ON THE ALGEBRAICITY OF CRITICAL L VALUES ATTACHED TO VECTOR VALUED SIEGEL CUSP FORMS

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1. Introduction

This paper is a summary of recent results of the author and collaborators [HPSS21, Hor22] on the algebraicity of critical L values attached to Siegel cusp forms. In this paper, we first review Shimura's result regarding the algebraicity of critical L values. Next, we discuss a generalization of it. The key tools of the proof of algebraicity are the pullback formula of Siegel Eisenstein series, the arithmeticity of Eisenstein series and the arithmeticity of certain operators.

2. Preliminaries

2.1. **Definitions.** Let $\mathfrak{H}_n$ be the Siegel upper half space of degree n. The symplectic group $G_n(\mathbb{R}) = \mathrm{Sp}_{2n}(\mathbb{R})$ acts on $\mathfrak{H}_n$ transitively. We denote by $K_{n,\infty} \cong \mathrm{U}(n)$ the stabilizer of $\mathbf{i}_n = \sqrt{-1}\,\mathbf{1}_n \in \mathfrak{H}_n$. Let (ρ, V) be a finite-dimensional representation of $\mathrm{GL}_n(\mathbb{C})$. Set

$$j(g,z) = cz + d, \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_n(\mathbb{R}), z \in \mathfrak{H}_n$$

and $j_\rho = \rho \circ j$.

Let Γ be a congruence subgroup of $G_n(\mathbb{Q}) = \mathrm{Sp}_{2n}(\mathbb{Q})$; i.e., Γ contains $\Gamma(N) = \{\gamma \in G_n(\mathbb{Z}) \mid \gamma \equiv \mathbf{1}_{2n} \bmod N\}$ as a finite index subgroup for some $N \in \mathbb{Z}$. For a V-valued C^∞-function f on $\mathfrak{H}_n$, we say that f is a C^∞-modular form of weight ρ with respect to Γ if f satisfies the following two conditions:

- f is slowly increasing.
- $f|_\rho\gamma = f$ for any $\gamma \in \Gamma$. Here, $f|_\rho\gamma(z) = j_\rho(\gamma, z)^{-1} f(\gamma(z))$.

Let $\mathfrak{g}_n = \mathrm{Lie}(G_n(\mathbb{R})) \otimes_\mathbb{R} \mathbb{C}$. We then have the well-known decomposition:

$$\mathfrak{g}_n = \mathfrak{k}_n \oplus \mathfrak{p}_{n,+} \oplus \mathfrak{p}_{n,-}, \qquad \mathfrak{k}_n = \mathrm{Lie}(K_{n,\infty}) \otimes_\mathbb{R} \mathbb{C}.$$

Fix a Cartan subalgebra of $\mathfrak{k}_n$ as usual. It is also a Cartan subalgebra of $\mathfrak{g}_n$. We fix a positive root system by

$$\{e_i - e_j \mid 1 \le i < j \le n\} \cup \{-(e_i + e_j) \mid 1 \le i \le j \le n\}. \tag{2.1.1}$$

Then, the subalgebra $\mathfrak{p} = \mathfrak{k}_n \oplus \mathfrak{p}_{n,-}$ contains the Borel subalgebra. By the choice of positive root system (2.1.1), a holomorphic discrete series representation is a highest weight representation.

For a V-valued C^∞-function f on $\mathfrak{H}_n$, we say that f is nearly holomorphic if $f^\rho(g) = f|_\rho g(\mathbf{i}_n)$ is $\mathfrak{p}_{n,-}$-finite under the right translation. Note that f is holomorphic if and only if $\mathfrak{p}_{n,-} \cdot f^\rho = 0$. If a C^∞-modular form f is nearly holomorphic, we call f a nearly holomorphic modular form. We denote by $N_\rho(\Gamma)$ (resp. $M_\rho(\Gamma)$) the space of nearly holomorphic modular forms (resp. holomorphic modular forms) of weight ρ with respect to Γ.

