$\tilde{A}$ and $\tilde{D}$ type cluster algebras: Triangulated surfaces, friezes and the cluster category

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By viewing $\tilde{A}$ and $\tilde{D}$ type cluster algebras as triangulated surfaces, we find all cluster variables in terms of either (i) the frieze pattern (or bipartite belt) or (ii) the periodic quantities previously found for the cluster map associated with these frieze patterns. We show that these cluster variables form friezes which are precisely the ones found in [1] by applying the cluster character to the associated cluster category.

This is a summary of the results of arXiv:2105.11682, as presented at a RIMS workshop in October 2021.

1 Introduction

The main results of [8] we wish to document here are that the cluster variables of $\tilde{A}_{q,p}$ type are given by the union of three sets

$$\{x_n \mid n \in \mathbb{Z}\} \cup \left\{ D^l(J_{jp}) \,\middle|\, \begin{matrix} j = 0, \dots, q-1 \\ l = 1, \dots, q-1 \end{matrix} \right\} \cup \left\{ D^l(\tilde{J}_{jq}) \,\middle|\, \begin{matrix} j = 0, \dots, p-1 \\ l = 1, \dots, p-1 \end{matrix} \right\}$$

each of which forms a frieze living on a cylinder. The x_n satisfy the recurrence

$$x_{n+p+q} x_n = x_{n+p} x_{n+q} + 1$$

which has periodic quantities [6, 5]

$$J_n = \frac{x_{n+2p} + x_n}{x_{n+p}}, \qquad \tilde{J}_n = \frac{x_{n+2q} + x_n}{x_{n+q}} \tag{1}$$

with period q and p respectively. The $D^l(J_{jp})$ and $D^l(\tilde{J}_{jq})$ are determinant functions of the J_n and $\tilde{J}_n$, respectively.

We also have that the $\tilde{D}_N$ cluster algebras have cluster variables

$$\left\{ X_n^i \,\middle|\, \begin{array}{l} i = 1, \ldots, N+1 \\ n \in \mathbb{Z} \end{array} \right\} \cup \left\{ D_1^l(J'_j) \,\middle|\, \begin{array}{l} j = 0, \ldots, N-3 \\ l = 1, \ldots, N-3 \end{array} \right\} \cup \Gamma_{\text{except}}$$

where the X_n^i satisfy a system of $N+1$ recurrences with a periodic quantity J'_n and the $D_1^l(J'_j)$ are determinant functions of the J'_n. The first set forms a frieze of the form $\tilde{D}_N \times \mathbb{Z}$, the second forming a frieze on a cylinder and the third set is a set of three exceptional cluster variables.

2 Quiver mutation and cluster mutation

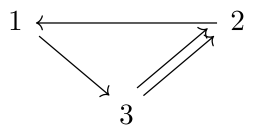
Let Q be a quiver without loops or 2 cycles. Quiver mutation (Fomin-Zelevinsky, [4]) at a vertex k, written μ_k, is defined as follows:

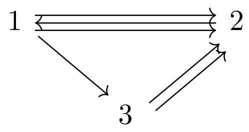
(i) For each length 2 path $i \to k \to j$ add a new arrow $i \to j$.

(ii) Reverse each arrow touching vertex k.

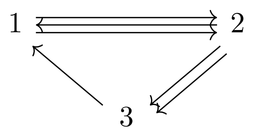
(iii) Delete all 2 cycles that have appeared.

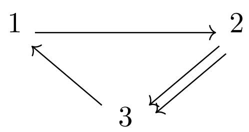
Example 2.1. *We mutate at vertex* 3 *of the following quiver on the left. The first step gives the right quiver*





and the second and third steps give the following left and right quivers, respectively.

