AN AVERAGE MANIN CONJECTURE WITH WEAK APPROXIMATION ON FANO HYPERSURFACES

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1. Introduction

In this note, we shall count the rational points on hypersurfaces in $\mathbb{P}^n(\mathbb{Q})$ defined over $\mathbb{Q}$. In other words, we are interested in the rational (or integral) solutions of the homogeneous equation

$$V\colon\quad f_{\mathbf{a}}(\mathbf{x}) \coloneqq \sum_{M\in\mathscr{M}_{d,n}} a_M M(\mathbf{X}) = 0, \tag{1}$$

where

- This equation is in $n+1$ variables $\mathbf{X}=(X_0,\dots,X_n)$ and of degree d.
- The set $\mathscr{M}_{d,n}$ is the set of all monomials
$$\mathscr{M}_{d,n} \coloneqq \{X_0^{i_0}\cdots X_n^{i_n} \mid i_0,\dots,i_n\in\mathbb{Z}_{\geq 0},\ i_0+\cdots+i_n=d\}$$
of degree d in the $(n+1)$-variables $\mathbf{X}=(X_0,\dots,X_n)$.
- The coefficient $\mathbf{a}=(a_M)_{M\in\mathscr{M}_{d,n}}$ is given by integers, i.e. $\mathbf{a}\in\mathbb{Z}^{\mathscr{M}_{d,n}}$.

By using the Veronese embedding

$$\nu_{d,n}\colon \mathbb{R}^{n+1}\to\mathbb{R}^{\mathscr{M}_{d,n}};\ \mathbf{x}\mapsto (M(\mathbf{x}))_{M\in\mathscr{M}_{d,n}},$$

we can write the equation (1) compactly as

$$f_{\mathbf{a}}(\mathbf{x}) = \langle \mathbf{a}, \nu_{d,n}(\mathbf{X})\rangle = 0,$$

where $\langle\bullet,\bullet\rangle$ is the usual Euclidean inner product.

We write $\overline{\mathbf{x}}$ for the image of the standard projection $K^{n+1}\to\mathbb{P}^n(K)$ of $\mathbf{x}\in K^{n+1}$. In this note, unless otherwise specified, for a given rational point of $\overline{\mathbf{x}}\in\mathbb{P}^n(\mathbb{Q})$, we associate the primitive vector $\mathbf{x}\in\mathbb{Z}^{n+1}_{\mathrm{prim}}$ which gives the homogeneous coordinate of $\overline{\mathbf{x}}$, i.e. $\overline{\mathbf{x}}=[x_0:\ldots:x_n]$. Here, the primitive vectors are the integral vectors with coordinates having the greatest common divisor 1:

$$\mathbb{Z}^N_{\mathrm{prim}} \coloneqq \{(x_1,\dots,x_N)\in\mathbb{Z}^N \mid \gcd(x_1,\dots,x_N)=1\}.$$

Note that the usual projection $\mathbb{Z}^{n+1}_{\mathrm{prim}}\to\mathbb{P}^n(\mathbb{Q})$ is then two-by-one correspondence.

To count the rational points, we use the naive height given by

$$H(\overline{\mathbf{x}}) \coloneqq \|\mathbf{x}\|,$$

where $\|\bullet\|$ is the usual Euclidean norm on $\mathbb{R}^{n+1}$.

We associate the primitive integral vector $\mathbf{a}_V$ to the hypersurface (1) which is two-by-one correspondence with the ambiguity of the sign of $\mathbf{a}_V$. Let

$$\mathbb{V}_{d,n} := \{V : \text{hypersurface of } \mathbb{P}^n(\mathbb{Q}) \text{ of degree } d \text{ defined over } \mathbb{Q}\}.$$

For $V \in \mathbb{V}_{d,n}$, our main counting function is

$$\begin{aligned} N_V(B) &:= \#\{\overline{\mathbf{x}} \in V(\mathbb{Q}) \mid H(\overline{\mathbf{x}}) \leq B\} \\ &= \tfrac{1}{2}\#\{\mathbf{x} \in \mathbb{Z}^{n+1}_{\mathrm{prim}} \mid \|\mathbf{x}\| \leq B \text{ and } \langle \mathbf{a}, \nu_{d,n}(\mathbf{x})\rangle = 0\}, \end{aligned}$$

which counts the rational points on V of height $\leq B$.

2. The Manin–Peyre conjecture for $N_V(B)$

What can we expect for the behavior of $N_V(B)$? In this section, we shall see some expectations on the behavior of $N_V(B)$.

We first look at a rough probabilistic heuristics:

Heuristics 1 (cf. 1.4 of Chapter II of [5]). *For some constant c, we have*

$$N_V(B) \approx \frac{c}{\|\mathbf{a}_V\|} B^{n+1-d} \quad (B \to \infty),$$

where $\approx$ stands for "heuristical" equality.