2.2. **Arithmeticity.** For the details of the ideas represented in this subsection, see [HPSS21]. Let (ρ, V) be a finite-dimensional representation of $\mathrm{GL}_n(\mathbb{C})$. We fix a $\mathbb{Q}$-rational structure $V_\mathbb{Q}$ of V; i.e., $V_\mathbb{Q}$ is a finite dimensional vector subspace of V over $\mathbb{Q}$ with $V_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{C} = V$ and $V_\mathbb{Q}$ is stable under $\mathfrak{k}_{n,\mathbb{Q}}$. For $\sigma \in \mathrm{Aut}(\mathbb{Q})$ and $v \in V$, we define $\sigma(v) \in V$ by $\sigma(v) = \sum_i v_i \otimes \sigma(c_i)$ where $v = \sum_i v_i \otimes c_i \in V_\mathbb{Q} \otimes \mathbb{C}$. For a holomorphic modular form f on $\mathfrak{H}_n$, we have a Fourier expansion,

$$f(z) = \sum_{h \in \mathrm{Sym}_n(\mathbb{Q})} c_h \exp(2\pi\sqrt{-1}\mathrm{tr}(hz)), \qquad c_h \in V.$$

Then, set

$${}^\sigma f(z) = \sum_{h \in \mathrm{Sym}_n(\mathbb{Q})} \sigma(c_h) \exp(2\pi\sqrt{-1}\mathrm{tr}(hz)).$$

Similarly, Shimura introduced the action of σ on nearly holomorphic modular forms.

Theorem 2.1 (Shimura)**.** Let f be a nearly holomorphic modular form. Then, f^σ is a nearly holomorphic modular form.

In the following, we reduce the algebraicity of critical L values to the arithmeticity of nearly holomorphic modular forms.

3. Critical values

In this section, we recall the results related to the algebraicity of critical values up to an explicit period.

3.1. **Statement for critical values.** For a Hecke eigen Siegel cusp form f and a Hecke character μ of $\mathbb{Q}^\times \mathbb{R}_+^\times \backslash \mathbb{A}_\mathbb{Q}^\times$, let $L(s, f, \mu)$ be the standard L function twisted by μ associated to f. Suppose that f is of weight ρ_λ, an irreducible finite dimensional representation of $\mathrm{GL}_n(\mathbb{C})$ with highest weight $\lambda = (\lambda_1, \ldots, \lambda_n)$. If $\lambda_1 = \cdots = \lambda_n$, we call f a scalar valued modular form. Take $\varepsilon \in \{0, 1\}$ so that $\mu_\infty = \mathrm{sgn}^\varepsilon$. Then, the critical points of $L(s, f, \mu)$ in the right half plane is the following set:

$$\{1 \leq m \leq \lambda_n - n \mid n + \varepsilon \equiv m \bmod 2\}.$$

Shimura established the algebraicity of critical values attached to scalar valued Siegel modular forms as follows:

Theorem 3.1 (Shimura, et. al)**.** Let k be a weight, f a holomorphic Hecke eigen Siegel cusp form of weight k, μ a Hecke character of $\mathbb{Q}^\times \mathbb{R}_+^\times \backslash \mathbb{A}_\mathbb{Q}^\times$, and $m \geq 1$ a critical point. Suppose

- every Fourier coefficient of f lies in a CM field.
- $k > 3n/2 + 1$.
- $m \neq 1$ if $\mu^2 = 1$.

We then have

$$\sigma\left(\frac{L(m, f, \mu)}{i^k \pi^{nk+(n+1)m} G(\mu)^{n+1} \langle f, f \rangle}\right) = \frac{L(m, {}^\sigma f, {}^\sigma \mu)}{i^k \pi^{nk+(n+1)m} G({}^\sigma \mu)^{n+1} \langle {}^\sigma f, {}^\sigma f \rangle}$$

for any $\sigma \in \mathrm{Aut}(\mathbb{C})$. Here, $G(\mu)$ is the Gauss sum of μ and $\langle f, f \rangle$ is the Petersson inner product of f normalized by $\mathrm{vol}(G_n(\mathbb{Z}) \backslash \mathfrak{H}_n) = 1$.

Remark 3.2. (1) By the functional equation, we can prove a similar result for critical values in the left half plane.
(2) The main theorem of [HPSS21] is a generalization of algebraicity to the vector valued case.
(3) The third condition "$m \neq 1$ if $\mu^2 = 1$" is essential. Indeed, in the proof, the algebraicity of the Eisenstein series is essential. The corresponding Eisenstein series is not nearly holomorphic. The algebraicity of such an Eisenstein series is unknown.

3.2. **Sketch of proof.** We give readers an idea of the proof. The key ingredients are

- the arithmeticity of Eisenstein series.
- the doubling method.
- holomorphic projection.

