Construction of fundamental solutions to Schrödinger equations on compact manifolds by Feynman path integral methods

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Abstract

We construct fundamental solutions to Schrödinger equations on compact Riemannian manifolds. We employ a time-slicing approximation, which is a mathematically rigorous method of defining the Feynman path integral. Our time-slicing approximation converges to a fundamental solution to the Schrödinger equation modified by the scalar curvature. The coefficient of the scalar curvature in the modified Schrödinger equation depends on the choice of the amplitude which appears in the definition of the time-slicing approximation.

1 Introduction

1.1 Feynman path integrals on curved spaces

We consider the Schrödinger equation

$$i\frac{\partial}{\partial t}u(t) = H_\lambda u(t), \quad u(0) = u_0, \tag{1.1}$$

on an oriented compact Riemannian manifold (M, g) with the Hamiltonian

$$H_\lambda := -\frac{1}{2}\triangle_g + V + \lambda R,$$

where

- $\triangle_g$ is the Laplacian associated with the metric g,

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- $V \in C^\infty(M;\mathbb{R})$ is the potential,
- $R \in C^\infty(M)$ is the scalar curvature of (M,g), and
- $\lambda\,(=0,1/6,1/12) \in \mathbb{R}$ is a real parameter.

Since H_λ is essentially self-adjoint on $L^2(M,g)$ (we also denote its closure by H_λ), the Schrödinger propagator e^{-itH_λ} exists. The aim of this paper is to represent e^{-itH_λ} by the Feynman path integral [3]. In the paper [3], Feynman states that the time-development of the quantum system is represented as the "integral"

$$K(t,s,x,y) := \int_{\Omega_{t,s,x,y}} e^{iS(\gamma)}\,\mathcal{D}\gamma, \tag{1.2}$$

where

- $\Omega_{t,s,x,y}$ is a space of all paths γ which satisfy $\gamma(s)=y$ and $\gamma(t)=x$,
- $S(\gamma)$ is an action of γ.

Concerning the formal expression (1.2), the following two problems arise.

(1) What is the mathematical definition of the "integral" (1.2)?

(2) Does $K(t,x,y) := K(t,0,x,y)$ correspond to the fundamental solution e^{-itH_λ} of the Schrödinger equation (1.1)?

Here we briefly describe our approach to the above questions in this paper. On the question (1), it is already known that one cannot realize the "integral" (1.2) as the Lebesgue integration by constructing a suitable measure on the space $\Omega_{t,s,x,y}$ [1]. An alternative method of the definition of (1.2) is the time-slicing approximation. In the time-slicing approximation, we regard (1.2) as a limit of oscillatory integrals on finite dimensional spaces, and we do not try to construct any measure on the space $\Omega_{t,s,x,y}$. This method is introduced in Feynman's original paper [3]. In this paper, we employ the time-slicing approximation for the definition of (1.2).

On the question (2), the amplitude function which appears in the definition of the time-slicing approximation affects the form of the Schrödinger equation (1.1). In the formal expression (1.2), the information of amplitudes is included in the "measure" $\mathcal{D}\gamma$. In this paper, the Schrödinger equations with $\lambda = 0, 1/6, 1/12$ are derived by the time-slicing approximation with the natural choices of the amplitudes. We remark that this change of the Schrödinger equations does not occur on the flat space ($R=0$) such as the Euclidean spaces.

1.2 Mathematical setting

In this Subsection, we describe our mathematical formulation of the problem in the previous Subsection. Let (M,g) is an n-dimensional oriented compact

Riemannian manifold. For a sufficiently small $\tau > 0$, we consider a short-time approximate solution $E(\tau)$ of the form

$$E(\tau)u(x) := \frac{1}{(2\pi i)^{n/2}} \int_M \chi(x,y)a(\tau,x,y)e^{iS(\tau,x,y)}u(y)\,\mathrm{vol}_g(y).$$

Here vol_g is the volume form associated with the metric g and the other functions $S(\tau,x,y)$, $\chi(x,y)$ and $a(\tau,x,y)$ are defined as follows.

