EIKONAL EQUATIONS ON METRIC SPACES

XIAODAN ZHOU
OKINAWA INSTITUTE OF SCIENCE AND TECHNOLOGY GRADUATE UNIVERSITY

In the Euclidean space, the eikonal equation has important applications in various fields such as geometric optics, electromagnetic theory and image processing. It is well known that the notion of viscosity solutions provides a nice framework for the well-posedness of the eikonal equation and more general fully nonlinear equations [CIL, BC].

In seeking to further develop various fields such as optimal transport, mean field games, topological networks etc., the study of eikonal equation and more general Hamilton-Jacobi equations has been extended to a metric space (X, d). In particular, different approaches of defining the solutions to the eikonal equation are introduced by Ambrosio and Feng [AF], Giga, Hamamuki, and Nakayasu [GHN], and Gangbo and Świech [GS].

Inspired by the work of Newcomb and Su [NS], Liu, Shanmugalingam and the author [LSZ] propose a definition called Monge solution to the eikonal equation in a complete and rectifiably connected metric space X. We show the equivalence between the Monge solution with the solution introduced in [GHN] and [GS] and obtain the uniqueness and existence to the associated Dirichlet boundary problem. These results also extend to a more general class of Hamilton-Jacobi equation.

In this short note, we focus on a simple example to explain the definition of Monge solution intuitively. Consider eikonal equation

$$|\nabla u|(x) = 1 \quad \text{in } \Omega, \tag{E}$$

where $\Omega = (-1, 1) \subset \mathbb{R}$ and u satisfying the boundary condition $u(-1) = u(1) = 0$.

One optimal control interpretation of the equation is the following: consider a point starting from a position $x \in [-1, 1]$ and this points moves at a speed no faster than 1. Let $u(x)$ be the least time it takes for the point to exit the interval $[-1, 1]$. One can verify that $u(x)$ satisfies the eikonal equation and the solution should be the distance of the point x to the boundary $\{1, -1\}$, that is,

$$u(x) = \inf \left\{ t : \xi(t) \in \{\pm 1\} \text{ with } \ \xi(0) = x,\ |\xi'| \leq 1 \ a.e. \right\} = 1 - |x|.$$

Let $z := \xi(t)$. We can also write down the dynamic programming principle as

$$u(x) = \inf\{u(z) + t : x - t \leq z \leq x + t\} \quad \text{for any } x \in (-1, 1),\ 0 \leq t \ll 1.$$

If u is touched by $\psi \in C^1$ at x from above ($u - \psi$ attains a maximum at x), then

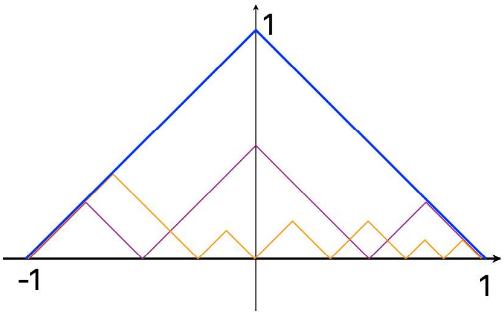
$$\psi(x) = u(x) \leq \inf_{|s| \leq t} \psi(x + s) + t \leq \psi(x) + \inf_{|s| \leq t} \psi_x(x)s + t + o(t)$$

$$\Rightarrow \ 0 \leq -|\psi_x(x)|t + t + o(t) \ \Rightarrow \ |\psi_x(x)| \leq 1.$$

Likewise, we can also get that $|\psi_x(x)| \geq 1$ if u is touched by $\psi \in C^1$ at x from below. The definition of viscosity solution to (E) is given as:

- An upper semicontinuous function u is called a viscosity subsolution to (E) if $|\psi_x(x)| \leq 1$ for all $\psi \in C^1(\Omega)$ such that $u - \psi$ attains a maximum at $x \in \Omega$;
- A lower semicontinuous function u is called a viscosity supersolution to (E) if $|\psi_x(x)| \geq 1$ for all $\psi \in C^1(\Omega)$ such that $u - \psi$ attains a minimum at $x \in \Omega$.

A function is called a viscosity soution to (E) if u is both a subsolution and a supersolution. The unique viscosity solution to (E) satisfying the boundary condition $u(-1) = u(1) = 0$ is $u(x) = 1 - |x|$ for $x \in [-1, 1]$. As shown by the top curve in the picture below.



The Euclidean definition of viscosity solution relies on a pointwise test using C^1 functions which are not available in general metric spaces. We are thus motivated to find a characterization of this solution requiring less structure of the underlying space.

One may attempt to use the local Lipschitz constant (or local slope) of a function u defined as

$$|\nabla u|(x) = \limsup_{y \to x} \frac{|u(y) - u(x)|}{d(x, y)},$$

and requires the solution to satisfy $|\nabla u|(x) = 1$ pointwise. However, as one can see from the example listed above, there are infinitely many functions satisfying this condition, as indicated by the different curves in the above picture. In other word, this definition of solution does not provide the uniqueness.

Then we consider the subslope of a locally Lipschitz function defined as

$$|\nabla^- u|(x) = \limsup_{y \to x} \frac{[u(y) - u(x)]_-}{d(x, y)}.$$

Here, $[a]_- := \max\{-a, 0\}$. Clearly, if u attains local minimum at $x_0 \in \Omega$, then $|\nabla^- u|(x_0) = 0$. In the above example, $u(x) = 1 - |x|$ is the only function satisfying that $|\nabla^- u|(x) = 1$ pointwise in the domain. Based on this observation, we call a function the Monge solution to (E) if

$$|\nabla^- u|(x) = \limsup_{y \to x} \frac{[u(y) - u(x)]_-}{d(x, y)} = 1 \qquad \text{for all } x \in \Omega.$$

One can show this definition is equivalent to classical viscosity solution in the Euclidean space. Since this definition involves only the distance of the underlying space, it can be generalized to metric spaces immediately.

References

[AF] L. Ambrosio and J. Feng. On a class of first order Hamilton-Jacobi equations in metric spaces. *J. Differential Equations*, 256(7):2194–2245, 2014.

[BC] M. Bardi and I. Capuzzo-Dolcetta. *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*. Systems & Control: Foundations & Applications. Birkhäuser Boston Inc., Boston, MA, 1997. With appendices by Maurizio Falcone and Pierpaolo Soravia.

[CIL] M. G. Crandall, H. Ishii, and P.-L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.)*, 27(1):1–67, 1992.

[GS] W. Gangbo and A. Święch. Metric viscosity solutions of Hamilton-Jacobi equations depending on local slopes. *Calc. Var. Partial Differential Equations*, 54(1):1183–1218, 2015.

[GHN] Y. Giga, N. Hamamuki, and A. Nakayasu. Eikonal equations in metric spaces. *Trans. Amer. Math. Soc.*, 367(1):49–66, 2015.

[LSZ] Q. Liu, N. Shanmugalingam, and X. Zhou. Equivalence of solutions of eikonal equation in metric spaces. *J. Differential Equations*, 272 (2021), 979-1014.

[NS] R. T. Newcomb, II and J. Su. Eikonal equations with discontinuities. *Differential Integral Equations*, 8(8):1947–1960, 1995.

Xiaodan Zhou, Analysis on metric spaces unit, Okinawa Institute of Science and Technology Graduate University, Okinawa 904-0495, Japan, xiaodan.zhou@oist.jp