We can construct Shimura's Siegel Eisenstein series from the datum (k, N, μ). More precisely, from (k, N, μ), we can construct a special kind of standard section. Here, k is a weight, N is a level and μ is a Hecke character. We denote the corresponding Eisenstein series by $E_k(z, s, \mu, N)$.

Proposition 3.3 (Arithmeticity of Eisenstein series)**.** With the above notation, let m_0 be a real number such that $0 \leq m_0 \leq k-(n+1)/2$ and $k-(n+1)/2-m_0 \in 2\mathbb{Z}$. Suppose $\mu_\infty = \mathrm{sgn}^k$, and $\mu \neq 1/2$ if $\mu^2 = 1$. We then have

$$^\sigma\left(\pi^{-c_0} E_k(z, m_0, \mu, N)\right) = \pi^{-c_0} E_k(z, m_0, {}^\sigma\mu, N)$$

with an explicit constant c_0 depending only on m_0 and n.

Proof. See [PSS18, Proposition 6.8]. □

For simplicity of notation, we will denote the Eisenstein series by $E(z, s, \mu)$. By using the doubling method, we have

$$\langle E\left(\left(\begin{smallmatrix} \cdot & \\ & z \end{smallmatrix}\right), s, \mu\right), f\rangle \approx L(s+1/2, f, \mu)({}^c f)(z). \tag{3.2.1}$$

Here, c is the complex conjugate. By (3.2.1), we thus have

$$\begin{aligned} L(m, {}^\sigma f, {}^\sigma\mu)({}^{c\circ\sigma} f)(z) &\approx \langle E\left(\left(\begin{smallmatrix} \cdot & \\ & z \end{smallmatrix}\right), m-1/2, {}^\sigma\mu\right), {}^\sigma f\rangle \\ &\approx \langle {}^\sigma E\left(\left(\begin{smallmatrix} \cdot & \\ & z \end{smallmatrix}\right), m-1/2, \mu\right), {}^\sigma f\rangle \end{aligned} \tag{3.2.2}$$

We restrict the Siegel Eisenstein series $E(z, m-1/2, \mu)$ on $\mathfrak{H}_{2n}$ to $\mathfrak{H}_n \times \mathfrak{H}_n$. Let $E(z_1, z_2, m-1/2, \mu)$ be the restriction. Let $\{f_1, \ldots, f_\ell\}$ be an orthogonal basis of cusp forms of degree n of weight k with respect to $\Gamma(N)$ and $\{h_1, \ldots, h_q\}$ a basis of orthogonal complements of cusp forms. We may assume that $f_1, \ldots, f_\ell$ are rational over some number fields and Hecke eigenforms. We then have

$$E(z_1, z_2, m-1/2, \mu) = \sum_i c_i f_i(z_1)({}^c f_i)(z_2) + \sum_{\alpha,\beta} d_{\alpha,\beta} h_\alpha(z_1) h_\beta(z_2).$$

By (3.2.1), c_i can be expressed in terms of L-values. More precisely, the coefficients c_i are equal to a constant multiple of $L(m, f, \mu)/\langle f, f\rangle$.

Assume $f_1 = f$. Then, by (3.2.2), one obtains

$$\begin{aligned} &L(m, {}^\sigma f, {}^\sigma\mu)({}^{c\circ\sigma} f)(z_2) \approx \langle {}^\sigma E_k(\,\cdot\,, z_2, m-1/2, \mu), {}^\sigma f\rangle \\ &= \sum_i \sigma(c_i)\langle {}^\sigma f_i, {}^\sigma f\rangle({}^{\sigma\circ c} f_i)(z_2) + \sum_{\alpha,\beta} \sigma(d_{\alpha,\beta})\langle {}^\sigma h_\alpha, {}^\sigma f\rangle({}^\sigma h_\beta)(z_2). \end{aligned}$$

Since the Fourier coefficients lie in CM fields, we have ${}^{\sigma\circ c}f_i = {}^{c\circ\sigma} f_i$. Moreover, we have

$$L(m, {}^\sigma f, {}^\sigma\mu)({}^{\sigma\circ c}f)(z_2) \approx \sum_i \sigma(c_i)\langle {}^\sigma f_i, {}^\sigma f\rangle({}^{\sigma\circ c}f_i)(z_2) + \sum_{\alpha,\beta} \sigma(d_{\alpha,\beta})\langle {}^\sigma h_\alpha, {}^\sigma f\rangle({}^\sigma h_\beta)(z_2).$$