$\boldsymbol{S(\tau,x,y)}$**: action along the lowest energy classical path.** Taking local coordinates $(x_1,\ldots,x_n)$, we define $g^* : T_x^*M \times T_x^*M \to \mathbb{R}$ by

$$g^*(\xi,\eta) := \sum_{j,k=1}^{n} g^{jk}(x)\xi_j\eta_k$$

where $(g^{jk}(x))_{j,k=1}^n$ is the inverse matrix of $(g_{jk}(x))_{j,k=1}^n$ defined by $g = \sum_{j,k=1}^n g_{jk}(x)dx_jdx_k$ and $\xi = \sum_{j=1}^n \xi_j dx_j$ and $\eta = \sum_{j=1}^n \eta_j dx_j$. We also define $|\xi|_{g^*}^2 := g^*(\xi,\xi)$. As the corresponding classical mechanics, we consider the Hamiltonian

$$H(x,\xi) := \frac{1}{2}|\xi|_{g^*}^2 + V(x)$$

for $(x,\xi) \in T^*M$. Let $\pi : T^*M \to M$ be the natural projection. We call $x(t) : [0,\tau] \to M$ a classical path from y to x in time τ if $x(t) = \pi(x(t),\xi(t))$ for some $(x(t),\xi(t)) : [0,\tau] \to T^*M$ which satisfies the Hamilton equation

$$\frac{dx_j}{dt}(t) = \frac{\partial H}{\partial \xi_j}(x(t),\xi(t)), \quad \frac{d\xi_j}{dt}(t) = -\frac{\partial H}{\partial x_j}(x(t),\xi(t)). \tag{1.3}$$

If $x(t) = \pi(x(t),\xi(t))$ is the classical path, then the energy $E = H(x(t),\xi(t))$ is a constant. We call $x(t)$ a classical path with the lowest energy from y to x in time τ if $x(t)$ has the smallest E among all classical paths satisfying the boundary condition $x(0) = y$, $x(\tau) = x$.

For the definition of the action function $S(\tau,x,y)$, we employ the following theorem.

Theorem 1.1. *There exist a small $\delta > 0$ and a small neighborhood $\mathcal{N}$ of the diagonal*

$$\mathrm{diag}\, M := \{(x,x) \in M \times M \mid x \in M\}$$

such that for any $(\tau,x,y) \in (0,\delta) \times \mathcal{N}$, there exists a unique classical path $x_s^\tau(x,y) \in M$ with the lowest energy from y to x in time τ.

Then we define the function $S(\tau,x,y)$ as follows.

Definition 1.2. Fix a small $\delta > 0$ and a small neighborhood $\mathcal{N}$ as in Theorem 1.1. For $(\tau,x,y) \in (0,\delta) \times \mathcal{N}$, we take the unique classical path $x_s^\tau(x,y)$ as in Theorem 1.1 and define

$$S(\tau,x,y) := \int_0^\tau \left(\frac{1}{2} \left| \frac{dx_s^\tau}{ds}(x,y) \right|_g^2 - V(x_s^\tau(x,y)) \right) ds.$$

$\boldsymbol{\chi(x, y)}$**: cutoff function.** In order to restrict (x, y) to $\mathcal{N}$, we introduce a cutoff function $\chi(x, y) \in C^\infty(M \times M)$ supported in $\mathcal{N}$. For technical reasons, we require the properties $\chi = 1$ near $\operatorname{diag} M$ and $0 \leq \chi \leq 1$ everywhere.

$\boldsymbol{a(\tau, x, y)}$**: amplitude.** In this paper, we consider three amplitude functions. The first one is

$$a(\tau, x, y) := \tau^{-n/2},$$

which is same as in the case of Euclidean spaces.

The second choice is the square root of the Morette-Van Vleck determinant:

$$a(\tau, x, y) := D(\tau, x, y)^{1/2}.$$

The Morette-Van Vleck determinant $D(\tau, x, y)$ is defined as

$$D(\tau, x, y) := g_\iota(x)^{-1/2} g_{\iota'}(y)^{-1/2} \det(-\partial_x \partial_y S(\tau, x, y))$$

by local coordinates with the positive orientation, where $g_\iota(x)$ and $g_{\iota'}(t)$ are positive functions defined by the relation $\operatorname{vol}_g(x) = g_\iota(x)^{1/2} dx_1 \wedge \cdots \wedge dx_n$ and $\operatorname{vol}_g(y) = g_{\iota'}(y)^{1/2} dy_1 \wedge \cdots \wedge dy_n$. $D(\tau, x, y)$ is independent of the choice of the local coordinates with the positive orientation around x and y.