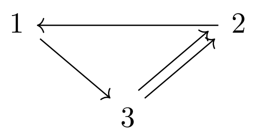




We consider x_i (the initial cluster variables) to be attached at each vertex i. Mutation at k will fix all variables x_i with $i \neq k$ but x_k becomes x'_k, a new cluster variable at k, defined by

$$x'_k = \frac{1}{x_k} \left(\prod_{j \to k} x_j + \prod_{j \leftarrow k} x_j \right).$$

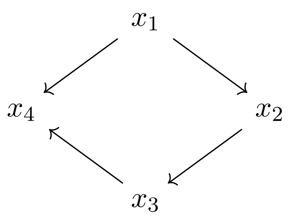
Example 2.2. *In our example above we mutated the following quiver at vertex* 3*, resulting in the new cluster variable* x'_3*:*



$$x'_3 = \frac{1}{x_3}(x_1 + x_2^2)$$

3 Dynamical systems from cluster mutation

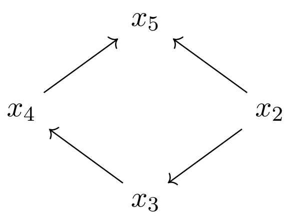
We do not give the general definition of how we can obtain a dynamical system from any acyclic quiver. Instead we work with an example: an orientation of an affine A type quiver, $\tilde{A}_{3,1}$



named since it has 3 clockwise arrows and 1 anticlockwise. Performing μ_1 gives a new cluster variable, let's call it x_5, defined by

$$x_5 x_1 = x_4 x_2 + 1.$$

and a new quiver:



Now μ_2 gives a new cluster variable, let's call it x_6, defined by

$$x_6 x_2 = x_5 x_3 + 1.$$

We can see that this formula is the same as the previous one, except the subscripts have increased by 1. We continue to mutate clockwise around the quiver, we define the resulting cluster variables as a sequence (x_n). since the mutation formula is the same each time we find the recurrence

$$x_{n+4} x_n = x_{n+3} x_{n+1} + 1$$

satisfied by the cluster variables x_n. More generally $\tilde{A}_{q,p}$ quivers are $\tilde{A}$ diagrams with q arrows pointing clockwise and p arrows pointing anticlockwise. As for $\tilde{A}_{3,1}$ they give the recurrences [6, 5]

$$x_{n+p+q} x_n = x_{n+q} x_{n+p} + 1. \tag{2}$$

4 Friezes

Friezes were defined by Coxeter [2], they were originally arrangements of integers in the plane like

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & 1 & & 1 & & 4 & & \dots & \\
\dots & & 2 & & 3 & & 15 & & \\
 & 5 & & 5 & & 11 & & \dots & \\
\dots & & 12 & & 18 & & 8 & & \\
 & 7 & & 43 & & 13 & & \dots & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

satisfying $\beta\gamma - \alpha\delta = 1$ for each diamond $\begin{array}{ccc} & \alpha & \\ \beta & & \gamma \\ & \delta & \end{array}$ and bounded above and below by a row of ones and then a row of zeroes. Due to (2) we have frieze diamonds

$$
\begin{array}{ccc}
 & x_{n+p} & \\
x_n & & x_{n+p+q} \\
 & x_{n+q} &
\end{array}
$$

for all $n \in \mathbb{Z}$. We can join many of these diamonds to form a frieze

$$
\begin{array}{ccccccccc}
 & & & \vdots & & \vdots & & & \\
 & x_{-2q} & & & x_{p-q} & & x_{2p} & & \\
\dots & & & x_{-q} & & x_p & & & \dots \\
 & x_{-q-p} & & & x_0 & & x_{p+q} & & \\
\dots & & & x_{-p} & & x_q & & & \dots \\
 & x_{-2p} & & & x_{q-p} & & x_{2q} & & \\
 & & & \vdots & & \vdots & & &
\end{array}
\tag{3}
$$

where we have set $n = 0$ for readability, as this picture is fixed by shifts in n. We note that starting at x_n and moving q steps north-east increases the subscript by q times p, so we land at x_{n+pq}. Conversely if we start at x_n and move p steps south-east we will also arrive at x_{n+pq}. By identifying the duplicate values that appear on (3) we obtain a frieze on a cylinder.

5 Dynamical systems (again)

In order to study the recurrences (2) more we use the following determinant theorem.

Theorem 5.1 (Desanot-Jacobi)**.** *Let M be an $n \times n$ matrix. Then*

$$
|M| = \frac{|NW||SE| - |NE||SW|}{|C|} \tag{4}
$$

where, for example, NW denotes the connected $(n-1)\times(n-1)$ matrix in the upper left (north-west) of M and C is the $(n-2)\times(n-2)$ matrix in the centre of M.