To simplify the RHS, Shimura introduced the holomorphic projection $\mathfrak{A}$. The projection $\mathfrak{A}$ defines a projection onto the space of holomorphic cusp forms. Hence, we may assume that f_i and h_j are holomorphic. Shimura then proved that ${}^\sigma h_j$ is orthogonal to cusp forms and

$$\langle {}^\sigma f_1, {}^\sigma f_i\rangle = 0 \qquad \text{if } i \neq 1.$$

We conclude that

$$L(m, {}^\sigma f, {}^\sigma\mu)({}^{\sigma\circ c}f)(z_2) \approx \sigma(c_1)\langle {}^\sigma f, {}^\sigma f\rangle({}^{\sigma\circ c}f_i)(z_2).$$

and

$$L(m, {}^\sigma f, {}^\sigma\mu) \approx \sigma(c_1)\langle {}^\sigma f, {}^\sigma f\rangle.$$

3.3. **Generalization to vector valued cases.** In [HPSS21], we prove the following:

Theorem 3.4 (H.-Pitale-Saha-Schmidt)**.** Let f be a holomorphic Hecke eigen Siegel cusp form of weight ρ_λ with $\lambda = (\lambda_1, \ldots, \lambda_n)$. Take a critical point m of $L(s, f, \mu)$ with $m \geq 1$. Suppose

- $\lambda_1 \equiv \cdots \equiv \lambda_n \pmod 2$.
- $\lambda_1 \equiv \varepsilon \pmod 2$.
- $m \neq 1$ if $\mu^2 = 1$.

We then have

$$\sigma\left(\frac{L(m, f, \mu)}{i^k \pi^{\sum_{j=1}^n \lambda_j + (n+1)m} G(\mu)^{n+1}\langle f, f\rangle}\right) = \frac{L(m, {}^\sigma f, {}^\sigma\mu)}{i^k \pi^{\sum_{j=1}^n \lambda_j + (n+1)m} G({}^\sigma\mu)^{n+1}\langle {}^\sigma f, {}^\sigma f\rangle}.$$

The reason we need the condition "$\lambda_1 \equiv \cdots \equiv \lambda_n \pmod 2$" is the following:

Lemma 3.5. Suppose that λ satisfies $\lambda_1 \equiv \cdots \equiv \lambda_n \pmod 2$. Then, a one dimensional representation of $K_{n,\infty}$ occurs in the holomorphic discrete series representation of weight λ.

The above lemma plays the crucial role in the computations of the archimedean zeta integral, .

Remark 3.6. Recently, Z. Liu [Liu21] compute the archimedean zeta integral at certain special value s.

In [HPSS21], the Hecke eigenform f as in the theorem is not holomorphic and thus the statement is slightly different. To prove the above theorem, we need the following kind of period relation.

In the following, we need several differential operators, related operators and spaces. Every definition is given in [Shi00]. For a representation ρ of $\mathrm{GL}_n(\mathbb{C})$ and $\ell \in \mathbb{Z}_{\geq 0}$, let D_ρ^ℓ and E^ℓ be Shimura's deferential operators. We then obtain the contraction operator θ. Let f be a holomorphic Siegel cusp form of weight ρ_λ. Take an irreducible representation ρ_ω of $K_{n,\infty}$ with highest weight $\omega = (\omega_1, \ldots, \omega_n)$.

Suppose that f is rational over a number field F and ρ_ω occur in the holomorphic discrete series representation $\mathcal{D}_\lambda$ of weight λ with multiplicity one. Then, $2\ell = \sum_{i=1}^n(\omega_i - \lambda_i)$ is a non-negative even integer. The representation ρ_ω occurs in $\tau^\ell \otimes \rho_\lambda$ with multiplicity one. Thus, we obtain the unique projection operator $\mathbf{pr}\colon \tau^\ell \otimes \rho_\lambda \longrightarrow \rho_\omega$ as the representation of $K_{n,\infty}$. Set $g = \pi^{-\ell}\,\mathbf{pr}\, D_\rho^\ell f$. The modular form g is rational over F by [Shi00, Theorem 14.9]. We then obtain the following period relation:

Proposition 3.7. With the above notation, there exists a rational number $c \in \mathbb{Q}$ such that

$$c\langle f, f\rangle = \pi^{2\ell}\langle g, g\rangle.$$

Proof. By [Shi00, Theorem 12.15], one has

$$\pi^{2\ell}\langle g, g\rangle = \langle \mathbf{pr}\, D_\rho^\ell f, \mathbf{pr}\, D_\rho^\ell f\rangle = \langle D_\rho^\ell f, \widetilde{\mathbf{pr}\, D_\rho^\ell f}\rangle = \langle f, (-1)^\ell \theta E^\ell(\widetilde{\mathbf{pr}\, D_\rho^\ell f})\rangle.$$