The third choice is the square root of Morette-Van Vleck determinant with an auxiliary term:

$$a(\tau, x, y) := D(\tau, x, y)^{1/2}(1 - i a_1(\tau, x, y)).$$

Here $a_1(\tau, x, y)$ is the solution to the transport equation

$$\frac{\partial a_1}{\partial t} + g(\operatorname{grad}_x S, \operatorname{grad}_x a_1) = -\frac{1}{2} D^{-1/2} \triangle_x D^{1/2}, \quad a_1(0, x, y) = 0. \tag{1.4}$$

Fix a fixed time $t > 0$. We call a multiple $\Delta := (\tau_1, \ldots, \tau_N)$ with $\tau_j > 0$ and $\tau_1 + \cdots + \tau_N = t$ a partition of t. The size of the partition $\Delta = (\tau_1, \ldots, \tau_N)$ is defined as $|\Delta| := \max_{1 \leq j \leq N} \tau_j$. For a partition $\Delta = (\tau_1, \ldots, \tau_N)$ of $t > 0$, we define the time-slicing approximation $\mathcal{E}(\Delta)$ as an iteration of the operators

$$\mathcal{E}(\Delta) := E(\tau_N) \cdots E(\tau_1).$$

Our main theorem states that the time-slicing approximation converges to the fundamental solution to the Schrödinger equation (1.1).

Theorem 1.3. *Let $a(\tau, x, y) = \tau^{-n/2}, D^{1/2}, D^{1/2}(1 - i a_1)$. In the case of $a = \tau^{-n/2}$, we further assume that the Ricci curvature of (M, g) is positive definite. For each amplitude, we set $\lambda \in \mathbb{R}$ in the modified Schrödinger equation* (1.1) *as*

$$\lambda = \begin{cases} 1/6 & \text{if } a = \tau^{-n/2}, \\ 1/12 & \text{if } a = D^{1/2}, \\ 0 & \text{if } a = D^{1/2}(1 - i a_1). \end{cases}$$

Then, for any $T > 0$ and $\varepsilon \in (0, 1/2]$, there exists a constant $C > 0$ such that the estimate

$$\|\mathcal{E}(\Delta) - e^{-itH_\lambda}\|_{H^{1+\varepsilon} \to L^2} \le C|\Delta|^\varepsilon \tag{1.5}$$

holds for all $t \in (0, T]$ and partition Δ of t.

Here $H^{1+\varepsilon} = H^{1+\varepsilon}(M)$ is the Sobolev space on the compact manifold M of order $1 + \varepsilon$.

The case of $a = D^{1/2}$ is proved in [6] and the other cases are in preparation. In this paper, we describe an outline of the proof of Theorem 1.3 from Section 2. We can refer to [6] for the detail of the proof.

Remark. In the case of $a = \tau^{-n/2}$, the positive Ricci curvature condition is just a sufficient condition and not a necessary condition. For example, the inequality (1.5) holds on the flat tori. In general, Theorem 1.3 with $a = \tau^{-n/2}$ is applicable if the inequality

$$\tau^n D(\tau, x, y) \ge 1 - C\tau$$

holds for all $(\tau, x, y) \in (0, \delta) \times \mathcal{N}$. Since $D(\tau, x, y)$ is expanded as

$$\tau^n D(\tau, x, y) = 1 + \frac{1}{6} \sum_{i,j=1}^{n} R_{ij}(y) x_i x_j + O(|x|^3 + \tau) \tag{1.6}$$

in normal coordinates centered at y where $R_{ij}(y)$ is the Ricci curvature tensor at y, the inequality (1.6) holds if (M, g) has the positive Ricci curvature.

Here we refer to the previous studies of the time-slicing approximations. On the Euclidean spaces, for example, Fujiwara [5] and Kumano-go [9] studied the time-slicing approximation in the case of at most quadratically increasing potential and proved that the time-slicing approximation converges to the fundamental solution to the Schrödinger equation. Ichinose [7] dealt with polynomially growing potentials and proved the convergence to the fundamental solution in the strong operator topology on the L^2 space.