Quasi-Proof. Consider the set $\mathscr{X} \subseteq \mathbb{Z}^{n+1}$ of all primitive vectors of height $\leq B$. We have roughly $\#\mathscr{X} \asymp B^{n+1}$ such primitive vectors. For each of $\mathbf{x} \in \mathscr{X}$, we have

$$\langle \mathbf{a}_V, \nu_{d,n}(\mathbf{x})\rangle \in [-c_1\|\mathbf{a}_V\|B^d, +c_1\|\mathbf{a}_V\|B^d] \tag{2}$$

for some constant c_1 by the Cauchy–Schwarz inequality and $\nu_{d,n}(\mathbf{x}) \asymp \|\mathbf{x}\|^d$. We now make a very rough assumption that the values $\langle \mathbf{a}_V, \nu_{d,n}(\mathbf{x})\rangle$ are well-distributed in the interval in (2). We then have $\langle \mathbf{a}_V, \nu_{d,n}(\mathbf{x})\rangle = 0$ with the probability $\asymp (\|\mathbf{a}_V\|B^d)^{-1}$. Therefore, we can expect

$$N_V(B) \approx \#\mathscr{X} \cdot \mathrm{Prob}(\langle \mathbf{a}_V, \nu_{d,n}(\mathbf{x})\rangle = 0) \approx \frac{c}{\|\mathbf{a}_V\|} B^{n+1-d}.$$

This is the claimed approximation. □

Heuristics 1 is too rough but shows us the importance of the exponent $n + 1 - d$. It is empirical rule that probabilistic heuristics is close to the truth only if the main term of the asymptotic formula has a certain amount of magnitude. We thus expect Heuristics 1 gives a nice approximation only when $n \geq d$. This range is corresponding to the range where our hypersurface V satisfies the condition so called being "Fano". In this note, we always assume that we are in the Fano range $n \geq d$ and also exclude the linear case and the case $(n, d) = (2, 2)$:

$$n \geq d \geq 2 \quad \text{and} \quad (n, d) \neq (2, 2).$$

The case $n = d$ and $n < d$ are corresponding to the case where V is of intermediate type and of general type, respectively, and in these cases, we have another type of conjectures of Batyrev–Manin and of Bombieri–Lang. (See Chapter II of [5].)

When V is Fano, we can apply the circle method formally to derive a more detailed hypothetical asymptotic formula, which coincides with the special case of the Manin–Peyre conjecture:

Conjecture 1 (Manin–Peyre conjecture for $N_V(B)$)**.** We have

$$N_V(B) = \mathfrak{S}(V)B^{n+1-d} + (\text{error}),$$

for smooth $V \in \mathbb{V}_{d,n}$ with $n \geq d$, where the singular series $\mathfrak{S}(V)$ is given by an Euler product. (For the detailed description of $\mathfrak{S}(V)$ and the general case of the Manin–Peyre conjecture, see Chapter II of [5].)

The factor $1/\|\mathbf{a}_V\|$ in Heuristics 1 is not visible in Conjecture 1 but we can expect that the archimedean factor in the singular series $\mathfrak{S}(V)$ is of the size $\asymp 1/\|\mathbf{a}_V\|$. Also, the "thinset" and the logarithmic factor in the general Manin–Peyre conjecture does not exists for most of the hypersurfaces V and so we ignored them (see the paragraph after formula (2.5) of [3]).

Indeed, when n is sufficiently larger than d, we can apply the circle method rigorously to prove the Manin–Peyre conjecture, which is the famous result of Birch [2]:

Theorem 1 (Birch [2])**.** *For a smooth $V \in \mathbb{V}_{d,n}$ with $n \geq 2^d(d-1)$, we have*

$$N_V(B) = \mathfrak{S}(V)B^{n+1-d} + O(B^{n+1-d-\delta})$$

with some positive constant $\delta > 0$.

Thus, the Manin–Peyre conjecture is proved for hypersurfaces in Theorem 1, but we need a very restrictive assumption that $n \geq 2^d(d-1)$. Thus, Theorem 1 cannot cover the remaining range $d \leq n < 2^d(d-1)$ of Conjecture 1. The power of 2 in Theorem 1 is coming from the application of the Weyl or van der Corput differencing, which are unavoidable today because the relevant exponential sum is not in flexible form as in the case of the Waring problem, etc.

3. Average Manin conjecture — le Boudec's result

As we saw above, we still have a big obstacle to prove the Manin–Peyre conjecture for all hypersurfaces in the Fano range $n \geq d$. However, recently, le Boudec [7] considered this problem with the average over hypersurfaces. Write

$$N_{d,n} \coloneqq \#\mathcal{M}_{d,n} = \binom{n+d}{d}, \quad \mathcal{B}_N(X) \coloneqq \{\mathbf{x} \in \mathbb{R}^N \mid \|\mathbf{x}\| \leq X\}, \quad V_N \coloneqq \mathrm{Vol}(\mathcal{B}_N(1))$$

and

$$\mathbb{V}_{d,n}(A) \coloneqq \{V \in \mathbb{V}_{d,n} \mid \|\mathbf{a}_V\| \leq A\}.$$

