TRANSCENDENTAL PARAMETERS OF ANALOGS OF SQUARES FOR SOLVING A CERTAIN SYSTEM OF EQUATIONS

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ABSTRACT. Let $S(\alpha) = \{\lfloor \alpha n^2 \rfloor : n = 1, 2, \ldots\} \setminus \{0\}$. Kanado and the author showed that for all rational numbers $\alpha \in (0,1)$ we can find infinitely many tuples (k, ℓ, m) of positive integers such that all of k, ℓ, m, $k+\ell$, $\ell+m$, $m+k$, $k+\ell+m$ are in $S(\alpha)$. In this short article, we construct a transcendental number $\alpha \in (0,1)$ satisfying this relation.

1. Introduction

Let $\mathbb{N}$ be the set of all positive integers. A rectangular cuboid is called an *Euler brick* if the edges and face diagonals have integral lengths. Further, an Euler brick is called a *perfect Euler brick* if the space diagonal also has integral length. It is known that there are infinitely many Euler bricks (see OEIS A031173, A031174, and A031175). However, the existence (or non-existence) of a perfect Euler brick is unknown. By the Pythagorean theorem, a perfect Euler brick exists if and only if there exists $(k, \ell, m) \in \mathbb{N}^3$ such that all of

$$k, \quad \ell, \quad m, \quad k+\ell, \quad \ell+m, \quad m+k, \quad k+\ell+m \tag{1.1}$$

are perfect squares. Instead of squares, Glasscock investigated Piatetski-Shapiro sequences [Gla17]. Let $\lfloor x \rfloor$ denote the integer part of x for all $x \in \mathbb{R}$. A sequence of positive integers of the form $\lfloor n^\alpha \rfloor$ is called a Piatetski-Shapiro sequence. Let $\mathrm{PS}(\alpha) = \{\lfloor n^\alpha \rfloor : n \in \mathbb{N}\}$. For a given set $X \subseteq \mathbb{N}$, we define $T(X)$ as the set of all tuples $(k, \ell, m) \in \mathbb{N}^3$ with $k \le \ell \le m$ such that all of (1.1) belong to X. Further, we say that X satisfies the *infinite PEB conditions* if $\#T(X) = \infty$.

Interestingly, Glasscock found that $\mathrm{PS}(\alpha)$ satisfies the infinite PEB conditions for almost all $\alpha \in (1,2)$ [Gla17, Corollary 1]. Note that $\mathrm{PS}(2)$ is equal to the set of all perfect squares. The author improved on this finding, showing that Glasscock's result remains true even if we replace "for almost all" with "for all"; that is, $\mathrm{PS}(\alpha)$ satisfies the infinite PEB conditions for all $\alpha \in (1,2)$ [Sai22, Corollary 1.2]. However, the gaps between $\lfloor n^\alpha \rfloor$ and n^2 are very large. Indeed, it is observed that for every fixed $\alpha \in (1,2)$

$$n^2/n^\alpha \to \infty \quad (\text{as } n \to \infty).$$

For solving this problem, Kanado and the author studied a set closer to squares satisfying the infinite PEB conditions. They proposed the following set: for every $\alpha \in (0,1)$

$$S(\alpha) := \{\lfloor \alpha n^2 \rfloor : n = 1, 2, \ldots\} \setminus \{0\}.$$

They showed

1. for all $\alpha \in (0,1) \cap \mathbb{Q}$, $S(\alpha)$ satisfies the infinite PEB conditions;
2. for almost all $\alpha \in (0,1)$, $S(\alpha)$ satisfies the infinite PEB conditions;
3. if $T(S(\alpha))$ is finite for some $\alpha \in (0,1)$, then there is no perfect Euler brick.

These results can be seen in [KS, Theorem 1.2, Theorem 1.4, Theorem 1.6], respectively. In the previous research, we did not get any concrete examples of irrational $\alpha \in (0,1)$ such that $S(\alpha)$ satisfies the infinite PEB conditions. In this short article, we concrete such α.