On the other hand, there are only a few mathematical studies of the time-slicing approximation on manifolds. Miyanishi [10, 11] studied the case of free particles on compact manifolds with a suitable symmetry. There are some studies of the imaginary-time path integrals, that is, roughly speaking, construction of the heat kernel. Inoue and Maeda [8] constructed the imaginary-time path integral for the free particle and the derived the heat equation modified by the scalar curvature. Fine and Sawin [4] constructed the imaginary-time path integrals for the supersymmetric quantum mechanics.

2 Reduction to stability and consistency

We reduce the proof of the main theorem (Theorem 1.3) to the analysis of the asymptotic behavior of the short-time approximate solution $E(\tau)$ as $\tau \to +0$.

Lemma 2.1. *Under the same assumption in Theorem 1.3, the following statements hold.*

(i) *(Stability) There exists a constant $C > 0$ such that the inequality*

$$\|E(\tau)\|_{L^2 \to L^2} \leq e^{C\tau}$$

holds for sufficiently small $\tau > 0$.

(ii) *(Consistency) For any $\varepsilon \in (0, 1/2]$ and $\lambda \in \mathbb{R}$ as in Theorem 1.3, there exists a constant $C > 0$ such that the inequality*

$$\left\| i\frac{\partial}{\partial \tau} E(\tau)u - H_\lambda E(\tau)u \right\|_{L^2} \leq C\tau^\varepsilon \|u\|_{H^{1+\varepsilon}}$$

holds for sufficiently small $\tau > 0$ and $u \in C^\infty(M)$.

We prove Theorem 1.3 by the above Lemma 2.1.

Proof of Theorem 1.3. Take an arbitrary $u \in C^\infty(M)$ and set

$$G(\tau)u := \begin{cases} i\partial_\tau E(\tau)u - H_\lambda E(\tau)u & \text{if } 0 < \tau \ll 1, \\ 0 & \text{if } \tau = 0. \end{cases} \tag{2.1}$$

Then $\tau \mapsto G(\tau)u$ is continuous in the L^2 topology at $\tau = 0$ by the consistency. Thus we can apply the Duhamel principle and obtain

$$E(\tau)u - e^{-i\tau H_\lambda}u = i\int_0^\tau e^{-i(\tau-\sigma)H_\lambda} G(\sigma)u \, d\sigma.$$

Hence we have the inequality

$$\|E(\tau)u - e^{-i\tau H_\lambda}u\|_{L^2} \leq \int_0^\tau \|G(\sigma)u\|_{L^2} \, d\sigma \leq C\tau^{1+\varepsilon}\|u\|_{H^{1+\varepsilon}}.$$

We introduce an operator $P := (i + H_\lambda)^{-(1+\varepsilon)/2}$. Since P and $e^{-i\tau H_\lambda}$ commute, we obtain

$$\begin{aligned} &\|\mathcal{E}(\Delta)P - e^{-itH_\lambda}P\|_{L^2 \to L^2} \\ &\leq \sum_{j=0}^N \| \underbrace{E(\tau_N)\cdots E(\tau_{j+1})}_{\text{stability}} \underbrace{(E(\tau_j) - e^{-i\tau_j H_\lambda})P}_{\text{consistency}} \underbrace{e^{-i(\tau_{j-1}+\cdots+\tau_1)H_\lambda}}_{\text{unitarity}} \|_{L^2 \to L^2} \\ &\leq \sum_{j=0}^N e^{C(\tau_N+\cdots+\tau_{j+1})} C\tau_j^{1+\varepsilon} \leq C|\Delta|^\varepsilon \end{aligned}$$

for any partition $\Delta = (\tau_1, \ldots, \tau_N)$ of $t \in (0, T]$. □

Theorem 1.3 and the stability in Lemma 2.1 implies the convergence in strong topology:

Corollary 2.2. *For each $u \in L^2(M)$, we have*

$$\lim_{|\Delta| \to 0} \mathcal{E}(\Delta)u = e^{-itH_\lambda}u$$

in the L^2 topology.

3 Classical mechanics

First we briefly describe the proof of Theorem 1.1, which states the unique existence of the classical path with the lowest energy.

