mathfrak{B}}$ with $R = R_+^\Pi \cup (-R_+^\Pi)$, where $R_+^\Pi := R \cap \mathrm{Span}_{\mathbb{Z}_{\geq 0}} \Pi$.
We call R *a finite generalized root system* if

(R0) $|R| < \infty$ and $\mathfrak{B} \neq \emptyset$.
(R1) $\forall \Pi \in \mathfrak{B}$, $\Pi \subset R$
(R2) $\forall \Pi \in \mathfrak{B}$, $\forall \alpha \in \Pi$, $\mathbb{Z}\alpha \cap R = \{\pm\alpha\}$
(R3) $\forall \Pi \in \mathfrak{B}$, $\forall \alpha \in \Pi$, $\exists \Pi^{(\alpha)} \in \mathfrak{B}$, $R_+^{\Pi^{(\alpha)}} \cap (-R_+^\Pi) = \{-\alpha\}$

Namely $\Pi^{(\alpha)} = \{-\alpha\} \cup \{\beta + N_{\alpha,\beta}^\Pi \alpha | \beta \in \Pi \setminus \{\alpha\}\}$,
where $N_{\alpha,\beta}^\Pi = \max\{k \in \mathbb{Z}_{\geq 0} | \beta + k\alpha \in R_+^\Pi\}$. Let $N_{\alpha,\alpha}^\Pi := -2$. Then $\Pi^{(\alpha)} = \{\beta + N_{\alpha,\beta}^\Pi \alpha | \beta \in \Pi\}$.

Rank-2-Hec-9



Let $\Pi^{(\beta_1,\ldots,\beta_k)}$ denote $\Pi^{(\beta_1)}$ (if $k=1$) and $(\Pi^{(\beta_1,\ldots,\beta_{k-1})})^{(\beta_k)}$ (if $k \geq 2$).

There exist bijections $\pi_\Pi : \{1,2,\ldots,n\} \to \Pi$ ($\Pi \in \mathfrak{B}$) such that for $1 \leq i \leq n$, $\pi_{\Pi^{(\pi(\Pi)(i))}}(j) = \pi_\Pi(j) + N^\Pi_{\pi(\Pi)(i),\pi(\Pi)(j)}\pi_\Pi(i)$.

In particular, if $\Pi^{(\beta_1,\ldots,\beta_k)} = \Pi$ happens,
then $\pi_{\Pi^{(\beta_1,\ldots,\beta_k)}} = \pi_\Pi$ as a map,
that is, $\pi_{\Pi^{(\beta_1,\ldots,\beta_k)}}(i) = \pi_\Pi(i)$ for all $1 \leq i \leq n$. Let $\mathfrak{B}^\sharp := \{\pi_\Pi | \Pi \in \mathfrak{B}\}$.
For $1 \leq i \leq n$, define the map $\tau_i : \mathfrak{B}^\sharp \to \mathfrak{B}^\sharp$ by $\tau_i(\pi_\Pi) := \pi_{\Pi^{(\pi(\Pi)(i))}}$.
Then $\tau_i^2 = \mathrm{id}$. Moreover

$$\underbrace{\tau_i\tau_j\tau_i\cdots}_{m_{ij}}(\pi_\Pi) = \underbrace{\tau_j\tau_i\tau_j\cdots}_{m_{ij}}(\pi_\Pi),$$

where $m_{ij} = m_{ij}^\Pi := |(\mathbb{Z}_{\geq 0}\pi_\Pi(i) \oplus \mathbb{Z}_{\geq 0}\pi_\Pi(j)) \cap R| < \infty$,for $1 \leq i \neq j \leq n$.

The Cayley graph of the Weyl groupoid $W(R)$ (defined by the next slide) is the graph $\Gamma(R) = (V(R), E(R))$ with $V(R) := \mathfrak{B}^\sharp$ and $E(R) := \{\{\pi, \tau_i(\pi)\} | \pi \in \mathfrak{B}^\sharp,\ 1 \leq i \leq n\}$. The (universal) Weyl groupoid $W(R)$ is the groupoid 'formed' by the base change matrices

$$S_i^{\pi,\tau_i(\pi)} \in \mathrm{GL}_n(\mathbb{Z}) \quad (1 \leq i \leq n,\ \pi \in \mathfrak{B}^\sharp)$$

defined by

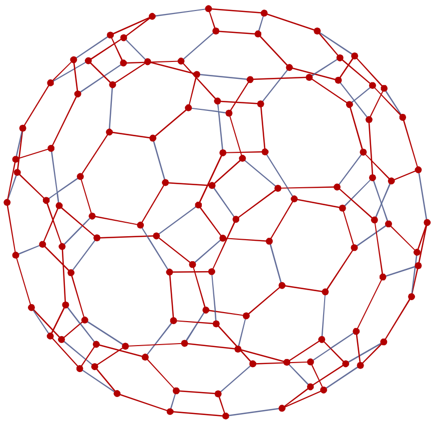
$$[\tau_i(\pi)(1), \ldots, \tau_i(\pi)(n)] = [\pi(1), \ldots, \pi(n)] S_i^{\pi, \tau_i(\pi)}.$$

We have $S_i^{\pi,\tau_i(\pi)} = (S_i^{\pi,\tau_i(\pi)})^{-1} = S_i^{\tau_i(\pi),\pi}$, $S_i^{\pi,\tau_i(\pi)} S_i^{\tau_i(\pi),\pi} = E$ and

$$\underbrace{S_i^{\pi,\tau_i(\pi)} S_j^{\tau_i(\pi),\tau_j\tau_i(\pi)} \cdots}_{m_{ij}} = \underbrace{S_j^{\pi,\tau_j(\pi)} S_i^{\tau_j(\pi),\tau_i\tau_j(\pi)} \cdots}_{m_{ij}} \quad (i \neq j).$$

These are exactly the defining relations of $W(R)$ (I. Heckenberger, H. Yamane, A generalization of Coxeter groups, root systems, and Matsumoto's theorem, Math. Z, 259 (2008), 255-276, see also H. Yamane, Generalized root systems and the affine Lie superalgebra $\mathrm{G}^{(1)}(3)$, São Paulo J. Math. Sci. 10 (2016), no. 1, 9-19.). Cuntz-Heckenberger classified Weyl groupoids. (M. Cuntz and I. Heckenberger, Finite Weyl groupoids, J. Reine Angew. Math. 702 (2015) 77-108.) (M. Cuntz and I. Heckenberger, Finite Weyl groupoids of rank three, Trans. Amer. Math. Soc. 364 (2012), 1369-1393) In Rank=3, 55-cases exists; among them, 3-cases are associated with simple complex Lie algebras, 3-cases are associated with simple complex Lie superalgebras of type $A-G$, and 18-cases are associated with irreducible generalized quantum groups.