Here, $\widetilde{\mathbf{pr}\, D_\rho^\ell f}$ is the $S_\ell(T)\otimes\rho_\lambda$-valued function defined by

$$\widetilde{\mathbf{pr}\, D_\rho^\ell f}(g) = \mathbf{pr}\, D_\rho^\ell f(g) \in \rho_\omega \subset S_\ell(T)\otimes\rho_\lambda.$$

Thus, it suffices to show that there exists a rational constant c such that

$$(-1)^\ell \theta E^\ell(\widetilde{\mathbf{pr}\, D_\rho^\ell f}) = cf.$$

By definition, we have

$$\theta E^\ell(\widetilde{\mathbf{pr}\, D_\rho^\ell f})(z) = \sum_v E^\ell(\widetilde{\mathbf{pr}\, D_\rho^\ell f})(\zeta_\nu, \eta_\nu)(z).$$

Here, $\{\zeta_\nu\}$ and $\{\eta_\nu\}$ are bases of $S_\ell(T)$, dual to each other, defined in [Shi00, §12.14]. For a V_ρ-valued function α on $\mathfrak{H}_n$, set $\alpha^\rho(g) = \alpha|_\rho g(\mathbf{i}_n)$, where $\mathbf{i}_n = \sqrt{-1}\,\mathbf{1}_n \in \mathfrak{H}_n$. We then have

$$\begin{aligned}
&\left(\sum_\nu E^\ell(\widetilde{\mathbf{pr}\, D_\rho^\ell f})(\zeta_\nu,\eta_\nu)\right)^\rho(g)\\
&= \rho(j(g,\mathbf{i}))^{-1}\sum_\nu E^\ell(\widetilde{\mathbf{pr}\, D_\rho^\ell f})(\zeta_\nu,\eta_\nu)(g(\mathbf{i}))\\
&= \rho(j(g,\mathbf{i}))^{-1}(\rho_\lambda\otimes\tau^\ell\otimes\sigma^\ell)(j(g,\mathbf{i}))\sum_\nu \left(E^\ell(\widetilde{\mathbf{pr}\, D_\rho^\ell f})\right)^{\rho_\lambda\otimes\tau^\ell\otimes\sigma^\ell}(\zeta_\nu,\eta_\nu)(g)\\
&= (\tau^\ell\otimes\sigma^\ell)(j(g,\mathbf{i}))\sum_\nu \left(E^\ell(\widetilde{\mathbf{pr}\, D_\rho^\ell f})\right)^{\rho_\lambda\otimes\tau^\ell\otimes\sigma^\ell}(\zeta_\nu,\eta_\nu)(g).
\end{aligned}$$

Since the image of the sum over (ζ_ν,η_ν) is invariant under the action of $\tau^\ell\otimes\sigma^\ell$, the above equals

$$\sum_\nu \left(E^\ell(\widetilde{\mathbf{pr}\, D_\rho^\ell f})\right)^{\rho_\lambda\otimes\tau^\ell\otimes\sigma^\ell}(\zeta_\nu,\eta_\nu)(g).$$

Note that the representation σ^ℓ is dual to τ^ℓ. Thus, the contraction operator $\theta\colon \tau^\ell\otimes\sigma^\ell \longrightarrow \mathbb{C}$ defines the unique invariant pairing. We thus have

$$\left(E^\ell(\widetilde{\mathbf{pr}\, D_\rho^\ell f})\right)^{\rho\otimes\sigma^\ell\otimes\tau^\ell}((\zeta_\nu,\eta_\nu))(g) = Y\cdot\left(\widetilde{\mathbf{pr}\, D_\rho^\ell f}\right)^{\rho_\lambda\otimes\tau^\ell}(\eta_\nu)(g)$$

for some $Y \in \mathcal{U}(\mathfrak{p}_{n,-})$ of homogeneous degree ℓ corresponding to ζ_ν by [Shi00, (A.8.6(b))].