Proof of Theorem 1.1. We introduce a scaling

$$\Theta_\tau : T^*M \to T^*M, \quad \Theta_\tau(x,\xi) := (x, \tau^{-1}\xi).$$

Then $(x(t), \xi(t))$ satisfies the Hamilton equation (1.3) if and only if $(\tilde{x}(s), \tilde{\xi}(s)) := \Theta_\tau(x(\tau s), \xi(\tau s))$ satisfies the Hamilton equation

$$\begin{aligned} &\frac{d\tilde{x}_j}{ds}(s) = \frac{\partial H_\tau}{\partial \xi_j}(\tilde{x}(s), \tilde{\xi}(s)), \quad \frac{d\tilde{\xi}_j}{ds}(s) = -\frac{\partial H_\tau}{\partial x_j}(\tilde{x}(s), \tilde{\xi}(s)), \\ &\tilde{x}(0) = y, \quad \tilde{x}(1) = x. \end{aligned} \tag{3.1}$$

where $H_\tau(x,\xi) := |\xi|_g^2/2 + \tau^2 V(x)$. Note that the problem (3.1) is extended naturally in the case of $\tau \leq 0$. If $\tau = 0$, then there exists a unique solution to (3.1) with the lowest energy for sufficiently close x and y by the existence of geodesically convex neighborhoods. For small $|\tau| \ll 1$, we consider the Hamiltonian flow $(\overline{q}_s^\tau(y,\eta), \overline{p}_s^\tau(y,\eta))$ with respect to the Hamiltonian H_τ:

$$\begin{aligned} &\frac{d\overline{q}_{s,j}^\tau}{ds}(s) = \frac{\partial H_\tau}{\partial \xi_j}(\overline{q}_s^\tau, \overline{p}_s^\tau), \quad \frac{d\overline{p}_{s,j}^\tau}{ds}(s) = -\frac{\partial H_\tau}{\partial x_j}(\overline{q}_s^\tau, \overline{p}_s^\tau), \\ &\overline{q}_0^\tau(y,\eta) = y, \quad \overline{p}_0^\tau(y,\eta) = \eta. \end{aligned} \tag{3.2}$$

We can apply the inverse function theorem at each point on

$$\{0\} \times \{(y, 0) \in T^*M \mid y \in M\}$$

to the function

$$(\tau, y, \eta) \longmapsto (\tau, \overline{q}_1^\tau(y,\eta), y).$$

Thus, we denote the inverse function of the above function by $(\tau, \eta(\tau,x,y), y)$ and set

$$(q_s^\tau(x,y), p_s^\tau(x,y)) := (\overline{q}_s^\tau(y, \eta(\tau,x,y)), \overline{p}_s^\tau(y, \eta(\tau,x,y))),$$

and we obtain the solution $(q_s^\tau(x,y), p_s^\tau(x,y))$ to the Hamilton equation (3.1). □

The action $S(\tau, x, y)$ defined in Definition 1.2 has following asymptotic behavior as $\tau \to +0$.

Theorem 3.1. *We set $\Phi(\tau,x,y) := \tau S(\tau,x,y)$ for $(\tau,x,y) \in (0,\delta)\times\mathcal{N}$. Then Φ is extended to a smooth function in $(-\delta,\delta)\times\mathcal{N}$. Moreover, as $\tau \to 0$, $\Phi(\tau,x,y)$ has the following asymptotic behavior*

$$\Phi(\tau,x,y) = \frac{1}{2} d(x,y)^2 + O(\tau^2)$$

where d stands for the distance function associated with the Riemannian metric g.

Remark. $\Phi(\tau,x,y)$ is equal to the action along the lowest energy classical path with respect to the scaled Hamiltonian H_τ from y to x in time 1.