Quite recently, I found Hamiltonian circuits for some of their classification of Rank=3 using Wolfram Mathemathica 12.3.



This is made by Wolfram Mathemathica 12.3. The 15-th generalized root system of the list of the Cuntz-Heckenberger's paper.

Generalized Quantum Group $U(\chi)$

Let $\mathbb{C}^\times := \mathbb{C}\setminus\{0\}$. Let I be a nonempty set. Let $\mathbb{Z}\Pi$ be a free $\mathbb{Z}$-module with a base $\Pi = \{\alpha_i | i \in I\}$. Then the rank of $\mathbb{Z}\Pi$ is $|I|$. Let $\chi : \mathbb{Z}\Pi \times \mathbb{Z}\Pi \to \mathbb{K}^\times$ be a map with

$$\chi(\lambda+\mu,\nu) = \chi(\lambda,\nu)\chi(\mu,\nu), \chi(\lambda,\mu+\nu) = \chi(\lambda,\mu)\chi(\lambda,\nu) \quad (\lambda,\mu,\nu \in \mathbb{Z}\Pi).$$

If $\chi(\alpha_i,\alpha_i) = q^{a_{ij}}$, it is usual quatum groups.
If $\chi(\alpha_i,\alpha_i) = (-1)^{p(i)p(j)}q^{a_{ij}}$, it is usual quatum supergroups.

To such χ, we can associate an associative $\mathbb{K}$-algebra (with 1) defined by the following axioms $(U1)$-$(U6)$.

$(U1)$ U has generators K_λ, L_λ(=almost $K_{-\lambda}$ ($\lambda \in \mathbb{Z}\Pi$), E_i, F_i ($i \in I$).

$(U2)$ The following equations hold. $K_0 = L_0 = 1$, $K_\lambda K_\mu = K_{\lambda+\mu}$, $L_\lambda L_\mu = L_{\lambda+\mu}$, $K_\lambda L_\mu = L_\mu K_\lambda$, $K_\lambda E_i = \chi(\lambda, \alpha_i) E_i K_\lambda$,
$K_\lambda F_i = \chi(\lambda, -\alpha_i) F_i K_\lambda$, $L_\lambda E_i = \chi(-\alpha_i, \lambda) E_i L_\lambda$, $L_\lambda F_i = \chi(\alpha_i, \lambda) F_i L_\lambda$, $E_i F_j - F_j E_i = \delta_{ij}(-K_{\alpha_i} + L_{\alpha_i})$.

$(U3)$ We have an injection $\varsigma_1 : \mathbb{Z}\Pi \times \mathbb{Z}\Pi \to U$, $(\lambda, \mu) \mapsto K_\lambda L_\mu$, and $\varsigma_1(\mathbb{Z}\Pi \times \mathbb{Z}\Pi)$ is an linearly independent set.

$(U4)$ Let $U^0 := \mathrm{Span}_{\mathbb{K}} \varsigma_1(\mathbb{Z}\Pi \times \mathbb{Z}\Pi)$. Let U^+ (resp. U^-) be the $\mathbb{K}$-subalgebra (with 1) of U generated by E_i (resp. F_i). Then we have the $\mathbb{K}$-linear isomorphism $\varsigma_2 : U^- \otimes U^0 \otimes U^+ \to U$ defined by $\varsigma_2(Y \otimes Z \otimes X) := YZX$.

$(U5)$ We have the $\mathbb{K}$-linear subspaces U_λ ($\lambda \in \mathbb{Z}\Pi$) of U. $U^0 \subset U_0$, $E_i \in U_{\alpha_i}$, $F_i \in U_{-\alpha_i}$ ($i \in I$), $U = \oplus_{\lambda \in \mathbb{Z}\Pi} U_\lambda$, $U_\mu U_\nu \subset U_{\mu+\nu}$ ($\mu, \nu \in \mathbb{Z}\Pi$).

$(U6)$ Let $U^\pm_\lambda := U_\lambda \cap U^\pm$ ($\lambda \in \mathbb{Z}\Pi$). Let $\mathbb{Z}\Pi^+ := \oplus_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$ ($\subset \mathbb{Z}\Pi$). Then for any $\lambda \in \mathbb{Z}\Pi^+ \setminus \{0\}$, we have $\{X \in U^+_\lambda | \forall i \in I, XF_i = F_i X\} = \{0\}$ and $\{Y \in U^-_{-\lambda} | \forall i \in I, YE_i = E_i Y\} = \{0\}$. (We have $\dim U^+_\lambda = \dim U^-_{-\lambda}$ ($\lambda \in \mathbb{Z}\Pi^+$))

Let $q_\lambda := \chi(\lambda, \lambda)$ ($\lambda \in \mathbb{Z}\Pi$). Define the map $\mathrm{ord} : \mathbb{Z}\Pi \to \mathbb{N} \cup \{\infty\}$ by $\mathrm{ord}(\lambda) := |\{\sum_{r=0}^k q_\lambda^r | k \in \mathbb{Z}_{\geq 0}\}|$. Let $U^{+,\prime} := (\sum_{\lambda \in \mathbb{Z}\Pi^+ \setminus \{0\}} U^+_\lambda) \setminus \{0\}$. Define the map $\deg : U^{+,\prime} \to \mathbb{Z}\Pi^+ \setminus \{0\}$ by $\deg(X) := \lambda$ ($X \in U^+_\lambda \cap U^{+,\prime}$, $\lambda \in \mathbb{Z}\Pi^+ \setminus \{0\}$).

