Improved error estimates for the Davenport–Heilbronn theorems

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Abstract

This is a résumé of the preprint [BTT] of Manjul Bhargava, Frank Thorne and the author, based on the author's talk at RIMS conference[1].

1 Introduction

The purpose of this article is to give an outline of the proof the following theorem, obtained by Bhargava, Thorne and the author [BTT, Theorem 1.1]:

Theorem 1 Let $N_3^{\pm}(X)$ denote the number of isomorphism classes of cubic fields F satisfying $0 < \pm\mathrm{Disc}(F) < X$. Then

$$N_3^{\pm}(X) = C^{\pm}\frac{1}{12\zeta(3)}X + K^{\pm}\frac{4\zeta(1/3)}{5\Gamma(2/3)^3\zeta(5/3)}X^{5/6} + O(X^{2/3+\epsilon})$$

where $C^+ = 1$, $C^- = 3$, $K^+ = 1$, and $K^- = \sqrt{3}$.

We briefly recall the history of this counting problem: The first main term is due to Davenport and Heilbronn [DH71], while the second main term was conjectured by Datskovsky and Wright [DW88, p. 125] and Roberts [Rob01] and proven in [BST13] and [TT13b]. The latter works in turn built on the successively improved error terms obtained in [DH71], [Bel97], and [BBP10].

We also obtain a variation of Theorem 1 that counts isomorphism classes of cubic fields satisfying certain specified sets of *local conditions*. For details, see [BTT, Theorems 1.4, 1.5].

2 Binary cubic forms and cubic rings

A *cubic ring* is a unitary commutative ring that is free of rank three as a $\mathbb{Z}$-module. Its *discriminant* is the determinant of the trace form $\langle x, y\rangle = \mathrm{Tr}(xy)$. The lattice of *integral binary cubic forms* is defined by $V(\mathbb{Z}) := \{au^3 + bu^2v + cuv^2 + dv^3 \mid a, b, c, d \in \mathbb{Z}\}$, and the *discriminant* of $f(u, v) = au^3 + bu^2v + cuv^2 + dv^3 \in V(\mathbb{Z})$ is defined by $\mathrm{Disc}(f) = b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd$. The group $\mathrm{GL}_2(\mathbb{Z})$ acts on $V(\mathbb{Z})$ by $(\gamma \cdot f)(u, v) = f((u, v) \cdot \gamma)/(\det \gamma)$.

The correspondence of Levi [Lev14] and Delone–Faddeev [DF64], as further extended by Gan, Gross, and Savin [GGS02] to include the degenerate case, is as follows:

Theorem 2 ([Lev14, DF64, GGS02]) There is a canonical, discriminant-preserving bijection between the set of $\mathrm{GL}_2(\mathbb{Z})$-orbits on $V(\mathbb{Z})$ and the set of isomorphism classes of cubic rings.

[1]RIMS Workshop 2022 "Analytic Number Theory and Related Topics", October 11 (Tue) – 14 (Fri), 2022.

We want to count cubic fields, which is equivalent to count maximal cubic domains. Let $N^{\pm}_{\max}(X)$ counts the number of maximal cubic rings R satisfying $0 < \pm\mathrm{Disc}(R) < X$. For a squarefree integer q, let $N^{\pm}(X;q)$ denotes the number of cubic rings R satisfying $0 < \pm\mathrm{Disc}(R) < X$ which are non maximal at all prime divisors of q. Then by inclusion-exclusion, we have

$$N^{\pm}_{\max}(X) = \sum_{q} N^{\pm}(X;q) = \sum_{q<Q} N^{\pm}(X;q) + \sum_{q\geq Q} N^{\pm}(X;q). \tag{1}$$

The latter sum is $O(X/Q^{1-\epsilon})$, since $N^{\pm}(X;q) = O(X/q^{2-\epsilon})$.

We use the correspondence of Theorem 2 to analyze $N^{\pm}(X;q)$ for small q. Let $\Psi_q \colon V(\mathbb{Z}) \to \{0,1\}$ be the indicator functions of cubic rings that are non maximal at all prime divisors of q. It was proved in [DH71] that Ψ_q factors through the reduction map $V(\mathbb{Z}) \to V(\mathbb{Z}/q^2\mathbb{Z})$.