4 Proof of stability and consistency

4.1 Proof of stability

Proof of Lemma 2.1 (i). We consider the operator $E(\tau)^*E(\tau)$. Regarding $\tau > 0$ as the semiclassical parameter, we can prove that $E(\tau)^*E(\tau)$ is a semiclassical τ-pseudodifferential operator with the principal symbol

$$\sigma(E(\tau)^*E(\tau)) = \frac{|b(\tau, \overline{q}_1^\tau(y,\eta), y)|^2}{|D_\Phi(\tau, \overline{q}_1^\tau(y,\eta), y)|}. \tag{4.1}$$

Here $\overline{q}_1^\tau(y,\eta)$ is the projection of the Hamiltonian flow with respect to the scaled Hamiltonian H_τ to the configuration space, which is defined in (3.2). $b(\tau, x, y)$ is defined as

$$b(\tau, x, y) := \tau^{n/2} a(\tau, x, y)\chi(x, y)\ (= O(1) \text{ as } \tau \to +0)$$

and $D_\Phi(\tau, x, y)$ is defined as

$$D_\Phi(\tau, x, y) := \tau^n D(\tau, x, y).$$

Then we have

$$\|\sigma(E(\tau)^*E(\tau))\|_{L^\infty(T^*M)} \leq 1 + C\tau$$

for all cases $a = \tau^{-n/2}$ (with the positive Ricci curvature condition) and $a = D^{1/2}$, $D^{1/2}(1 - ia_1)$. Thus the L^2-boundedness theorem (see [2, Proposition E.24] for example) of pseudodifferential operators implies

$$\|E(\tau)^*E(\tau)\|_{L^2\to L^2} \leq \|\sigma(E(\tau)^*E(\tau))\|_{L^\infty(T^*M)} + O(\tau) \leq 1 + C\tau.$$

We roughly describe the derivation of the principal symbol (4.1). In local coordinates, the integral kernel $K(\tau, x, y)$ of $E(\tau)^*E(\tau)$ in the sense that

$$E(\tau)^*E(\tau)u(x) = \int_{\mathbb{R}^n} K(\tau, x, y)u(y)\, dy_1 \cdots dy_n$$

is

$$K(\tau, x, y) = \frac{g(y)^{1/2}}{(2\pi\tau)^n} \int_{\mathbb{R}^n} \overline{b(\tau, z, x)} b(\tau, z, y) e^{i(-\Phi(\tau,z,x)+\Phi(\tau,z,y))/\tau} g(z)^{1/2} dz_1 \cdots dz_n,$$

where $g(y)$ is the volume density:

$$\mathrm{vol}_g(y) = g(y)^{1/2} dy_1 \cdots \wedge dy_n,$$

and $\Phi(\tau, x, y)$ is defined in Theorem 3.1. We approximate the phase function $-\Phi(\tau, z, x) + \Phi(\tau, z, y)$ as

$$-\Phi(\tau, z, x) + \Phi(\tau, z, y) = \eta \cdot (x - y), \quad \eta = -\frac{\partial \Phi}{\partial y}(\tau, z, y) + O(|x - y|).$$

We change the variables $z \mapsto \eta$. Since $\Phi(\tau, x, y)$ generates the Hamiltonian flow $(\overline{q}_1^\tau(y,\eta), \overline{p}_1^\tau(y,\eta))$ in the sense that

$$\overline{p}_1^\tau(y,\eta) = \frac{\partial \Phi}{\partial x}(\tau, \overline{q}_1^\tau(y,\eta), y), \quad \eta = -\frac{\partial \Phi}{\partial y}(\tau, \overline{q}_1^\tau(y,\eta), y),$$

we observe that the inverse function of $z \mapsto -\partial_y \Phi(\tau, z, y)$ is approximately equal to $\eta \mapsto \overline{q}_1^\tau(y,\eta)$. Thus we have

$$K(\tau, x, y) = \frac{1}{(2\pi\tau)^n} \int_{\mathbb{R}^n} p(\tau, x, \eta, y) e^{i\eta\cdot(x-y)/\tau} \, d\eta$$

where

$$\begin{aligned} &p(\tau, x, \eta, y) \\ &:= \overline{b(\tau, \overline{q}_1^\tau(y,\eta), x)} b(\tau, \overline{q}_1^\tau(y,\eta), y) \\ &\quad \times \left| \det \frac{\partial^2 \Phi}{\partial x \partial y}(\tau, \overline{q}_1^\tau(y,\eta), y) + O(|x-y|) \right|^{-1} g(\overline{q}_1^\tau(y,\eta))^{1/2} g(y)^{1/2}. \end{aligned}$$