The result of le Boudec is the following:

Theorem 2 (le Boudec (2022) [7, Theorem 3])**.** *For $n \geq d \geq 2$, we have*

$$\frac{1}{|\mathbb{V}_{d,n}(A)|}\sum_{V \in \mathbb{V}_{d,n}(A)} N_V(B) = C_{d,n}\frac{B^{n+1-d}}{A}\left(1 + O\left(\frac{B}{A} + \frac{\log B}{B}\right)\right) \tag{3}$$

provided $A \geq B$, where the constant $C_{d,n}$ is given by

$$C_{d,n} \coloneqq \frac{1}{2\zeta(n+1)}\frac{V_{N_{d,n}-1}}{V_{N_{d,n}}}\frac{\zeta(N_{d,n})}{\zeta(N_{d,n}-1)}\int_{\mathcal{B}_{n+1}(1)}\frac{d\mathbf{x}}{\|\nu_{d,n}(\mathbf{x})\|}$$

and the implicit constant depends on d, n.

The left-hand side of (3) can be thought as the average of $N_V(B)$ over V with the "height" up to A. The right-hand side has the form $C_{d,n} \cdot B^{n+1-d}/A$, which reminds us the main term of Heuristics 1 and Conjecture 1. Thus, we can regard le Boudec's result Theorem 2 as the proof of the Manin–Peyre conjecture *on average* over V. Notably, le Boudec achieved the result in the full (!) Fano range $n \geq d$.

Remark 1. In this note, we use the term "on average" to indicate that we consider the L^1 moment. The Manin–Peyre conjecture on average with the L^2 moment (and with a modified main term) is proved by le Boudec, Browning and Sawin [3]. Their result [3, Theorem 1.1] implies that the Hasse principle holds for almost all hypersurfaces with $n \geq d \geq 2$ and $(n,d) \neq (3,3)$, which solved a conjecture of Poonen–Voloch [9, Conjecture 2.2-(ii)] for these cases.

We can use le Boudec's result to obtain some consequences of the Manin–Peyre conjeccture for almost all hypersurface V. For example, when $B = o(A^{\frac{1}{n+1-d}})$ as $A \to \infty$, Heuristics 1 implies that $N_V(B) = o(1)$. However, $N_V(B)$ counts the *number* of rational points and so a reasonable interpretation of $N_V(B) = o(1)$ is $N_V(B) = 0$. Namely, the least height of rational points on V should be of the size $\geq \varepsilon A^{\frac{1}{n+1-d}}$:

Heuristics 2. *For sufficiently large $\|\mathbf{a}_V\|$, we have*

$$\mathfrak{M}(V) := \min\{H(\overline{\mathbf{x}}) \mid \overline{\mathbf{x}} \in V\} \geq \psi(\|\mathbf{a}_V\|^{\frac{1}{n+1-d}})$$

for any function $\psi\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ with $\psi(u) = o(u)$ as $u \to \infty$.

By Theorem 2 with taking

$$B = (\varepsilon A)^{\frac{1}{n+1-d}} \leq A,$$

we can get the following approximation of Heuristics 2:

Corollary 1 (le Boudec (2022) [7, Theorem 1])**.** *Assume $n \geq d \geq 2$. For $0 < \varepsilon < 1$,*

$$\mathfrak{M}(V) \geq (\varepsilon\|\mathbf{a}_V\|)^{\frac{1}{n+1-d}} \tag{4}$$

for all but $\ll \varepsilon|\mathbb{V}_{d,n}(A)|$ hypersurfaces $V \in \mathbb{V}_{d,n}(A)$.

Corollary 1 is the result valid for almost all V if we take $\mathbb{V}_{d,n}(A)$ as the total space. However, when V is not locally soluble, then the inequality (4) vacuously true (we use the convention $\min \varnothing = +\infty$). Therefore, if almost all $V \in \mathbb{V}_{d,n}(A)$ are not locally soluble, then Corollary 1 is meaningless as it is. Thus, it's better to take the set of locally soluble hypersurfaces

$$\mathbb{V}_{d,n}^{\mathrm{loc}}(A) := \{V \in \mathbb{V}_{d,n}(A) \mid V(\mathbb{Q}_p) \neq \varnothing\}$$

as the total space. Indeed, there is no problem to use $\mathbb{V}_{d,n}(A)$ as the total space when $(n,d) \neq (2,2)$, i.e. positive proportion of $V \in \mathbb{V}_{d,n}(A)$ are locally soluble:

Theorem 3 (Poonen–Voloch [9, Theorem 3.6])**.** *We have*

$$|\mathbb{V}_{d,n}^{\mathrm{loc}}(A)| \sim c_{d,n}|\mathbb{V}_{d,n}(A)| \quad (A \to \infty)$$

for $n \geq d \geq 2$ with $(n,d) \neq (2,2)$, where

$$c_{d,n} := \prod_{p \in M_{\mathbb{Q}}} \mathrm{Prob}(\overline{\mathbf{a}} \in \mathbb{P}^{N_{d,n}-1}(\mathbb{Q}_p) \mid \langle \mathbf{a}, \nu_{d,n}(\mathbf{X})\rangle = 0 \text{ is soluble in } \mathbb{Q}_p).$$

Remark 2. In the definition of $c_{d,n}$ above, hypersurfaces in $\mathbb{P}^n(\mathbb{Q}_p)$ are identified with the point of $\mathbb{P}^{N_{d,n}-1}(\mathbb{Q}_p)$ corresponding to the coefficient vector and the probability measure on $\mathbb{P}^{N_{d,n}-1}(\mathbb{Q}_p)$ is obtained by normalizing a p-adic canonical measure.