Looking again at (2) we see that the following determinant is zero

$$\begin{vmatrix} x_n & x_{n+p} & x_{n+2p} \\ x_{n+q} & x_{n+p+q} & x_{n+2p+q} \\ x_{n+2q} & x_{n+p+2q} & x_{n+2p+2q} \end{vmatrix} = 0$$

because each of NW, SE, NE and SW are of the form

$$\begin{vmatrix} x_n & x_{n+p} \\ x_{n+q} & x_{n+p+q} \end{vmatrix} = 1$$

up to a shift in n. Taking a kernel vector we have (after scaling)

$$\begin{pmatrix} x_n & x_{n+p} & x_{n+2p} \\ x_{n+q} & x_{n+p+q} & x_{n+2p+q} \\ x_{n+2q} & x_{n+p+2q} & x_{n+2p+2q} \end{pmatrix} \begin{pmatrix} 1 \\ -J_n \\ 1 \end{pmatrix} = 0$$

Here

$$J_n = \frac{x_{n+2p} + x_n}{x_{n+p}} \tag{5}$$

is period q. The kernel vector on the other side gives period p

$$\tilde{J}_n = \frac{x_{n+2q} + x_n}{x_{n+q}}$$

These immediately give linear relations between the cluster variables x_n:

$$x_{n+2p} - J_n x_{n+p} + x_n = 0, \qquad x_{n+2q} - \tilde{J}_n x_{n+q} + x_n = 0$$

with periodic coefficients. In [5] these are exploited to give the constant coefficient linear relation

$$x_{n+2pq} - \mathcal{K} x_{n+pq} + x_n = 0$$

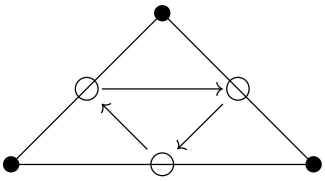
At this point we can pause to ask two questions. We took a specific sequence of mutations to generate the cluster variables x_n. What variables lie outside of this sequence? Are the J_n and $\tilde{J}_n$ cluster variables? This is the starting point for the preprint [8] that this summary is based on. To answer these questions we utilize a different point of view.

6 Triangulated surfaces and cluster algebras

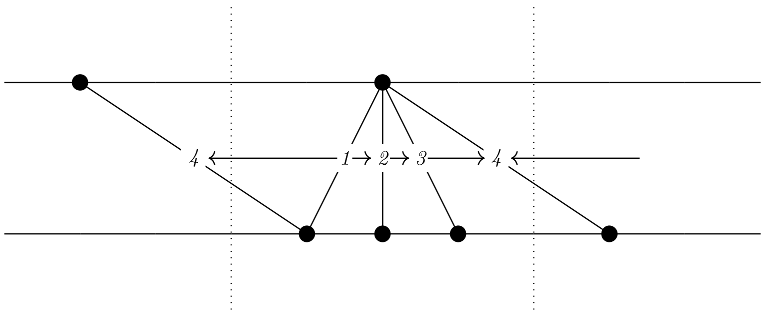
Following [3] we consider triangulations of surfaces with marked points on boundary components. Each triangulation gives rise to a quiver as follows:

(i) Attach a vertex to every (non-boundary) arc.

(ii) For every pair of vertices i and j that are part of the same triangle we draw an arrow $i \mapsto j$ if, while travelling clockwise around the triangle, j comes directly after i. If no arcs lie on a boundary then the situation looks as follows:



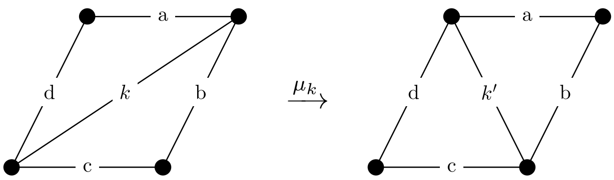
Example 6.1. *For the following triangulation of an annulus with* 3 *and* 1 *marked point(s) on each boundary component, we find the* $\tilde{A}_{3,1}$ *quiver.*



(6)

The annulus is given by glueing along the dotted lines, the two copies of "4" *should be identified.*

We glue cluster variables x_i to each arc i and assign boundary arcs the value 1. In this picture the analogue of mutation μ_k is given by flips:

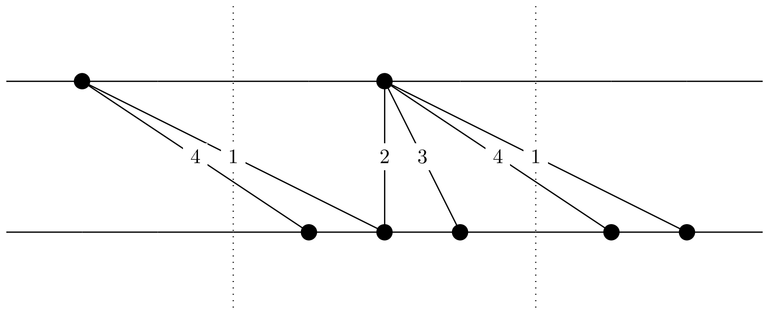


with a new cluster variable satisfying the Ptolemy relation

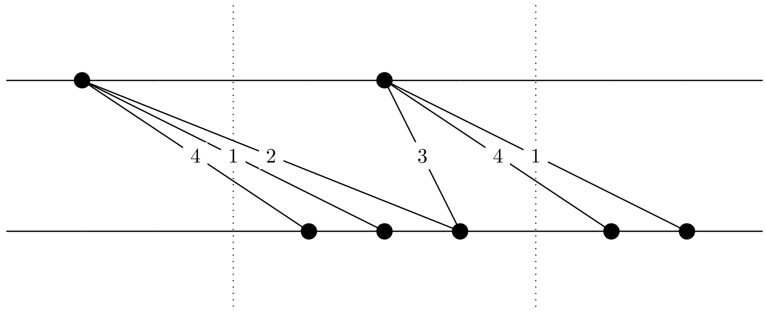
$$x_{k'}x_k = x_a x_c + x_b x_d. \tag{7}$$

If we can construct an initial quiver from a surface triangulation, then the arcs obtained by sequences of flips (this gives all possible arcs) are in bijection with the cluster variables obtained by sequences of mutations (all cluster variables in the cluster algebra).

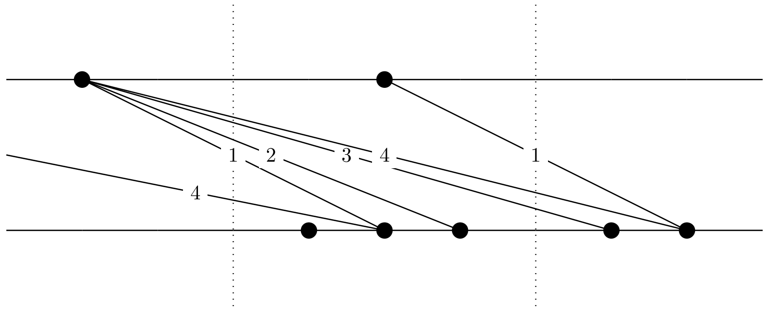
The above example giving the $\tilde{A}_{3,1}$ quiver can be generalised to an annulus with q and p marked points on either boundary, which gives $\tilde{A}_{q,p}$. With this in mind, if we wish to solve our earlier questions for $\tilde{A}_{q,p}$ type quivers, we need to identify the x_n as arcs, and see what other arcs are possible. To avoid cumbersome diagrams we just demonstrate with the $\tilde{A}_{3,1}$ quiver (6). The effect of μ_1 is:



Next μ_2 gives:



Finally we show the composition $\mu_4\mu_3$:



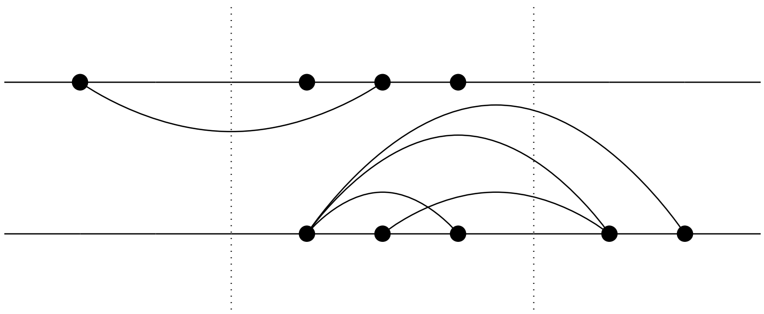
We define this composition as $\mu := \mu_4\mu_3\mu_2\mu_1$. As can be observed in these pictures the effect of μ on the initial quiver is:

(i) Move the top of each arc one node to the left

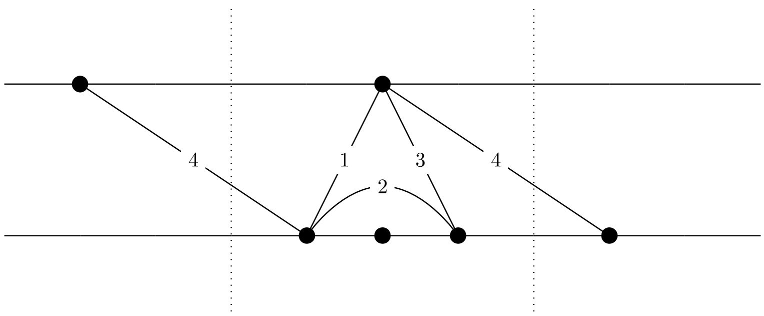
(ii) Move the bottom of each arc one node to the right

Recall that the sequence of cluster variables x_n (2) is given by applying μ^n for $n \in \mathbb{Z}$. In our annulus picture μ^n will give all arcs connecting the top boundary to the bottom, so these arcs correspond to the x_n. We should now ask which possible arcs are not obtained by μ^n, i.e the cluster variables that lie outside of the x_n.

Of course, the remaining possibilities are the "curved" arcs connecting one boundary component to itself, for example these:



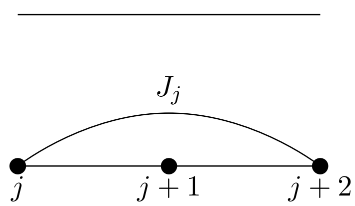
If we return to our initial quiver (6) and instead mutate at 2 we obtain one of these curved arcs:



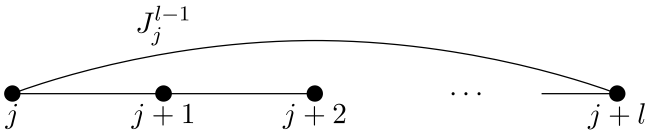
and by the Ptolemy relation (7) this new arc is given by

$$\frac{x_3 + x_1}{x_2}$$

which is J_1 (see (5)). The other J_n can be obtained similarly they are the arcs that connect the bottom boundary to itself and "jump" one marked point. The $\tilde{J}_n$ look the same but on the top boundary (note that in our $\tilde{A}_{3,1}$ example we have no curved arcs on the top boundary, as there aren't enough marked points). We label the bottom boundary such that these arcs, J_j for $j = 1, 2, \ldots, q$, look like



The remaining curved arcs are those that jump more that one marked point, which we call J_j^{l-1} (starting at j and jumping $l-1$ points):



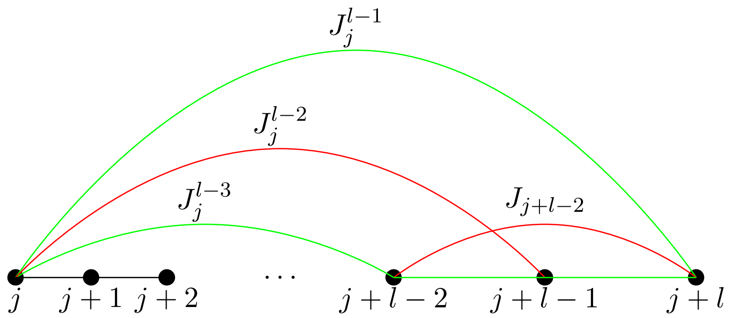
We can calculate these in terms of the J_n by the following result.

Lemma 6.2. *The arcs J_j^l satisfy the recurrence relation*

$$J_j^{l-1} = J_{j+l-2} J_j^{l-2} - J_j^{l-3} \tag{8}$$

for $l = 3, 4, \ldots, q$, with initial values $J_j^0 = 1$ and $J_j^1 = J_j$.