Hence the principal symbol is

$$\begin{aligned} &\sigma(E(\tau)^* E(\tau))(\tau, y, \eta) = p(\tau, y, \eta, y) \\ &= |b(\tau, \overline{q}_1^\tau(y,\eta), y)|^2 \left| \det \frac{\partial^2 \Phi}{\partial x \partial y}(\tau, \overline{q}_1^\tau(y,\eta), y) \right|^{-1} g(\overline{q}_1^\tau(y,\eta))^{1/2} g(y)^{1/2} \\ &= \frac{|b(\tau, \overline{q}_1^\tau(y,\eta), y)|^2}{|D_\Phi(\tau, \overline{q}_1^\tau(y,\eta), y)|}. \end{aligned}$$

□

4.2 Proof of consistency

Proof of Lemma 2.1 (ii). We only consider the case of $a = D^{1/2}$ in this paper. The proof in the case of $a = D^{1/2}(1 - ia_1)$ is similar to that of $a = D^{1/2}$. On the other hand, more detailed analysis is needed in the case of $a = \tau^{-n/2}$.

Let $G(\tau)$ be the operator defined by (2.1). We can decompose $G(\tau)$ into the sum of two operators

$$G(\tau) = G_1(\tau) + \tau^{-1} G_2(\tau) \tag{4.2}$$

where $G_1(\tau)$ and $G_2(\tau)$ locally satisfy

$$G_j(\tau)^* G_j(\tau) u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} p_j\left(\tau, \frac{x+y}{2}, \tau\eta\right) e^{i\eta\cdot(x-y)} u(y) \, dy d\eta$$

with symbols $p_j(\tau, x, \xi)$ such that

- $p_1(0, y, 0) = 0$ for all $y \in M$ and
- $p_2(\tau, y, \eta) = 0$ near $\{0\} \times \{(y, 0) \in T^*M \mid y \in M\}$.

The family of symbols $\{\tau^{-2\varepsilon} p_1(\tau, y, \tau\eta)\}_{0<\tau<1}$ and $\{\tau^{-2-2\varepsilon} p_2(\tau, y, \tau\eta)\}_{0<\tau<1}$ are bounded in the class $S^{2\varepsilon}_{0,0}(T^*\mathbb{R}^{2n})$ and $S^{2\varepsilon+2}_{0,0}(T^*\mathbb{R}^{2n})$ respectively where

$$S^m_{0,0}(\mathbb{R}^{2n}) := \left\{ a \in C^\infty(\mathbb{R}^{2n}) \,\middle|\, \begin{array}{l} \langle\xi\rangle^{-m} \partial_x^\alpha \partial_\xi^\beta a(x,\xi) \in L^\infty(\mathbb{R}^{2n}) \\ \text{for all multiindices } \alpha, \beta \end{array} \right\}.$$

with the seminorms

$$|a|_{\alpha\beta} := \| \langle\xi\rangle^{-m} \partial_x^\alpha \partial_\xi^\beta a(x,\xi)\|_{L^\infty(\mathbb{R}^{2n})}.$$

Thus, by the continuity of pseudodifferential operators on the Sobolev spaces, we have

$$\|G_1(\tau)^* G_1(\tau)\|_{H^\varepsilon \to H^{-\varepsilon}} \leq C\tau^{2\varepsilon}$$

and

$$\|G_2(\tau)^* G_2(\tau)\|_{H^{1+\varepsilon} \to H^{-1-\varepsilon}} \leq C\tau^{2+2\varepsilon}$$

for some $C > 0$ independent of $\tau > 0$. Thus we obtain

$$\|G_1(\tau)\|^2_{H^\varepsilon \to L^2} \leq C\|G_1(\tau)^* G_1(\tau)\|_{H^\varepsilon \to H^{-\varepsilon}} \leq C\tau^{2\varepsilon}$$

and

$$\|G_2(\tau)\|^2_{H^{1+\varepsilon} \to L^2} \leq C\|G_2(\tau)^* G_2(\tau)\|_{H^{1+\varepsilon} \to H^{-1-\varepsilon}} \leq C\tau^{2+2\varepsilon}.$$

Substituting them to (4.2), we obtain the desired estimate

$$\|G(\tau)\|_{H^{1+\varepsilon} \to L^2} \leq C\tau. \qquad \square$$

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