Let $C \in \mathbb{N}$ be a large integer which determined later. We define $h = h_C : \mathbb{N} \to \mathbb{N}$ as

$$h(1) = C, \quad h(n+1) = C^{h(n)} \quad (n = 1, 2, \ldots).$$

Then we obtain the following result:

Theorem 1.1. *Let $C \geq 21410$ be an integer, and let $\gamma = \sum_{n=1}^{\infty} C^{-h(n)}$. Then γ is a Liouville number and $S(\gamma)$ satisfies the infinite PEB conditions.*

Note that for every integer $C \geq 2$ the series $\sum_{n=1}^{\infty} C^{-h(n)}$ converges in $(0,1)$. Indeed, $\gamma > 0$ is trivial. By Lemma 2.1, $h(n) \geq 2h(n-1) \geq \cdots \geq 2^{n-1}h(1) \geq 2^n$. Therefore, $h(n) > n$. We observe that

$$\sum_{n=1}^{\infty} C^{-h(n)} < \sum_{n=1}^{\infty} 2^{-h(n)} < \sum_{n=1}^{\infty} 2^{-n} = 1.$$

Hence, γ exists and belongs to $(0,1)$.

Remark 1.2. A real number α is called a *Liouville number* if for all $n \in \mathbb{N}$ there exist $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that

$$0 < \left| \alpha - \frac{p}{q} \right| < \frac{1}{q^n}. \tag{1.2}$$

It is well-known that every Liouville number is transcendental.

Notation 1.3. For $x \in \mathbb{R}$, let $\{x\}$ denote the fractional part of x. For all intervals $I \subset \mathbb{R}$, let $I_{\mathbb{Z}} = I \cap \mathbb{Z}$. For all $x \in \mathbb{R}$, let $\lceil x \rceil = -\lfloor -x \rfloor$, which means the least integer greater than or equal to x.

2. Lemmas

Lemma 2.1. *Let $C \geq 2$. Then $h(n+1) \geq 2h(n)$ for all $n \in \mathbb{N}$.*

Proof. If $n = 1$, then $2h(1) = 2C \leq C \cdot C = C^2 \leq C^C = h(2)$. Assume that

$$h(n+1) \geq 2h(n)$$

for some $n \in \mathbb{N}$. Then since $2x \leq x^2$ for all $x \geq 2$, we have

$$2h(n+1) = 2C^{h(n)} \leq 2C^{h(n+1)/2} \leq C^{h(n+1)} = h(n+2).$$

□

Lemma 2.2. *There exist positive integers $q_1 < q_2 < \cdots$ and $p_1, p_2, \ldots$ such that for all $n \in \mathbb{N}$,*

$$0 < \gamma - \frac{p_n}{q_n} < \frac{2}{C^{q_n}}. \tag{2.1}$$

Proof. For sufficiently large $n \in \mathbb{N}$, we have

$$0 < \gamma - \sum_{j=1}^{n} \frac{1}{C^{h(j)}} = \sum_{j=n+1}^{\infty} C^{-h(j)} = C^{-h(n+1)} \left(1 + \sum_{j=n+2}^{\infty} C^{-h(j)+h(N+1)} \right) \tag{2.2}$$

By Lemma 2.1, we obtain $h(j) \geq h(n+2) \geq 2h(n+1)$ for every $j \geq n+2$. Therefore the most right-hand side of (2.2) is

$$< C^{-h(n+1)}\left(1 + \sum_{j=n+2}^{\infty} C^{-h(j)/2}\right) \leq 2C^{-h(n+1)}$$

for sufficiently large $n \in \mathbb{N}$, since $\sum_{j=1}^{\infty} C^{-h(j)/2}$ is convergent series. Since C is a positive integer, setting $q_n = h(n+1)(= C^{h(n)})$, there exists $p_n \in \mathbb{Z}$ such that $\sum_{j=1}^{n} 1/C^{h(j)} = p_n/q_n$. Hence, we conclude $0 < \gamma - p_n/q_n < 2C^{-q_n}$. □

Lemma 2.3. *Let $(P_n)_{n=0}^{\infty}$ be the sequence defined by*

$$P_0 = 0,\ P_1 = 1,\ P_{n+2} = 2P_{n+1} + P_n \quad (n = 0, 1, 2, \ldots).$$

Then for every $n \in \mathbb{Z}_{\geq 0}$, $P_n = \frac{1}{2\sqrt{2}}(\phi_P^n - (-\phi_P)^{-n})$, where $\phi_P = 1 + \sqrt{2}$. Further, for all integers $1 \leq p < q$, there exists $r \in [2, \lceil \sqrt{2}q \rceil]_{\mathbb{Z}}$ such that

$$\left(\left\lfloor \frac{p}{q}\, P_{2r}^2 \right\rfloor, \left\lfloor \frac{p}{q}\, P_{2r}^2 \right\rfloor, \left\lfloor \frac{p}{q}\, (P_{2r}^2/2 - 1)^2 \right\rfloor\right) \in T(S(p/q)), \tag{2.3}$$

$$\left\{\frac{p}{q}\, P_{2r}^2\right\} = 0, \quad \left\{\frac{p}{q}\, (P_{2r}^2/2 - 1)^2\right\} = \frac{p}{q}. \tag{2.4}$$