Remark 3. When $(n,d)=(2,2)$, the result of Serre [11, Example 4] gives $\#\mathbb{V}^{\mathrm{loc}}_{d,n}(A)=o(\mathbb{V}_{d,n}(A))$ as $A\to\infty$, i.e. almost all $V\in\mathbb{V}_{d,n}$ is not locally soluble and so the statement of Corollary 1 is not so meaningful as it is.

We can sharpen Corollary 1 by using Theorem 3 as

Corollary 2 (le Boudec)**.** *Assume $n\geq d\geq 2$ and $(n,d)\neq(2,2)$. For $0<\varepsilon<1$,*

$$\mathfrak{M}(V)\geq(\varepsilon\|\mathbf{a}_V\|)^{\frac{1}{n+1-d}}$$

for all but $\ll\varepsilon|\mathbb{V}^{\mathrm{loc}}_{d,n}(A)|$ hypersurfaces $V\in\mathbb{V}^{\mathrm{loc}}_{d,n}(A)$.

4. Counting with weak approximation

Since the Manin–Peyre conjecture is proved on average (in the L^1 moment sense) by the result of le Boudec, we now explore some finer result on the Manin–Peyre conjecture on average. For such a refinement, we focus on the following principle called *weak approximation.* Let us write $M_{\mathbb{Q}}$ for the set of all places of $\mathbb{Q}$ including the archimedean place.

Principle 1 (Weak approximation)**.** The image of the diagonal map

$$\mathrm{diag}\colon V(\mathbb{Q})\to\prod_{p\in M_{\mathbb{Q}}}V(\mathbb{Q}_p)$$

is dense with respect to the product of the p-adic topologies for "nice" $V\in\mathbb{V}_{d,n}$. (Note that the space of adelic points is now given by the usual direct product since we are using *projective* varieties V.)

This principle motivates us to count the rational points approximating a given adelic point $\boldsymbol{\xi}\in\prod_p V(\mathbb{Q}_p)$. However, since we shall take the average over hypersurface V, we use $\prod_p\mathbb{P}^n(\mathbb{Q}_p)$ as the range of adelic point $\boldsymbol{\xi}$ instead of $\prod_p V(\mathbb{Q}_p)$. We then take a (basic) neighborhood $U\subseteq\prod_p\mathbb{P}^n(\mathbb{Q}_p)$ of $\boldsymbol{\xi}$ and consider the counting function with weak approximation

$$N_V(B;U)\coloneqq\#\{\overline{\boldsymbol{x}}\in V(\mathbb{Q})\mid H(\overline{\boldsymbol{x}})\leq B\text{ and }\mathrm{diag}(\overline{\boldsymbol{x}})\in U\}.$$

When we have $N_V(B;U)>0$, it means that we have a rational point of V close to $\boldsymbol{\xi}$ and we can conclude the weak approximation at $\boldsymbol{\xi}$.

For this counting function $N_V(B;U)$, we have another more detailed principle due to Peyre [8]. Note that we can normalize a canonical p-adic measure on $V(\mathbb{Q}_p)$ to obtain a probability measure $\mu_{V,p}$ on $V(\mathbb{Q}_p)$. We then consider the product measure $\mu_V\coloneqq\prod_p\mu_{V,p}$ defined on $\prod_p V(\mathbb{Q}_p)$.

Principle 2 (Peyre [8, Subsection 3.2])**.** We have

$$N_V(B;U)\sim\mu_V(U)N_V(B)\quad(B\to\infty)$$

for any "nice" $V\in\mathbb{V}_{d,n}$ and any Borel set $U\subseteq\prod_p V(\mathbb{Q}_p)$.

What we shall try is to prove Principle 2 on average (again in the L^1 sense), i.e. to introduce the weak approximation into le Boudec's result (Theorem 2). It is relatively easy just to introduce the weak approximation and so we want to obtain the result as uniform as possible with respect to the range of weak approximation U. In order to measure such uniformity, we use a more specific family of U: We use the "adelic" balls of the projective space $\mathbb{P}^n$ defined as follows.