Kharchenko-PBW Theorem (1999) There exist a nonempty subset S of $U^{+,\prime}$ and a total order $\preceq$ satisfying:
$\{X_1^{n_1} X_2^{n_2} \cdots X_k^{n_k} | X_t \in S, 0 \leq n_t < \mathrm{ord}(\deg(X_t)) \, (1 \leq t \leq k), X_1 < X_2 < \ldots < X_k\}$ form a $\mathbb{K}$-basis of U^+. Note that $\alpha_i \in R^+_\chi$ ($i \in I$).

Moreover the set $R^+_\chi := \deg(S) (\subset \mathbb{Z}\Pi^+ \setminus \{0\})$ is independent from S.

From now on, we assume $|R_\chi^+| < \infty$. Then $\deg_{|S}$ is injective.

Weyl-Kac-type typical irreducible character formulas of $U(\chi)$

Assume that $q_\beta := \chi(\beta,\beta) \neq 1$ for all $\beta \in R_\chi$, where this condition is not essential since $\mathrm{Char}(\mathbb{C}) = 0$. Let $\mathrm{ord}(\beta) := \min\{k \in \mathbb{N} | 1 + q_\beta + \cdots + q_\beta^k = 0\}$; if $\mathrm{ord}(\beta)$ can not be defined, let $\mathrm{ord}(\beta) := \infty$.

Let $R_\chi^{+,\mathrm{fin}} := \{\beta \in R_\chi^+ | \mathrm{ord}(\beta) < \infty\}$. Let $R_\chi^{+,\mathrm{inf}} := \{\beta \in R_\chi^+ | \mathrm{ord}(\beta) = \infty\}$. Then $R_\chi^+ = R_\chi^{+,\mathrm{fin}} \cup R_\chi^{+,\mathrm{inf}}$.

For $\beta \in R_\chi^{+,\mathrm{inf}}$ and $\lambda \in \mathbb{Z}\Pi$, we have $k_\lambda^\beta \in \mathbb{Z}$ with $\chi(\beta,\lambda)\chi(\lambda,\beta) = q_\beta^{k_\lambda^\beta}$.
For $\beta \in R_\chi^{+,\mathrm{inf}}$, define the $\mathbb{Z}$-module $s_\beta : \mathbb{Z}\Pi \to \mathbb{Z}\Pi$ by $s_\beta(\lambda) := \lambda - k_\lambda^\beta \beta$ ($\lambda \in \mathbb{Z}\Pi$).
Let $W := \langle s_\beta | \beta \in \dot{R}_\chi^{+,\mathrm{inf}} \rangle$. In fact, W is a Weyl group. We have the group homomorphism $\mathrm{sgn} : W \to \{\pm 1\} (\subset \mathbb{Z})$ with $\mathrm{sgn}(s_\beta) = -1$.
Define the $\mathbb{Z}$-module homomorphism $\hat{\rho} : \mathbb{Z}\Pi \to \mathbb{K}^\times$ by $\hat{\rho}(\alpha_i) := q_{\alpha_i}$ ($i \in I$).
For $\beta \in R_\chi^{+,\mathrm{inf}}$, we have $r_\beta \in \mathbb{Z}$ with $\hat{\rho}(\beta) = q_\beta^{r_\beta}$.

Let $\Lambda : U^0 \to \mathbb{C}$ be a $\mathbb{C}$-algebra homomorphism.

Let $\mathcal{L}(\Lambda)$ be a left irreducible U-module for which we have $v_\Lambda \in \mathcal{L}(\Lambda) \setminus \{0\}$ with $E_i v_\Lambda = 0$ ($i \in I$), $Z v_\Lambda = \Lambda(Z) v_\Lambda$ ($Z \in U^0$) and letting $\mathcal{L}(\Lambda)_\lambda := U_\lambda v_\Lambda$ ($\lambda \in \mathbb{Z}\Pi$), we also have $\mathcal{L}(\Lambda) = \oplus_{\lambda \in \mathbb{Z}\Pi^+} \mathcal{L}(\Lambda)_{-\lambda}$.

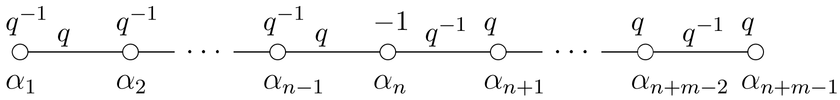
From now on, assume $\dim \mathcal{L}(\Lambda) < \infty$.
For $\beta \in R_\chi^{+,\mathrm{inf}}$, we have $n_\beta^\Lambda \in \mathbb{Z}_{\geq 0}$ with $\Lambda(K_\beta L_{-\beta}) = q_\beta^{n_\beta^\Lambda}$.
Then W has another action $\cdot$ on $\mathbb{Z}\Pi$ defined by $s_\beta \cdot \lambda := s_\beta(\lambda) - (r_\beta + n_\beta^\Lambda)\beta$ ($\beta \in R_\chi^{+,\mathrm{inf}}$, $\lambda \in \mathbb{Z}\Pi$).

Theorem (H. Yamane, J. Algebra Appl. 20(1) (2021) 2140014) Assume that $\hat{\rho}(\beta)\Lambda(K_\beta L_{-\beta}) \notin \{q_\beta^t | 1 \leq t \in \mathrm{ord}(\beta) - 1\}$ for all $\beta \in R_\chi^{+,\mathrm{fin}}$. Then we have

$$\dim \mathcal{L}(\Lambda)_\lambda = \sum_{w \in W} \mathrm{sgn}(w) \dim U^-_{\lambda - w\cdot 0}.$$

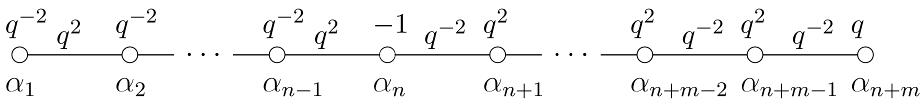
for $\lambda \in \mathbb{Z}\Pi$.