Next, we consider the projection **pr**. Recall that **pr** is the projection of $S_\ell(T)\otimes\rho_\lambda$ onto ρ_ω. Consider the action of $\mathcal{Z}(\mathfrak{k}_n)$ on $S_\ell(T)\otimes\rho_\lambda$. The highest weights under the action of $\mathrm{GL}_n(\mathbb{C})$ on $S_\ell(T)\otimes\rho_\lambda$ are of the form

$$(\alpha_1,\ldots,\alpha_n)$$

such that $\alpha_i \geq \lambda_i > n$ and $\alpha_i \in \mathbb{Z}$ for any i. Thus, an isotypic component under the action of $\mathcal{Z}(\mathfrak{k}_n)$ is isotypic under $\mathrm{GL}_n(\mathbb{C})$. It follows that there exists $\Omega \in \mathcal{Z}(\mathfrak{k}_n)$ such that Ω equals **pr**. Moreover, Ω is rational, i.e., $\Omega \in \mathcal{U}(\mathfrak{k}_{n,\mathbb{Q}})$ as in [HPSS21, §3.4]. We thus have

$$Y\cdot\left(\widetilde{\mathbf{pr}\,D^\ell_\rho f}\right)^{\rho_\lambda\otimes\tau^\ell}(\eta_\nu)(g) = Y\cdot\Omega\left(D^\ell_\rho f\right)^{\rho_\lambda\otimes\tau^\ell}(\eta_\nu)(g) = Y\Omega X\cdot f^\rho(g)$$

for some $X\in\mathcal{U}(\mathfrak{p}_{n,+})$.

Let $X_\nu \in \mathcal{U}(\mathfrak{p}_{n,+})$ and $Y_\nu \in \mathcal{U}(\mathfrak{p}_{n,-})$ be elements corresponding to η_ν and ζ_ν. We may assume that X_ν and Y_ν are rational. By the formula

$$\left(\theta E^\ell(\widetilde{\mathbf{pr}\,D^\ell_\rho f})\right)^{\rho_\lambda}(g) = \sum_\nu Y_\nu\Omega X_\nu f^\rho(g)$$

and the multiplicity one property of the highest weight $K_{n,\infty}$-type in $\mathcal{D}_\lambda$, there exists a constant $c\in\mathbb{C}$ such that

$$\sum_\nu Y_\nu\Omega X_\nu\cdot f^\rho(g) = cf.$$

One can regard Ω as an element of $\mathrm{End}_\mathbb{Q}(V_\mathbb{Q})$. For any $\sigma\in\mathrm{Aut}(\mathbb{C})$, we have

$$\sigma(c)\cdot{}^\sigma f = {}^\sigma(cf) = {}^\sigma\left(\theta E^\ell(\widetilde{\mathbf{pr}\,D^\ell_\rho f})\right)(z) = \theta\pi^\ell E^\ell(\mathbf{pr}\,\widetilde{\pi^{-\ell}D^\ell_\rho{}^\sigma f})(z) = c\cdot{}^\sigma f.$$

Hence $c\in\mathbb{Q}$. This completes the proof. □

4. Classification of $(\mathfrak{g},K)$-modules generated by nearly holomorphic Siegel modular forms and applications

To avoid the obstruction from Lemma 3.5, we need to investigate computations of (archimedean) zeta integrals. The integral is a certain integral consisting of a section of degenerate principal series and an element of holomorphic discrete series $\mathcal{D}_\lambda$. Thus, the choices of $K_{2n,\infty}$-types corresponding to f_s and $K_{n,\infty}$-types of a vector in $\mathcal{D}_\lambda$ are important. In this section, we give candidates of such $K_{2n,\infty}$-types and $K_{n,\infty}$-types.

4.1. **Main theorem.** For $1\leq i\leq n$ and $(\lambda_1,\ldots,\lambda_{n-i})\in\mathbb{Z}^{n-i}$ with $\lambda_1\geq\cdots\geq\lambda_{n-i}$, set

$$\rho_{i,n} = n-(i-1)/2,\quad \rho_n=\rho_{n,n},\quad \lambda=(\lambda_1,\ldots,\lambda_{n-i},\rho_{i,n}+1,\ldots,\rho_{i,n}+1)$$

and

$$\lambda' = (\lambda_1,\ldots,\lambda_{n-i},\rho_{i,n}-1,\ldots,\rho_{i,n}-1).$$

Suppose $\lambda_{n,i}\geq\rho_{i,n}+1$. Let $L(\lambda)$ be the irreducible highest weight representation of weight λ.