For $p \in M_{\mathbb{Q}}$, we define a metric d_p on $\mathbb{P}^n(\mathbb{Q}_p)$ by:

- Case I. $p < +\infty$. Take $\overline{\mathbf{x}}, \overline{\mathbf{y}} \in \mathbb{P}^n(\mathbb{Q}_p)$ with $\mathbf{x}, \mathbf{y} \in \mathbb{Q}_p^{n+1} \setminus \{0\}$. We then define

$$d_p(\overline{\mathbf{x}}, \overline{\mathbf{y}}) \coloneqq \frac{\|\mathbf{x} \wedge \mathbf{y}\|_p}{\|\mathbf{x}\|_p \cdot \|\mathbf{y}\|_p} \in [0, 1],$$

  where $\|\bullet\|$ on $\mathbb{Q}_p^{n+1}$ is the max norm with the p-adic absolute value and $\|\bullet\|$ on $\bigwedge^2 \mathbb{Q}_p^{n+1}$ is the max norm with the p-adic absolute value and with respect to the basis $(\mathbf{e}_i \wedge \mathbf{e}_j)_{0 \le i < j \le n}$, where $(\mathbf{e}_0, \ldots, \mathbf{e}_n)$ is the standard basis of $\mathbb{Q}_p^{n+1}$. (Note that such defined metrics d_p are ultrametrics.)
- Case II. $p = +\infty$. Take $\overline{\mathbf{x}}, \overline{\mathbf{y}} \in \mathbb{P}^n(\mathbb{R})$ with $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{n+1} \setminus \{0\}$. We then define

$$d_\infty(\overline{\mathbf{x}}, \overline{\mathbf{y}}) \coloneqq \frac{\|\mathbf{x} \wedge \mathbf{y}\|}{\|\mathbf{x}\| \cdot \|\mathbf{y}\|} \in [0, 1],$$

  where $\|\bullet\|$ on $\mathbb{R}^{n+1}$ is the usual Euclidean norm and $\|\bullet\|$ on $\bigwedge^2 \mathbb{R}^{n+1}$ is the Euclidean norm with respect to the basis $(\mathbf{e}_i \wedge \mathbf{e}_j)_{0 \le i < j \le n}$, where $(\mathbf{e}_0, \ldots, \mathbf{e}_n)$ is the standard basis of $\mathbb{R}^{n+1}$.

Consider the tuples

$$\overline{\boldsymbol{\xi}} = (\overline{\xi}_p) \in \prod_{p \in M_{\mathbb{Q}}} \mathbb{P}^n(\mathbb{Q}_p) \quad \text{and} \quad \boldsymbol{\sigma} = (\sigma_p) = (p^{-e_p}) \in \prod_{p \in M_{\mathbb{Q}}} \{p^{-e_p} \mid e_p \ge 0\}$$

such that $\sigma_p = 1$ for all but finitely many $p \in M_{\mathbb{Q}}$. We then define the "adelic" ball $\mathcal{B}_{\mathbb{A}}(\overline{\boldsymbol{\xi}}, \boldsymbol{\sigma})$ centered at $\overline{\boldsymbol{\xi}}$ of radius $\boldsymbol{\sigma}$ by

$$\mathcal{B}_{\mathbb{A}}(\overline{\boldsymbol{\xi}}, \boldsymbol{\sigma}) \coloneqq \prod_{p \in M_{\mathbb{Q}}} \mathcal{B}_p(\overline{\xi}_p, \sigma_p) \quad \text{and} \quad \mathcal{B}_p(\overline{\xi}_p, \sigma_p) \coloneqq \{\overline{\mathbf{x}} \in \mathbb{P}^n(\mathbb{Q}_p) \mid d_p(\overline{\mathbf{x}}, \overline{\xi}_p) \le \sigma_p\}.$$

When $U = \mathcal{B}_{\mathbb{A}}(\overline{\boldsymbol{\xi}}, \boldsymbol{\sigma})$, we write

$$N_V(B; \overline{\boldsymbol{\xi}}, \boldsymbol{\sigma}) \coloneqq N_V(B; \mathcal{B}_{\mathbb{A}}(\overline{\boldsymbol{\xi}}, \boldsymbol{\sigma})).$$

For a given radius $\boldsymbol{\sigma}$ of adelic ball, we let

$$q \coloneqq \prod_{p < \infty} \left(\frac{1}{\sigma_p}\right) = \prod_{p < \infty} p^{e_p} \quad \text{and} \quad \mathfrak{q} \coloneqq \prod_{p \in M_{\mathbb{Q}}} \left(\frac{1}{\sigma_p}\right) = \frac{q}{\sigma_\infty}.$$

With the above setting, our first main result is the following:

Main Theorem A (Matsuzawa–S.). *Under the above setting and $\varepsilon > 0$, we have*

$$\sum_{V \in \mathbb{V}_{d,n}(A)} N_V(B; \overline{\boldsymbol{\xi}}, \boldsymbol{\sigma}) = \widetilde{C}_{d,n}(\overline{\boldsymbol{\xi}}, \boldsymbol{\sigma}) A^{N_{d,n}-1} B^{n+1-d}(1 + R_{d,n}(A, B; \overline{\boldsymbol{\xi}}, \boldsymbol{\sigma}))$$

provided

$$A \ge \mathfrak{q} \quad \textit{and} \quad B \ge \mathfrak{q}^{\frac{n}{n+1-d}},$$

where

$$\widetilde{C}_{d,n}(\overline{\boldsymbol{\xi}},\boldsymbol{\sigma}) := \frac{1}{4\zeta(n+1)}\frac{V_{N_{d,n}-1}}{\zeta(N_{d,n}-1)}\frac{\varphi(q)}{J_{n+1}(q)}\int_{\mathcal{B}_{n+1}(1)\cap\mathscr{C}_{n+1}(\xi_\infty,\sigma_\infty)}\frac{d\mathbf{x}}{\|\nu_{d,n}(\mathbf{x})\|},$$

$$J_N(q) := q^N\prod_{p|q}\left(1-\frac{1}{p^N}\right),\quad \varphi(q) := J_1(q),$$

$$\mathscr{C}_N(\xi,\sigma) := \{\mathbf{x}\in\mathbb{R}^N \mid d_\infty(\overline{\mathbf{x}},\overline{\xi})\le\sigma\}$$

and the error term $R_{d,n}(A,B;\overline{\boldsymbol{\xi}},\boldsymbol{\sigma})$ *is estimated as*

$$R_{d,n}(A,B;\overline{\boldsymbol{\xi}},\boldsymbol{\sigma}) \ll (\mathfrak{q}B^{-1}+\mathfrak{q}^nB^{-(n+1-d)}+\mathfrak{q}A^{-1}+\mathfrak{q}^{\frac{2}{2n+1}}A^{-1}B^{\frac{3}{2n+1}})(\mathfrak{q}B)^\varepsilon + \widetilde{R}_{d,n}(A,B;\overline{\boldsymbol{\xi}},\boldsymbol{\sigma})$$

with $\widetilde{R}_{d,n}(A,B;\overline{\boldsymbol{\xi}},\boldsymbol{\sigma})=0$ *for* $A\ge B$ *and*

$$\widetilde{R}_{d,n}(A,B;\overline{\boldsymbol{\xi}},\boldsymbol{\sigma}) \ll (1+A^{-n}B^d)(\mathfrak{q}A^{-(n+\frac{1}{2})}B^{\frac{3}{2}}+\mathfrak{q}^{n-1}A^{-(n-1)})(\mathfrak{q}B)^\varepsilon$$

for $A\le B$ *and the implicit constant depends only on* d,n,ε.

As for the size of the coefficient $\widetilde{C}_{d,n}(\overline{\boldsymbol{\xi}},\boldsymbol{\sigma})$, we can easily prove that

Proposition 1. *The constant* $\widetilde{C}_{d,n}(\overline{\boldsymbol{\xi}},\boldsymbol{\sigma})$ *satisfies*

$$\widetilde{C}_{d,n}(\overline{\boldsymbol{\xi}},\boldsymbol{\sigma}) \asymp \mathfrak{q}^{-n}\prod_{p|q}\left(1-\frac{1}{p}\right),$$

where the implicit constant depends only on d,n.

As we saw in the previous section, we need to pay some attention on the total space to do statistics. We required the hypersurfaces in the total space is locally soluble in the previous section but now, we also require hypersurfaces to have a rational points approximating $\overline{\boldsymbol{\xi}}$, i.e. in the given ball $\mathcal{B}_{\mathbb{A}}(\overline{\boldsymbol{\xi}},\boldsymbol{\sigma})$. We are thus led to the set

$$\mathbb{V}^{\mathrm{loc}}_{d,n}(A;\overline{\boldsymbol{\xi}},\boldsymbol{\sigma}) := \{V\in\mathbb{V}_{d,n}(A) \mid \mathrm{diag}(V(\mathbb{Q}))\cap\mathcal{B}_{\mathbb{A}}(\overline{\boldsymbol{\xi}},\boldsymbol{\sigma})\neq\varnothing\}$$

as the total space of the statistics. As a counterpart of Theorem 3, we need an asymptotic formula for $|\mathbb{V}^{\mathrm{loc}}_{d,n}(A;\overline{\boldsymbol{\xi}},\boldsymbol{\sigma})|$ with taking care of the uniformity over the weak approximation. Such a result is given as follows:

Main Theorem B (Matsuzawa–S.). *For* $n\ge d\ge 2$ *with* $(n,d)\neq(2,2)$, *we have*

$$|\mathbb{V}^{\mathrm{loc}}_{d,n}(A;\overline{\boldsymbol{\xi}},\boldsymbol{\sigma})| = c_{d,n}(\overline{\boldsymbol{\xi}},\boldsymbol{\sigma})|\mathbb{V}_{d,n}(A)|\left(1+O\left(\frac{1}{\log\log A\log\log\log A}\right)\right) \asymp \mathfrak{q}^{-1}|\mathbb{V}_{d,n}(A)|$$

provided

$$A\ge\mathfrak{q}\log 3\mathfrak{q},$$

where

$$c_{d,n}(\overline{\boldsymbol{\xi}},\boldsymbol{\sigma}) := \prod_{p\in M_{\mathbb{Q}}}\mathrm{Prob}(\overline{\mathbf{a}}\in\mathbb{P}^{N_{d,n}-1}(\mathbb{Q}_p) \mid \langle\mathbf{a},\nu_{d,n}(\mathbf{X})\rangle=0\text{ has a solution in }\mathcal{B}_p(\overline{\xi}_p,\sigma_p))$$

and the implicit constant depends on d,n.