Classification of finite dimensional $\mathcal{L}(\Lambda)$
Let $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$. Let $\mathbb{C}^\times_{<\infty} := \{\zeta \in \mathbb{C}^\times | 2 \leq \exists k < \infty, \zeta^k = 1\}$ and $\mathbb{C}^\times_{\infty} := \{q \in \mathbb{C}^\times | \forall k <\in \mathbb{N}, q^k \neq 1\}$. Let $R^{+,\mathrm{fin}}_\chi := \{\alpha \in R^+_\chi | \chi(\alpha, \alpha) \in \mathbb{C}^\times_{<\infty}\}$ and $R^{+,\mathrm{inf}}_\chi := \{\alpha \in R^+_\chi | \chi(\alpha, \alpha) \in \mathbb{C}^\times_{\infty}\}$. Let $q \in \mathbb{C}^\times_{<\infty}$.
(1) Let U be the one associated with the Lie superalgbra $A(m-1, n-1)$. Then $\Pi = \{\alpha_i := \epsilon_i - \epsilon_{i+1} | 1 \leq i \leq n+m-1\}$, $R^{+,\mathrm{fin}}_\chi = \{\epsilon_i - \epsilon_j | 1 \leq i \leq n, n+1 \leq j \leq n+m\}$ and $R^{+,\mathrm{inf}}_\chi = \{\epsilon_i - \epsilon_j | 1 \leq i < j \leq n\}$. Then $\dim \mathcal{L}(\Lambda) < \infty$ if and only if the following conditions $(a1)$-$(a2)$ is satisfied.
$(a1)$ $1 \leq \forall i \leq n-1$, $\exists k_i \in \mathbb{Z}_{\geq 0}$, $\Lambda(K_{\alpha_i} L_{-\alpha_i}) = q^{-k_i}$
$(a2)$ $n+1 \leq \forall i \leq n+m-1$, $\exists k_i \in \mathbb{Z}_{\geq 0}$, $\Lambda(K_{\alpha_i} L_{-\alpha_i}) = q^{k_i}$



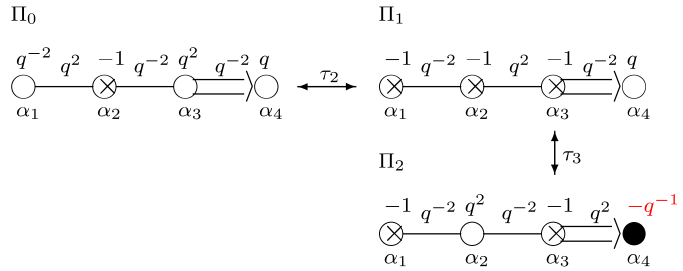
Generalized Dynkin diagram of $A(m-1, n-1)$

(2) Let U be the one associated with the Lie superalgbra $B(m, n)$. Then $\Pi = \{\alpha_i := \epsilon_i - \epsilon_{i+1} | 1 \leq i \leq n+m-1\} \cup \{\alpha_{n+m} := \epsilon_{n+m}\}$, $R^{+,\mathrm{fin}}_\chi = \{\epsilon_i \pm \epsilon_j | 1 \leq i \leq n, n+1 \leq j \leq n+m\}$ and $R^{+,\mathrm{inf}}_\chi = \{\epsilon_i | 1 \leq i \leq n+m\} \cup \{\epsilon_i \pm \epsilon_j | 1 \leq i < j \leq n\} \cup \{\epsilon_i \pm \epsilon_j | n+1 \leq i < j \leq n+m\}$. Then $\dim \mathcal{L}(\Lambda) < \infty$ if and only if the following conditions $(b1)$-$(b5)$ is satisfied. Let $\beta := \alpha_n + \alpha_{n+1} + \cdots + \alpha_{m+n}$.
$(b1)$ $1 \leq \forall i \leq n-1$, $\exists k_i \in \mathbb{Z}_{\geq 0}$, $\Lambda(K_{\alpha_i} L_{-\alpha_i}) = q^{-2k_i}$
$(b2)$ $n+1 \leq \forall i \leq n+m-1$, $\exists k_i \in \mathbb{Z}_{\geq 0}$, $\Lambda(K_{\alpha_i} L_{-\alpha_i}) = q^{2k_i}$
$(b3)$ $\exists k \in \mathbb{Z}_{\geq 0}$, $\Lambda(K_{\alpha_{m+n}} L_{-\alpha_{m+n}}) = q^k$
$(b4)$ $\Lambda(K_\beta L_\beta^{-1}) \in \{q^{-2k} | 0 \leq k \leq m-1\} \cup \{(-q)^{-t-2m} | t \in \mathbb{Z}_{\geq 0}\}$
$(b5)$ If $\Lambda(K_\beta L_\beta^{-1}) = q^{-2k}$ for some $0 \leq k \leq m-1$, then $\Lambda(K_{\alpha_i} L_{\alpha_i}^{-1}) = 1$ for all $n+k+1 \leq i \leq n+m$.



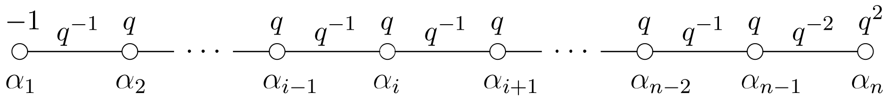
Generalized Dynkin diagram of $B(m, n)$

If t of $(b4)$ is an odd integer, we cannot take $q \to 1$. In that case Λ is typical.



$w_0 = s_3^{B_0} s_4^{B_0} s_3^{B_0} s_4^{B_0} s_1^{B_0} s_2^{B_1} s_3^{B_2} s_4^{B_2} s_3^{B_1} s_2^{B_0} s_1^{B_0} s_2^{B_1} s_3^{B_2} s_4^{B_2} s_3^{B_1} s_2^{B_0}$