Proof. We apply the Ptolemy relation to the green quadrilateral with red diagonals J_j^{l-2} and J_{j+l-2}



□

We also note a determinant formula for J_j^l:

$$J_j^l = D^l(J_j) := \begin{vmatrix} J_j & 1 & 0 & & & \\ 1 & J_{j+1} & 1 & 0 & & \\ 0 & 1 & J_{j+2} & 1 & 0 & \\ & 0 & 1 & J_{j+3} & \ddots & \\ & & 0 & \ddots & \ddots & 1 \\ & & & & 1 & J_{j+l-1} \end{vmatrix} \tag{9}$$

since, by expanding along the last row, we obtain the same recurrence (8).

As noted above every arc either connects the two boundaries (one of x_n) or connects one boundary to itself (a $D^l(J_j)$ or $D^l(\tilde{J}_j)$) so we how have a description of every cluster variable, as follows.

Theorem 6.3. *[8] The cluster variables for $\tilde{A}_{q,p}$ cluster algebras are*

$$\{x_n \mid n \in \mathbb{Z}\} \cup \left\{ D^l(J_{jp}) \,\middle|\, \begin{array}{l} j = 0, \dots, q-1 \\ l = 1, \dots, q-1 \end{array} \right\} \cup \left\{ D^l(\tilde{J}_{jq}) \,\middle|\, \begin{array}{l} j = 0, \dots, p-1 \\ l = 1, \dots, p-1 \end{array} \right\}$$

where the determinants $D^l(J_{jp})$ and $D^l(\tilde{J}_{jq})$ are as in (9).

7 Friezes (again)

We have seen previously that the cluster variables x_n form a frieze on a cylinder. What about the other cluster variables? We apply the Desanot-Jacobi identity (4) to the determinants (9) to see that

$$D^l(J_j)D^{l-2}(J_{j+1}) = D^{l-1}(J_j)D^{l-1}(J_{j+1}) - 1$$

giving frieze diamonds of the form

$$\begin{array}{ccc} & D^{l-2}(J_{j+1}) & \\ D^{l-1}(J_j) & & D^{l-1}(J_{j+1}) \\ & D^l(J_j) & \end{array}$$

which we can glue together to form the frieze

$$\begin{array}{cccccccccc} & 1 & & & 1 & & & 1 & & \\ \cdots & & J_1 & & & J_2 & & & J_3 & \\ & D^2(J_0) & & & D^2(J_1) & & & D^2(J_2) & & \cdots \\ & & D^3(J_0) & & & D^3(J_1) & & & D^3(J_2) & \\ & \vdots & & & \vdots & & & \vdots & & \\ & \vdots & & & \vdots & & & \vdots & & \\ \cdots & & D^{q-1}(J_{-1}) & & & D^{q-1}(J_0) & & & D^{q-1}(J_1) & \end{array}$$

which is finite in the vertical direction and periodic in the horizontal, so again we can consider it as lying on a cylinder.

8 Remarks

The cluster frieze was defined by Assem-Dupont [1] and is a structure on the cluster category. For $\tilde{A}$ type this structure agrees with the frieze structure we have found. There the injective modules correspond to x_n for $n < 0$, the projective modules correspond to x_n for $n > N$ and the shifted projective modules $P_i[1]$ correspond to $x_0, \dots, x_N$. The remainder, the regular modules, correspond to the $D^l(J_{jp})$ and $D^l(\tilde{J}_{jq})$.

For $\tilde{D}_N$ type instead of the single recurrence (2) we have a system of $N+1$ recurrences in the cluster variables X_n^i where i runs over the vertices and $n \in \mathbb{Z}$. Nonetheless a periodic quantity J'_n was obtained [7] for this system. The result given in [8] gives a similar structure for the cluster variables.

Theorem 8.1. *The $\tilde{D}_N$ cluster algebras have cluster variables*

$$\left\{ X_n^i \,\middle|\, \begin{array}{l} i = 1, \dots, N+1 \\ n \in \mathbb{Z} \end{array} \right\} \cup \left\{ D_1^l(J_j') \,\middle|\, \begin{array}{l} j = 0, \dots, N-3 \\ l = 1, \dots, N-3 \end{array} \right\} \cup \Gamma_{\text{except}}$$

with the first set forming a frieze of the form $\tilde{D}_N \times \mathbb{Z}$, the second forming a frieze of the form (7) *and the third set is a set of three exceptional cluster variables.*

The two friezes here again match the cluster frieze of [1].

Acknowledgements

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