3 Shintani zeta fucntions and Landau's method

Our proof will apply the theory of *Shintani zeta functions* [Shi72] associated with the space of binary cubic forms. Shintani introduced the following Dirichelt series:

$$\xi^{\pm}(s) := \sum_{\substack{x\in \mathrm{GL}_2(\mathbb{Z})\backslash V(\mathbb{Z})\\ \pm\mathrm{Disc}(x)>0}} \frac{|\mathrm{Stab}(x)|^{-1}}{|\mathrm{Disc}(x)|^s}.$$

More generally, we consider

$$\xi^{\pm}_q(s) := \sum_{\substack{x\in \mathrm{GL}_2(\mathbb{Z})\backslash V(\mathbb{Z})\\ \pm\mathrm{Disc}(x)>0}} \Psi_q(x)\frac{|\mathrm{Stab}(x)|^{-1}}{|\mathrm{Disc}(x)|^s} =: \sum_{n\geq 1}\frac{a^{\pm}_q(n)}{n^s}.$$

Then $N^{\pm}(X;q) = \sum_{n<X} a^{\pm}_q(n)$. By applying Perron's formula,

$$N^{\pm}(X;q) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \xi^{\pm}_q(s)\frac{X^s}{s}ds.$$

The Shintani zeta function $\xi^{\pm}_q(s)$ enjoys an analytic continuation and a functional equation. It has simple poles at $s=1$ and $5/6$, and the explicit formulas of the respective residues $r^{\pm}_1(q)$ and $r^{\pm}_{5/6}(q)$ are obtained by Datskovsky and Wright [DW86, Proposition 5.3 and Theorem 6.2]. Let

$$E^{\pm}(X;q) := N^{\pm}(X;q) - r^{\pm}_1(q)X - r^{\pm}_{5/6}(q)\frac{X^{5/6}}{5/6}.$$

If we formally shift the contour to the left, we get these two main terms by the residue theorem and also get an integral expression of $E^{\pm}(X;q)$. There is a convergence problem, so we consider the Riesz mean. Then we come back the original count by the finite differencing. This was established in the classical work of Landau [Lan12, Lan15], and we have $E^{\pm}(X;q) = O_q(X^{3/5})$ where the implied constant depends on q. For our purpose we need an estimate of $E^{\pm}(X;q)$ uniform in q, which we now discuss.

4 Uniform Landau and estimating the error term

Let

$$\hat{d}(\Psi_q) := q^8 \cdot \sup_N \frac{1}{N} \sum_{\substack{y\in \mathrm{GL}_2(\mathbb{Z})\backslash V^*(\mathbb{Z})\\ 0<|\mathrm{Disc}^*(y)|<N}} |\widehat{\Psi_q}(y)|.$$

Here V^* is the dual space and Disc^* is an invariant polynomial on V^*. The following estimate follows from [LDTT22] by Lowry-Duda, Thorne and the author:

Theorem 3 ([LDTT22]) We have

$$\sum_{q\in[Q,2Q]} |E(X;q)| \ll X^{3/5} \left(\sum_{q\in[Q,2Q]} r_1(q)\right)^{3/5} \left(\sum_{q\in[Q,2Q]} \hat{d}(\Psi_q)\right)^{2/5},$$

provided that

$$\left(\sum_{q\in[Q,2Q]} \hat{d}(\Psi_q)\right)^{2/5} \ll X \left(\sum_{q\in[Q,2Q]} r_1(q)\right)^{3/5}.$$

We know $r_1(q) \asymp q^{-2}$, so the problem is reduced to establish an estimate of

$$\frac{1}{N} \sum_{q\in[Q,2Q]} \sum_{\substack{y\in \mathrm{GL}_2(\mathbb{Z})\backslash V^*(\mathbb{Z}) \\ 0<|\mathrm{Disc}^*(y)|<N}} |\widehat{\Psi_q}(y)|. \tag{2}$$

By the Chinese remainder theorem, $\widehat{\Psi_q} = \prod_{p|q} \widehat{\Psi_p}$. Thorne and the author [TT13a] gave an explicit formula of $\widehat{\Psi_p}$, and in particular obtained the following upper bound:

Propositoin 4 ([TT13a]) We have

$$\widehat{\Psi_p}(y) \ll \begin{cases} p^{-2} & p^2 \mid y, \\ p^{-3} & p^4 \mid \mathrm{Disc}^*(y), \\ p^{-4} & p^3 \mid \mathrm{Disc}^*(y), \\ p^{-5} & p^2 \mid \mathrm{Disc}^*(y). \end{cases}$$

Moreover, $\widehat{\Psi_p}(y) = 0$ if $p^2 \nmid \mathrm{Disc}^*(y)$.