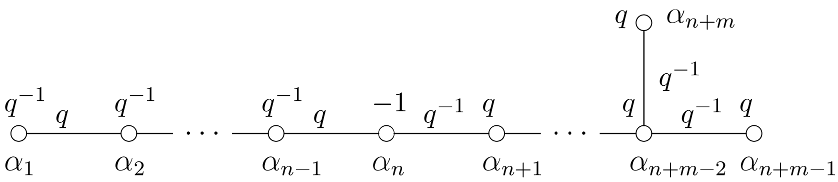
(3) Let U be the one associated with the Lie superalgbra $C(n)$. Then $\Pi = \{\alpha_i := \epsilon_i - \epsilon_{i+1} | 1 \leq i \leq n-1\} \cup \{\alpha_n := 2\epsilon_n\}$, $R_\chi^{+,\mathrm{fin}} = \{\epsilon_1 \pm \epsilon_j | 2 \leq j \leq n\}$ and $R_\chi^{+,\mathrm{inf}} = \{\epsilon_i \pm \epsilon_j | 2 \leq i < j \leq n\} \cup \{2\epsilon_i | 2 \leq i \leq n\}$. Then $\dim \mathcal{L}(\Lambda) < \infty$ if and only if the following conditions $(a1)$-$(a2)$ is satisfied.
$(c1)$ $2 \leq \forall i \leq n-1$, $\exists k_i \in \mathbb{Z}_{\geq 0}$, $\Lambda(K_{\alpha_i} L_{-\alpha_i}) = q^{k_i}$.
$(c2)$ $\exists k_n \in \mathbb{Z}_{\geq 0}$, $\Lambda(K_{\alpha_n} L_{-\alpha_n}) = q^{2k_n}$.



Generalized Dynkin diagram of $C(n)$

(4) Let U be the one associated with the Lie superalgbra $D(m,n)$. Then $\Pi = \{\alpha_i := \epsilon_i - \epsilon_{i+1} | 1 \leq i \leq n+m-1\} \cup \{\alpha_{n+m} := \epsilon_{n+m-1} + \epsilon_{n+m}\}$, $R_\chi^{+,\mathrm{fin}} = \{\epsilon_i \pm \epsilon_j | 1 \leq i \leq n, n+1 \leq j \leq n+m\}$ and $R_\chi^{+,\mathrm{inf}} = \{\epsilon_i \pm \epsilon_j | 1 \leq i < j \leq n\} \cup \{\epsilon_i \pm \epsilon_j | n+1 \leq i < j \leq n+m\}$. Then $\dim \mathcal{L}(\Lambda) < \infty$ if and only if the following conditions $(d1)$-$(d5)$ is satisfied. Let $\beta := 2\alpha_n + 2\alpha_{n+1} + \cdots + 2\alpha_{n+m-2} + \alpha_{n+m-1} + \alpha_{m+n}$ and $\gamma := \alpha_n + \alpha_{n+1} + \cdots + \alpha_{n+m-1}$.
$(d1)$ $1 \leq \forall i \leq n-1$, $\exists k_i \in \mathbb{Z}_{\geq 0}$, $\Lambda(K_{\alpha_i} L_{-\alpha_i}) = q^{-k_i}$
$(d2)$ $n+1 \leq \forall i \leq n+m$, $\exists k_i \in \mathbb{Z}_{\geq 0}$, $\Lambda(K_{\alpha_i} L_{-\alpha_i}) = q^{k_i}$
$(d3)$ $\exists k \in \mathbb{Z}_{\geq 0}$, $\Lambda(K_\beta L_{-\beta}) = q^{-2k}$
$(d4)$ If $\Lambda(K_\beta L_\beta^{-1}) = q^{-2k}$ for some $0 \leq k \leq m-2$, then $\Lambda(K_\gamma L_\gamma^{-1}) = q^{-k}$ and $\Lambda(K_{\alpha_i} L_{\alpha_i}^{-1}) = 1$ for all $n+k+1 \leq i \leq n+m$.

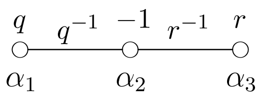
($d5$) If $\Lambda(K_\beta L_\beta^{-1}) = q^{-2(m-1)}$, then $\Lambda(K_\gamma L_\gamma^{-1}) = q^{-(m-1)}$.



Generalized Dynkin diagram of $D(m,n)$

Assume $k \geq m$ for ($d3$). Then $\Lambda(K_{\alpha_n} L_{-\alpha_n}) = \pm q^{-r}$ for some $r \in \mathbb{N}$. If $\Lambda(K_{\alpha_n} L_{-\alpha_n}) = -q^{-r}$, we cannot take $q \to 1$, and Λ is typical. Let $k \in \mathbb{N}$ and $u \in \{0,1\}$. Assume $\Lambda(K_{\alpha_i} L_{-\alpha_i}) = q^{-d_i}$ ($1 \leq i \leq N-m-1$, $\exists d_i \in \mathbb{Z}_{\geq 0}$). Assume $\Lambda(K_{\alpha_{N-m+1}} L_{-\alpha_{N-m+1}}) = \cdots = \Lambda(K_{\alpha_N} L_{-\alpha_N}) = q^k$. Assume $\Lambda(K_{\alpha_{N-m}} L_{-\alpha_{N-m}}) = (-1)^u q^{-m(k+1)}$.
Let $\lambda := \alpha_{N-m} + \alpha_{N-m+1} + \cdots + \alpha_{N-2} + \alpha_{N-1} + \alpha_N$.