By dividing Main Theorem A by Main Theorem B, we obtain

Main Theorem A'. *For $n \geq d \geq 2$ with $(n,d) \neq (2,2)$ and $\varepsilon > 0$, we have*

$$\frac{1}{|\mathbb{V}^{\mathrm{loc}}_{d,n}(A;\overline{\boldsymbol{\xi}},\boldsymbol{\sigma})|}\sum_{V\in\mathbb{V}_{d,n}(A)} N_V(B;\overline{\boldsymbol{\xi}},\boldsymbol{\sigma}) \ll \mathfrak{q}^{-(n-1)}\frac{B^{n+1-d}}{A}(1+R_{d,n}(A,B;\overline{\boldsymbol{\xi}},\boldsymbol{\sigma}))$$

provided

$$A \geq \mathfrak{q}\log 3\mathfrak{q} \quad \text{and} \quad B \geq \mathfrak{q}^{\frac{n}{n+1-d}},$$

where the implicit constant depends only on d, n, ε.

Note that the exponent of the factor $\mathfrak{q}^{-(n-1)}$ coincides with the dimension of our hypersurfaces. As a counterpart of Corollary 1 or Corollary 2, by taking

$$B = (\varepsilon\mathfrak{q}^{n-1}A)^{\frac{1}{n+1-d}} \quad \text{and assuming} \quad A \geq \mathfrak{q}\log 3\mathfrak{q},$$

we obtain the following corollary:

Corollary A. *For $n \geq d \geq 2$ with $(n,d) \neq (2,2)$ and $0 < \varepsilon < 1$, we have*

$$\mathfrak{M}(V;\overline{\boldsymbol{\xi}},\boldsymbol{\sigma}) := \min\{H(\overline{\mathbf{x}}) \mid \overline{\mathbf{x}} \in V \text{ and } \mathrm{diag}(\overline{\mathbf{x}}) \in \mathcal{B}_{\mathbb{A}}(\overline{\boldsymbol{\xi}},\boldsymbol{\sigma})\} \geq (\varepsilon\mathfrak{q}^{n-1}\|\mathbf{a}_V\|)^{\frac{1}{n+1-d}}$$

for all but $\ll \varepsilon|\mathbb{V}^{\mathrm{loc}}_{d,n}(A;\overline{\boldsymbol{\xi}},\boldsymbol{\sigma})|$ hypersurfaces $V \in \mathbb{V}^{\mathrm{loc}}_{d,n}(A;\overline{\boldsymbol{\xi}},\boldsymbol{\sigma})$ provided

$$A \geq \mathfrak{q}^{\theta(n,d)+\varepsilon}$$

with

$$\theta(n,d) = \max(1,\theta_1(n,d),\theta_2(n,d),\theta_3(n,d),\theta_4(n,d))$$

where

$$\theta_1 := \frac{4n-d-2}{(2n+1)(n+1-d)-3}, \quad \theta_2 := \frac{2(4n-d-2)}{(2n+1)(n+1-d)-3},$$
$$\theta_3 := \frac{5n-1+(2n-4)d}{(4n+2)(n+1)-(4n+4)d-3}, \quad \theta_4 := \frac{n^2-1}{(2n-1)(n+1-d)-d}.$$

Note that we have

$$\theta(n,d) = 1 \quad \text{if} \quad n \geq c_0 \cdot d$$

with some constant $c_0 \geq 1$ and

$$\theta(n,d) \leq n+1 \quad \text{for all } (n,d) \text{ with } n \geq d \geq 2 \text{ and } (n,d) \neq (2,2).$$

It is reasonable to expect Corollary A holds with $\theta = 1$ for all (n,d) but our method seems not strong enough to prove such a result.

5. Sketch of the proof

We sketch the proof of main result with indicating some key ingredients.