This proposition shows that the function $\widehat{\Psi_p}$ takes mostly quite small values, and has a thin support. Thus by switching the sum in (2), we can estimate the quantity rather effectively. As a consequence we have $\ll Q^{-6+\epsilon}$ for (2). Thus the total error in the squarefree sieve (1) is $\ll X/Q^{1-\epsilon} + X^{3/5}Q^{1/5+\epsilon}$. Choosing $Q = X^{1/3-\epsilon'}$, we have the desired bound.

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References

[Bel97] K. Belabas. A fast algorithm to compute cubic fields. *Math. Comp.*, 66(219):1213–1237, 1997.

[BBP10] Karim Belabas, Manjul Bhargava, and Carl Pomerance. Error estimates for the Davenport-Heilbronn theorems. *Duke Math. J.*, 153(1):173–210, 2010.

[BST13] Manjul Bhargava, Arul Shankar, and Jacob Tsimerman. On the Davenport-Heilbronn theorems and second order terms. *Invent. Math.*, 193(2):439–499, 2013.

[BTT] Manjul Bhargava, Takashi Taniguchi and Frank Thorne, Improved error estimates for the Davenport-Heilbronn theorems, preprint, 2021, `https://arxiv.org/abs/2107.12819`.

[DF64] B. N. Delone and D. K. Faddeev. *The theory of irrationalities of the third degree.* Translations of Mathematical Monographs, Vol. 10. American Mathematical Society, Providence, R.I., 1964.

[DH71] H. Davenport and H. Heilbronn. On the density of discriminants of cubic fields. II. *Proc. Roy. Soc. London Ser. A*, 322(1551):405–420, 1971.

[DW86] Boris Datskovsky and David J. Wright. The adelic zeta function associated to the space of binary cubic forms. II. Local theory. *J. Reine Angew. Math.*, 367:27–75, 1986.

[DW88] Boris Datskovsky and David J. Wright. Density of discriminants of cubic extensions. *J. Reine Angew. Math.*, 386:116–138, 1988.

[GGS02] Wee Teck Gan, Benedict Gross, and Gordan Savin. Fourier coefficients of modular forms on G_2. *Duke Math. J.*, 115(1):105–169, 2002.

[Lan12] Edmund Landau. Über die Anzahl der Gitterpunkte in gewissen Bereichen. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, pages 687–770, 1912.

[Lan15] Edmund Landau. Über die Anzahl der Gitterpunkte in gewissen Bereichen. Zweite abhandlung. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, pages 209–243, 1915.

[LDTT22] David Lowry-Duda, Takashi Taniguchi, and Frank Thorne. Uniform bounds for lattice point counting and partial sums of zeta functions. *Math. Z.*, 300(3):2571–2590, 2022.

[Lev14] F. Levi. Kubische Zahlkörper und binäre kubische Formenklassen. *Leipz. Ber.*, 66:26–37, 1914.

[Rob01] David P. Roberts. Density of cubic field discriminants. *Math. Comp.*, 70(236):1699–1705, 2001.

[Shi72] Takuro Shintani. On Dirichlet series whose coefficients are class numbers of integral binary cubic forms. *J. Math. Soc. Japan*, 24:132–188, 1972.

[TT13a] Takashi Taniguchi and Frank Thorne. Orbital L-functions for the space of binary cubic forms. *Canad. J. Math.*, 65(6):1320–1383, 2013.

[TT13b] Takashi Taniguchi and Frank Thorne. Secondary terms in counting functions for cubic fields. *Duke Math. J.*, 162(13):2451–2508, 2013.