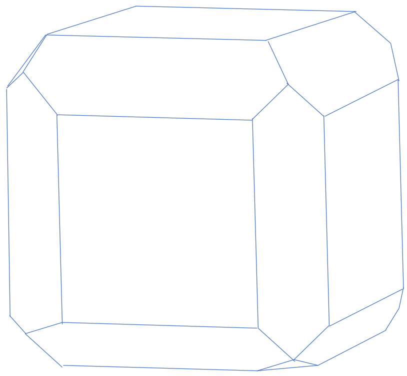
If $u = 0$, then $\mathcal{L}(\Lambda)$ is NOT typical and $\dim \mathcal{L}(\Lambda)_{-\lambda} < \dim \mathcal{M}(\Lambda)_{-\lambda}$. If $u = 1$, then $\mathcal{L}(\Lambda)$ is typical and $\dim \mathcal{L}(\Lambda)_{-\lambda} = \dim \mathcal{M}(\Lambda)_{-\lambda}$. (5) Let U be the one associated with the Lie superalgbra $D(2,1;\alpha)$. Let $q \in \mathbb{C}^\times_\infty$. Let $q_\beta := \chi(\beta,\beta)$. Let $\Lambda_\beta := \Lambda(K_\beta L_\beta^{-1})$.
(i) $r \in \mathbb{C}^\times_\infty$ and $qr \in \mathbb{C}^\times_\infty$.
(ii) $r \in \mathbb{C}^\times_\infty$ and $qr \in \mathbb{C}^\times_{<\infty}$.
(iii) $r \in \mathbb{C}^\times_{<\infty}$ and $qr \in \mathbb{C}^\times_\infty$.
Let $\alpha_0 := \alpha_1 + 2\alpha_2 + \alpha_3$. Then $q_{\alpha_0} = q^{-1}r^{-1}$, $q_{\alpha_1} = q$, $q_{\alpha_2} = -1$, $q_{\alpha_3} = r$, Then $\dim \mathcal{L}(\Lambda) < \infty$ if and only if the following conditions ($da1$)-($da3$) is satisfied.
($da1$) For $i \in \{0,1,3\}$, if $q_{\alpha_i} \in \mathbb{C}^\times_\infty$, then $\Lambda_{\alpha_i} = q_{\alpha_i}^{k_i}$ for some $k_i \in \mathbb{Z}_{\geq 0}$.
($da2$) If $q_{\alpha_i} \in \mathbb{C}^\times_\infty$ and $k_0 = 0$, then $\Lambda_{\alpha_j} = 1$ for $j \in \{1,2,3\}$.
($da3$) If $q_{\alpha_i} \in \mathbb{C}^\times_\infty$ and $k_0 = 1$, then $\Lambda_{\alpha_2} = 1$ or $\Lambda_{\alpha_1+\alpha_2} = q^{-1}$.



Generalized Dynkin diagram of $D(2,1;\alpha)$

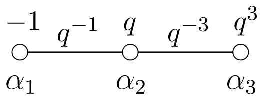
The following a shape of a finite-dimensional typical irreducible module of U

of $D(2,1;\alpha)$..



(6) Let U be the one associated with the Lie superalgbra $G(3)$.

Let $\alpha_0 := \alpha_1+2\alpha_2+\alpha_3$. Then $q_{\alpha_0} = -q^{-1}$, $q_{\alpha_1} = -1$, $q_{\alpha_2} = q$, $q_{\alpha_3} = q^3$. Then $\dim \mathcal{L}(\Lambda) < \infty$ if and only if the following conditions $(g1)$-$(g3)$ is satisfied.
$(g1)$ For $i \in \{0, 2, 3\}$, then $\Lambda_{\alpha_i} = q_{\alpha_i}^{k_i}$ for some $k_i \in \mathbb{Z}_{\geq 0}$.
$(g2)$ $k_0 \in \{0, 4\}$ or $k_0 \geq 6$.
$(g3)$ If $k_0 = 0$, then $\Lambda_{\alpha_j} = 1$ for $j \in \{1, 2, 3\}$.
$(g4)$ If $k_0 = 4$, then $\Lambda_{\alpha_2} = 1$.

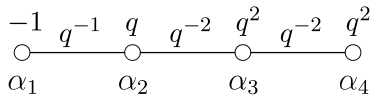


Generalized Dynkin diagram of $G(3)$

(7) Let U be the one associated with the Lie superalgbra $F(4)$.

Let $\alpha_0 := 2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4$. Then $q_{\alpha_0} = q^{-3}$, $q_{\alpha_1} = -1$, $q_{\alpha_2} = q$, $q_{\alpha_3} = q_{\alpha_4} = q^2$. Then $\dim \mathcal{L}(\Lambda) < \infty$ if and only if the following conditions $(f1)$-$(f4)$ is satisfied.
$(f1)$ For $i \in \{0, 2, 3, 4\}$, then $\Lambda_{\alpha_i} = q_{\alpha_i}^{k_i}$ for some $k_i \in \mathbb{Z}_{\geq 0}$.

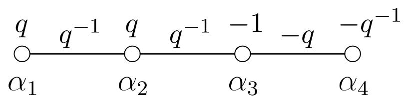
$(f2)$ $k_0 \in \{0,2,3\}$ or $k_0 \geq 6$.
$(f3)$ If $k_0 = 0$, then $\Lambda_{\alpha_j} = 1$ for $j \in \{1,2,3,4\}$.
$(f4)$ If $k_0 = 2$, then $\Lambda_{\alpha_2} = \Lambda_{\alpha_4} = 1$ and $\Lambda_{\alpha_1+\alpha_3} = q^{-3}$.
$(f4)$ If $k_0 = 3$, then $\Lambda_{\alpha_1+\alpha_2+2\alpha_4} = q^{-6}$ and $\Lambda_{\alpha_2} = q\Lambda_{\alpha_4}$.



Generalized Dynkin diagram of $F(4)$

(8) Let U be the one associated with the Nichols algebra Hec-14.

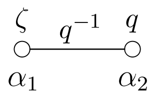
Let $\alpha_0 := \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4$. Then $q_{\alpha_0} = -q^{-1}$, $q_{\alpha_1} = q_{\alpha_2} = q$, $q_{\alpha_3} = -1$, $q_{\alpha_4} = -q^{-1}$. Then $\dim \mathcal{L}(\Lambda) < \infty$ if and only if the following conditions $(h1)$-$(h4)$ is satisfied.
$(h1)$ For $i \in \{0,1,2,4\}$, then $\Lambda_{\alpha_i} = q_{\alpha_i}^{k_i}$ for some $k_i \in \mathbb{Z}_{\geq 0}$.
$(h2)$ If $k_0 = 0$, then $\Lambda_{\alpha_j} = 1$ for $j \in \{1,2,3,4\}$.
$(h3)$ If $k_0 = 1$, then $\Lambda_{\alpha_2} = \Lambda_{\alpha_3} = 1$.
$(h4)$ If $k_0 = 1$, then $\Lambda_{\alpha_3} = 1$ or $\Lambda_{\alpha_1+\alpha_3+\alpha_4} = 1$



Generalized Dynkin diagram of Hec-14

(9) Let U be the one associated with the Nichols algebra $\mathbb{Z}/3\mathbb{Z}$-Quantum Group.