Proof. See [KS, (2.3), Lemma 3.1, Lemma 3.3, Proof of Theorem 1.2]. □

3. Proof of Theorem 1.1

Let $C \geq 21410$ be an integer, and γ be in Theorem 1.1. By Lemma 2.2, there exist positive integers $q_1 < q_2 < \cdots$ and $p_1, p_2, \ldots$ satisfying (2.1). It is clear that γ is a Liouville number. Indeed, fix any $N \in \mathbb{N}$. For sufficiently large $n \in \mathbb{N}$, one has $C^{q_n} \geq 2q_n^N$. Therefore,

$$0 < \left|\gamma - \frac{p_n}{q_n}\right| < \frac{2}{C^{q_n}} \leq \frac{1}{q_n^N},$$

which implies that γ is a Liouville number.

Let $r_n = r(q_n)$ be as in Lemma 2.3. Let us show that for every sufficiently large $n \in \mathbb{N}$

$$\left\lfloor \frac{p_n}{q_n}\, P_{2r_n}^2 \right\rfloor = \left\lfloor \gamma\, P_{2r_n}^2 \right\rfloor, \tag{3.1}$$

$$\left\lfloor \frac{p_n}{q_n}\, (P_{2r_n}^2/2 - 1)^2 \right\rfloor = \left\lfloor \gamma\, (P_{2r_n}^2/2 - 1)^2 \right\rfloor. \tag{3.2}$$

By combining (2.3), (3.1), and (3.2), we conclude that $T(S(\gamma))$ is infinite.

Let us fix any sufficiently large $n \in \mathbb{N}$. Let $p = p_n$, $q = q_n$, and $r = r_n$. It follows that

$$\frac{p}{q}\, P_{2r}^2 < \gamma P_{2r}^2 < \frac{p}{q}\, P_{2r}^2 + \frac{1}{C^q} P_{2r}^2.$$

By Lemma 2.3, $0 < C^{-q}P_{2r}^2 \leq C^{-q}(1+\sqrt{2})^{4r} \leq C^{-q}(1+\sqrt{2})^{4\sqrt{2}q+4}$. From numerical calculation, we see that $(1+\sqrt{2})^{4\sqrt{2}} < 147 \leq C$. Therefore, if n is sufficiently large, then $0 < C^{-q}P_{2r}^2 < 1$. By (2.4), we obtain (3.1).

The remaining part is to prove (3.2). We see that

$$\frac{p}{q}\, (P_{2r}^2/2 - 1)^2 < \gamma (P_{2r}^2/2 - 1)^2 < \frac{p}{q}\, (P_{2r}^2/2 - 1)^2 + \frac{1}{C^q}(P_{2r}^2/2 - 1)^2.$$

The fractional part of the most left-hand side is p/q. Therefore, it suffices to show that $p/q + C^{-q}(P_{2r}^2/2-1)^2 < 1$. By Lemma 2.3,

$$p/q + C^{-q}(P_{2r}^2/2-1)^2 < \gamma + C^{-q}(1+\sqrt{2})^{8r} \le \gamma + C^{-q}(1+\sqrt{2})^{8\sqrt{2}q+8}.$$

From numerical calculation, we have $(1+\sqrt{2})^{8\sqrt{2}} < 21410 \le C$. Therefore, if n is sufficiently large, then $p/q + C^{-q}(P_{2r}^2/2-1)^2 < 1$. Therefore, we obtain (3.2). We complete the proof of Theorem 1.1.

The key point of this proof is that γ is extremely near to rational numbers. If $\gamma \in (0,1)$ is an algebraic irrational number, then γ is not near to rational numbers from Roth's theorem. Thus, we lastly propose the following question.

Question 3.1. *Can we construct an algebraic irrational $\gamma \in (0,1)$ such that $S(\gamma)$ satisfies the infinite PEB conditions?*

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