Lemma 4.1. We have

$$\dim_{\mathbb{C}} \operatorname{Ext}^1_{\mathcal{O}^{\mathfrak{k}_n \oplus \mathfrak{p}_{n,-}}}(L(\lambda), L(\lambda')) = 1$$

and a non-split exact sequence,

$$0 \longrightarrow L(\lambda') \longrightarrow N(\lambda')^\vee \longrightarrow L(\lambda) \longrightarrow 0.$$

Proof. See [Hor22, Lemma 3.5.1]. □

This lemma plays a crucial role in the following main theorem.

Theorem 4.2. Let M be an indecomposable $(\mathfrak{g}_n, K_{n,\infty})$-module generated by nearly holomorphic Hilbert-Siegel modular forms with the ground field F.

(1) If $F \neq \mathbb{Q}$, the module M is irreducible.
(2) If $F = \mathbb{Q}$, the length of the module M is at most 2. More precisely, if M is reducible, there exists i and $(\lambda_1, \ldots, \lambda_{n-i})$ with $\lambda_{n-i} \geq \rho_{n,i} + 1$ such that

$$M \cong N(\lambda')^\vee.$$

Proof. See [Hor22, Theorem 6.5.1]. □

Remark 4.3. We will not prove the existence of f such that f is isomorphic to $N(\lambda)^\vee$ for any λ.

Set $G_n = \operatorname{Res}_{F/\mathbb{Q}} \operatorname{Sp}_{2n}$ for a totally real field F. We give a sketch of proof. For the space of automorphic forms $\mathcal{A}(G_n)$, one has a decomposition according to the cuspidal components:

$$\mathcal{A}(G_n) = \bigoplus_{(M,\pi)} \mathcal{A}(G_n)_{(M,\pi)}.$$

First, let us study the intersection $\mathcal{N}_{(M,\pi)} = \mathcal{N}(G_n) \cap \mathcal{A}(G_n)_{(M,\pi)}$. Here, $\mathcal{N}(G_n)$ is the space of $\mathfrak{p}_{n,-}$-finite automorphic forms.

Proposition 4.4. If $\mathcal{N}_{(M,\pi)} \neq 0$, we have

- M is associated to $\mathrm{GL}_1 \times \cdots \times \mathrm{GL}_1 \times G_{n-i}$.
- Let $\pi = \mu_1 \boxtimes \cdots \boxtimes \mu_i \boxtimes \tau$. Then, $\mu_1 = \cdots = \mu_i$ and τ_∞ is a highest weight representation.

Proof. See [Hor20, Proposition 5.14]. □

Consider the constant terms of the elements in $\mathcal{N}_{(M,\pi)}$. Let $Q_{i,n}$ (resp. $P_{i,n}$) be the standard parabolic subgroup of G_n with the Levi subgroup $\mathrm{GL}_1^i \times G_{n-i}$ (resp. $\mathrm{GL}_i \times G_{n-i}$). We may assume that M is associated to $Q_{i,n}$ and $\pi = \mu \boxtimes \cdots \boxtimes \mu \boxtimes \tau$. Then, the constant term along $P_{i,n}$ induces the inclusion,

$$\mathcal{N}_{(M,\pi)} \longhookrightarrow \bigoplus_{s_0 \in \mathbb{Z} + \rho_{n,i}} \operatorname{Ind}_{P_{i,n}(\mathbb{A}_\mathbb{Q})}^{G_n(\mathbb{A}_\mathbb{Q})} \mu|\cdot|^{s_0} \boxtimes \tau$$

by [Hor20, Proposition 5.9]

For an infinitesimal character χ, let $\mathcal{N}(\chi)_{(M,\pi)}$ be the χ-isotypic component of $\mathcal{N}_{(M,\tau)}$ and $\mathcal{N}^2(\chi)_{(M,\tau)}$ the subspace consisting of square-integrable automorphic forms. Then, there exists $s_0 \geq 0$ such that

$$\mathcal{N}^2(\chi)_{(M,\pi)} \backslash \mathcal{N}(\chi)_{(M,\pi)} \longhookrightarrow \operatorname{Ind}_{P_{i,n}(\mathbb{A}_\mathbb{Q})}^{G_n(\mathbb{A}_\mathbb{Q})} \mu|\cdot|^{s_0} \boxtimes \tau.$$