Let $\zeta \in \mathbb{C}^\times$ be such that $\zeta^2 + \zeta + 1 = 0$.
Let $\alpha_0 := 2\alpha_2 + \alpha_1$. Then $q_{\alpha_0} = \zeta q^{-1}$, $q_{\alpha_0} = q$. Then $\dim \mathcal{L}(\Lambda) < \infty$ if and only if the following conditions $(z1)$-$(z2)$ is satisfied.
$(z1)$ For $i \in \{0,2\}$, then $\Lambda_{\alpha_i} = q_{\alpha_i}^{k_i}$ for some $k_i \in \mathbb{Z}_{\geq 0}$.
$(z2)$ $k_0 \neq 1$.
$(z2)$ If $k_0 = 0$, then $\Lambda_{\alpha_j} = 1$ for $j \in \{1,2\}$.



$\zeta^2 + \zeta + 1 = 0$

Generalized Dynkin diagram of $\mathbb{Z}/3\mathbb{Z}$-Quantum Group

Theorem (Angiono-Y. 2015) Explicit construction of the universal R-matrix $\mathcal{R} \in U \otimes U$. ($\mathcal{R}\Delta(X)\mathcal{R}^{-1} = \Delta'(X)$ $(X \in U)$, $(\Delta \otimes I)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}$, $(I \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}$. $\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$.) "$\mathcal{R} = (\prod_{k=1}^{r} \exp_q(\grave{E}_k \otimes \grave{F}_k)) \exp(ht_0)$"

Shapovalov determinants (Heckenberger-Y. 2010)
Best ressult from Weyl groupoids In 2010, Heckenberger and I gave a factorization formula of the Shapovalov determinants of the Lusztig's small quantum groups at ANY root of unity, and the quantum superalgebras, the generalized quantum groups.

$$S : U^+ \otimes U^- \overset{\text{'identity'}}{\longrightarrow} U = U^- \otimes U^0 \otimes U^+ \overset{\text{'projection'}}{\longrightarrow} U^0$$

Let $S_\lambda := S_{|U^+_\lambda \otimes U^-_{-\lambda}}$.

$$\det \mathrm{Im} S_\lambda = z \cdot \prod_{\alpha \in R^+} \prod_{t_\alpha = 1}^{\infty} (-\hat{\rho}(\alpha)\chi(\alpha,\alpha)^{-t_\alpha} K_\alpha + L_\alpha)^{P_\lambda(\alpha, t_\alpha)},$$

where $\hat{\rho} = \hat{\rho}^{\chi,\Pi} : \mathbb{Z}\Pi \to \mathbb{C}^\times$ be the group homomorphism defined by $\hat{\rho}(\alpha_i) := \chi(\alpha_i, \alpha_i)$ $(i \in I)$.
Example for $\lambda = 2\alpha_1 + \alpha_2$ Denote 'projection' by Ω. Let $K_i := K_{\alpha_i}$ and $L_i := L_{\alpha_i}$. $S(E_1 \otimes F_1) = \Omega(E_1F_1) = \Omega(F_1E_1 - K_1 + L_1) = -K_1 + L_1$. $S(E_1 \otimes F_2) = \Omega(E_1F_2) = \Omega(F_2E_1) = 0$. $\Omega(E_1E_2F_1F_2) = \Omega((-K_1 + L_1)E_2F_2) = (-K_1 + L_1)(-K_2 + L_2)$. $\Omega(E_1E_2F_2F_1) = \Omega(E_1(-K_2 + L_2)F_1) = \Omega(E_1F_1(-q_{21}^{-1}K_2 + q_{12}L_2)) = (-K_1 + L_1)(-q_{21}^{-1}K_2 + q_{12}L_2)$.

$$\begin{aligned}
&\begin{vmatrix} \Omega(E_1E_2F_1F_2) & \Omega(E_1E_2F_2F_1) \\ \Omega(E_2E_1F_1F_2) & \Omega(E_2E_1F_2F_1) \end{vmatrix} \\
&= \begin{vmatrix} (-K_1 + L_1)(-K_2 + L_2) & (-K_1 + L_1)(-q_{21}^{-1}K_2 + q_{12}L_2) \\ (-K_2 + L_2)(-q_{12}^{-1}K_1 + q_{21}L_1) & (-K_2 + L_2)(-K_1 + L_1) \end{vmatrix} \\
&= (-K_1 + L_1)(-K_2 + L_2) \begin{vmatrix} -K_2 + L_2 & -q_{21}^{-1}K_2 + q_{12}L_2 \\ -q_{12}^{-1}K_1 + q_{21}L_1 & -K_1 + L_1 \end{vmatrix} \\
&= (-K_1 + L_1)(-K_2 + L_2)((1 - q_{12}^{-1}q_{21}^{-1})K_1K_2 + (1 - q_{12}q_{21})L_1L_2) \\
&= (1 - q_{12}q_{21})(-K_1 + L_1)(-K_2 + L_2)(-q_{12}^{-1}q_{21}^{-1}K_1K_2 + L_1L_2).
\end{aligned}$$

Skew center (Batra-Y. 2018)
Let $\varpi : \mathbb{Z}\Pi \to \mathbb{K}^\times$ be a $\mathbb{Z}$-module homomorphism. Define the $\mathbb{K}$-subspace $\mathfrak{Z}_\varpi = \mathfrak{Z}_\varpi(\chi, \Pi)$ of U_0 by

$$\mathfrak{Z}_\varpi = \mathfrak{Z}_\varpi(\chi, \Pi) := \{\, Z \in U_0 \,|\, \forall \alpha \in \mathbb{Z}\Pi,\, \forall X \in U_\alpha,\, ZX = \varpi(\alpha) XZ \,\},$$

where $U = \oplus_{\lambda \in \mathbb{Z}\Pi} U_\lambda$ is the $\mathbb{Z}\Pi$-grading with K_λ, $L_\mu \in U_0$, $E_i \in U_{\alpha_i}$ and $F_i \in U_{-\alpha_i}$. Let

$$\mathfrak{H}\mathfrak{C}_\varpi = \mathfrak{H}\mathfrak{C}_\varpi^{\chi,\Pi} : \mathfrak{Z}_\varpi(\chi, \Pi) \to U^0 = \oplus_{\lambda,\mu \in \mathbb{Z}\Pi} \mathbb{K} K_\lambda L_\mu.$$

be the Harish-Chandra map, which is injective.
Im$\mathfrak{H}\mathfrak{C}_\varpi$ is formed by the elements

$$\sum_{(\lambda,\mu) \in \mathbb{Z}\Pi \times \mathbb{Z}\Pi} a_{(\lambda,\mu)} K_\lambda L_\mu$$

with $a_{(\lambda,\mu)} \in \mathbb{K}$ satisfying the following equations $(e1)$-$(e4)$.

Let $q_\alpha := \chi(\alpha, \alpha)$, $\varpi^\chi_{\lambda,\mu;\beta} := \varpi(\beta) \cdot \frac{\chi(\beta,\mu)}{\chi(\lambda,\beta)}$ for β, λ, $\mu \in \mathbb{Z}\Pi$. Let

$$\kappa(q_\beta) := \begin{cases} \min\{x \in \mathbb{N} | q_\beta^x = 1\} & \text{if } \exists y \in \mathbb{N},\, q_\beta^y = 1, \\ 0 & \text{if } \forall z \in \mathbb{N},\, q_\beta^z \neq 1. \end{cases}$$

$(e1)$ For $(\lambda, \mu) \in \mathbb{Z}\Pi \times \mathbb{Z}\Pi$, $\beta \in R^+$ and $t \in \mathbb{Z} \setminus \{0\}$, if $q_\beta \neq 1$, $\kappa(q_\beta) = 0$ and $\varpi_{\lambda,\mu;\beta} = q_\beta^t$, then the equation $a_{(\lambda+t\beta,\mu-t\beta)} = \hat{\rho}(\beta)^t \cdot a_{(\lambda,\mu)}$ holds.

$(e2)$ For $(\lambda, \mu) \in \mathbb{Z}\Pi \times \mathbb{Z}\Pi$, if there exists $\beta \in R^+$ satisfying the condition that $\kappa(q_\beta) = 0$ and $\varpi_{\lambda,\mu;\beta} \neq q_\beta^t$ for all $t \in \mathbb{Z}$, then the equation $a_{(\lambda,\mu)} = 0$ holds.
$(e3)$ For $(\lambda, \mu) \in \mathbb{Z}\Pi \times \mathbb{Z}\Pi$, $\beta \in R^+$ and $1 \leq t \leq \kappa(q_\beta) - 1$, if $\kappa(q_\beta) \geq 2$ and $\varpi_{\lambda,\mu;\beta} = q_\beta^t$, then the equation

$$\sum_{x=-\infty}^{+\infty} a_{(\lambda+(\kappa(q_\beta)x+t)\beta,\mu-(\kappa(q_\beta)x+t)\beta)} \hat{\rho}(\beta)^{-(\kappa(q_\beta)x+t)} = \sum_{y=-\infty}^{+\infty} a_{(\lambda+\kappa(q_\beta)y\beta,\mu-\kappa(q_\beta)y\beta)} \hat{\rho}(\beta)^{-\kappa(q_\beta)y}$$

holds.

$(e4)$ For $(\lambda,\mu)\in\mathbb{Z}\Pi\times\mathbb{Z}\Pi$ and $\beta\in R^+$, if $\kappa(q_\beta)\geq 2$ and $\varpi_{\lambda,\mu;\beta}\neq q_\beta^m$ for all $0\leq m\leq\kappa(q_\beta)-1$, then the $\kappa(q_\beta)-1$ equations

$$\begin{aligned}&\sum_{x=-\infty}^{+\infty} a_{(\lambda+(\kappa(q_\beta)x+t)\beta,\mu-(\kappa(q_\beta)x+t)\beta)}\hat{\rho}(\beta)^{-(\kappa(q_\beta)x+t)}\\ &\quad=\sum_{y=-\infty}^{+\infty} a_{(\lambda+\kappa(q_\beta)y\beta,\mu-\kappa(q_\beta)y\beta)}\hat{\rho}(\beta)^{-\kappa(q_\beta)y}\quad(1\leq t\leq\kappa(q_\beta)-1)\end{aligned}$$

hold.
For $(\lambda,\mu)\in\mathbb{Z}\Pi\times\mathbb{Z}\Pi$, define $\Lambda^\chi_{\lambda,\mu;\varpi}\in\mathrm{Hom}_{\mathbb{K}-\mathrm{alg}}(U^0,\mathbb{K})$ by

$$\Lambda^\chi_{\lambda,\mu;\varpi}(K_{\lambda'}L_{\mu'}):=\chi(\lambda,\mu')\chi(\alpha,\mu)\varpi(\lambda')\quad(\lambda',\mu'\in\mathbb{Z}\Pi).$$

Let $\mathrm{Fin}^\chi_\varpi:=\{(\lambda,\mu)\in\mathbb{Z}\Pi\times\mathbb{Z}\Pi|\dim\mathcal{L}(\Lambda^\chi_{\lambda,\mu;\varpi})<\infty\}$. For $(\lambda,\mu)\in\mathrm{Fin}^\chi_\varpi$, there exists a unique $Z^\chi_{\lambda,tmu;\varpi}\in\mathfrak{Z}_\varpi(\chi,\Pi)$ such that

$$\mathfrak{H}\mathfrak{C}_\varpi(Z^\chi_{\lambda,tmu;\varpi})=\sum_{\nu\in\mathbb{Z}_{\geq 0}\Pi}\hat{\rho}^{\chi,\Pi}(\nu)\breve{m}_\alpha K_{\lambda+\nu}L_{\mu-\nu},$$

where $\breve{m}_\nu:=\dim\mathcal{L}(\Lambda^\chi_{\lambda,\mu;\varpi})_{-\nu}$.
Conjecture: (Batra-Y. 2019) The elements $Z^\chi_{\lambda,tmu;\varpi}\in\mathfrak{Z}_\varpi(\chi,\Pi)$ with $(\lambda,\mu)\in\mathrm{Fin}^\chi_\varpi$ form a $\mathbb{K}$-basis of $\mathfrak{Z}_\varpi(\chi,\Pi)$.
This fits into:
A. Sergeev, A. Veselov, Grothendieck rings of basic classical Lie superalgebras, Ann. Math. 173 (2011), 663-703.
The conjecture might be proved via Radford-Schneider theory for the correspondece between bicharacters and irreducible modules of the generalized quantum groups.

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