By computing the Eisenstein series, zeta integrals, et.c., we have the splitting,

$$\left(\mathrm{Ind}_{P_{i,n}(\mathbb{A}_\mathbb{Q})}^{G_n(\mathbb{A}_\mathbb{Q})}\mu|\cdot|^{s_0}\boxtimes\tau\right)_{\mathfrak{p}_{n,-}\text{-fin}} \xrightarrow{\sim} \mathcal{N}^2(\chi)_{(M,\pi)}\backslash\mathcal{N}(\chi)_{(M,\pi)}$$

under several conditions related to the Harish-Chandra parameters of χ and the infinitesimal character of π. If a splitting exists, we have

$$\mathcal{N}(\chi)_{(M,\pi)} = \mathcal{N}^2(\chi)_{(M,\pi)} \oplus \left(\mathrm{Ind}_{P_{i,n}(\mathbb{A}_\mathbb{Q})}^{G_n(\mathbb{A}_\mathbb{Q})}\mu|\cdot|^{s_0}\boxtimes\tau\right)_{\mathfrak{p}_{n,-}\text{-fin}}.$$

Note that the space

$$\left(\mathrm{Ind}_{P_{i,n}(\mathbb{A}_\mathbb{Q})}^{G_n(\mathbb{A}_\mathbb{Q})}\mu|\cdot|^{s_0}\boxtimes\tau\right)_{\mathfrak{p}_{n,-}\text{-fin}}$$

is isotypic for some irreducible $(\mathfrak{g}_n, K_{n,\infty})$-modules if $s_0 \geq 0$. With a few exceptions, we may assume $s_0 = 1$ and $F = \mathbb{Q}$. We thus have

$$\mathcal{N}(\chi)_{(M,\pi)} \hookrightarrow \bigoplus_{s_0=\pm 1}\left(\mathrm{Ind}_{P_{i,n}(\mathbb{A}_\mathbb{Q})}^{G_n(\mathbb{A}_\mathbb{Q})}\mu|\cdot|^{s_0}\boxtimes\tau\right)_{\mathfrak{p}_{n,-}\text{-fin}}. \tag{4.1.1}$$

By Lemma 4.1, we can classify indecomposable modules in (4.1.1). Thus, an indecomposable reducible module is isomorphic to $N(\lambda')^\vee$.

4.2. **Candidates of K-types.** Here, we give a weight that behaves like a scalar valued case. Let χ be a regular integral infinitesimal character. Further, suppose χ is the infinitesimal character of a holomorphic discrete series representation D_λ. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $j = \#\{i \mid \lambda_1 \equiv \lambda_i \pmod 2\}$. The following statement follows from the classification and computations of the multiplicities of the highest weight modules with the given infinitesimal characters.

Theorem 4.5. With the above notation, put $\rho = \det^{\lambda_1 - 1} \otimes \wedge^j \mathrm{std}_{\mathrm{GL}_n(\mathbb{C})}$.

(1) One has $\dim_\mathbb{C} \mathrm{Hom}_{\mathfrak{k}_n}(\rho, D_\lambda|_{\mathfrak{k}_n}) = 1$.
(2) If $\lambda_n \neq n+1$, one has $N_\rho(\Gamma, \chi) = N_\rho(\Gamma)[D_\lambda]$, the isotypic component of $\mathcal{D}_\lambda$.
(3) If $\lambda_n = n+1$, for any $f \in N_\rho(\Gamma, \chi)$, then f generates D_λ or an indecomposable module $N(\lambda_1, \ldots, \lambda_{n-1}, n-1)^\vee$.

We expect the weight of the form $\det^{\lambda_1 - 1} \otimes \wedge^j \mathrm{std}_{\mathrm{GL}_n(\mathbb{C})}$ to behave like a scalar weight k. Note that if λ satisfies the parity condition, then $j = n$ and $\rho = \det^{\lambda_1}$. Consider the weights,

$$\sigma = (\overbrace{k, \ldots, k}^{n+j}, \overbrace{k-2, \ldots, k-2}^{n-j})$$

and

$$\tau = (\overbrace{k+1, \ldots, k+1}^{j}, \overbrace{k-1, \ldots, k-1}^{2n-j}).$$

For $\rho = \det^{k-1} \otimes \wedge^j \mathrm{std}_{\mathrm{GL}_n(\mathbb{C})}$, the representation $\rho \boxtimes \rho$ occurs in

$$\sigma \downarrow \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) \quad \text{and} \quad \tau \downarrow \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$$

with multiplicity one.

The author expects that the arithmeticity of critical values attached to vector valued Siegel modular forms follows from the arithmeticity of the Siegel Eisenstein series of weight $(\ell+2, \ldots, \ell+2, \ell, \ldots, \ell)$.

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