5.1. **Proof of Main Theorem A.** The proof of Main Theorem A mainly follows the method of le Boudec [7] combined with the idea of le Boudec, Browining and Sawin [3] (cf. Lemma 4.7 of [3]). Of course, we need to introduce some new idea to get a better uniformity on $(\overline{\boldsymbol{\xi}},\boldsymbol{\sigma})$. We first rephrase our main sum over rational points in the projective spaces in terms of the sum over the associated primitive vectors:

$$\sum_{V\in\mathbb{V}_{d,n}(A)} N_V(B;\overline{\boldsymbol{\xi}},\boldsymbol{\sigma}) = \frac{1}{4}\sum_{\substack{\mathbf{a}\in\mathbb{Z}^{\mathscr{M}_{d,n}}_{\mathrm{prim}}\\ \|\mathbf{a}\|\leq A}}\ \sideset{}{^*}\sum_{u \pmod q}\ \sum_{\substack{\mathbf{x}\in\mathbb{Z}^{n+1}_{\mathrm{prim}}\\ \langle\mathbf{a},\nu_{d,n}(\mathbf{x})\rangle=0\\ \mathbf{x}\equiv u\mathbf{c} \pmod q\\ \mathbf{x}\in\mathscr{C}_{n+1}(\xi_\infty,\sigma_\infty)}} 1,$$

where the integral vector $\mathbf{c}$ is defined in terms of ξ_p and σ_p over finite places $p \in M_{\mathbb{Q}}$. We can then swap the summation to obtain

$$\sum_{V \in \mathbb{V}_{d,n}(A)} N_V(B; \overline{\boldsymbol{\xi}}, \boldsymbol{\sigma}) = \frac{1}{4} \sideset{}{^*}\sum_{u \pmod q} \sum_{\substack{\mathbf{x} \in \mathbb{Z}^{n+1}_{\mathrm{prim}} \\ \mathbf{x} \equiv u\mathbf{c} \pmod q \\ \mathbf{x} \in \mathscr{C}_{n+1}(\xi_\infty, \sigma_\infty)}} \#(\Lambda_{\nu_{d,n}(\mathbf{x})} \cap \mathbb{Z}^{\mathscr{M}_{d,n}}_{\mathrm{prim}} \cap \mathcal{B}_{N_{d,n}}(A)),$$

where $\Lambda_{\nu_{d,n}(\mathbf{x})}$ is the orthogonal lattice for the vector $\nu_{d,n}(\mathbf{x})$:

$$\Lambda_{\nu_{d,n}(\mathbf{x})} \coloneqq \{\mathbf{a} \in \mathbb{Z}^{\mathcal{M}_{d,n}} \mid \langle \mathbf{a}, \nu_{d,n}(\mathbf{x}) \rangle = 0\}.$$

For the cardinality

$$\#(\Lambda_{\nu_{d,n}(\mathbf{x})} \cap \mathbb{Z}^{\mathscr{M}_{d,n}}_{\mathrm{prim}} \cap \mathcal{B}_{N_{d,n}}(A)),$$

we use the usual lattice point counting with successive minima and the Möbius inversion (cf. Lemma 3.4 and Lemma 3.6 of [3]). Let $\lambda_i(\Lambda)$ be the i-th successive minima of a lattice $\Lambda \subseteq \mathbb{R}^N$:

$$\lambda_i(\Lambda) \coloneqq \min\{\lambda > 0 \mid \mathrm{span}_{\mathbb{R}}(\Lambda \cap \mathcal{B}_N(X)) \geq i\}.$$

This primitive lattice point counting results in

$$\begin{aligned} &\#(\Lambda_{\nu_{d,n}(\mathbf{x})} \cap \mathbb{Z}^{\mathscr{M}_{d,n}}_{\mathrm{prim}} \cap \mathcal{B}_{N_{d,n}}(A)) \\ &= \frac{V_{N_{d,n}}}{\zeta(N_{d,n})} \frac{A^{N_{d,n}-1}}{\det(\Lambda_{\nu_{d,n}(\mathbf{x})})} + O\Bigg(\sum_{1 \leq \nu \leq N_{d,n}-2} \frac{A^\nu}{\lambda_1(\Lambda_{\nu_{d,n}(\mathbf{x})}) \cdots \lambda_\nu(\Lambda_{\nu_{d,n}(\mathbf{x})})} \Bigg), \end{aligned}$$

where $\det(\Lambda)$ is the volume of the fundamental domain of the lattice Λ. Since $\det(\Lambda_{\nu_{d,n}(\mathbf{x})}) = \|\nu_{d,n}(\mathbf{x})\|$ for primitive $\mathbf{x}$ (cf. Lemma 4 of [7]), on inserting this lattice point counting formula into our main sum, we obtain

$$\sum_{V \in \mathbb{V}_{d,n}(A)} N_V(B; \overline{\boldsymbol{\xi}}, \boldsymbol{\sigma}) = T + O(E),$$

where

$$T \coloneqq \frac{V_{N_{d,n}}}{4\zeta(N_{d,n})} A^{N_{d,n}-1} \sideset{}{^*}\sum_{u \pmod q} \sum_{\substack{\mathbf{x} \in \mathbb{Z}^{n+1}_{\mathrm{prim}} \\ \mathbf{x} \equiv u\mathbf{c} \pmod q \\ \mathbf{x} \in \mathscr{C}_{n+1}(\xi_\infty, \sigma_\infty)}} \frac{1}{\|\nu_{d,n}(\mathbf{x})\|},$$

$$E \coloneqq \sideset{}{^*}\sum_{u \pmod q} \sum_{\substack{\mathbf{x} \in \mathbb{Z}^{n+1}_{\mathrm{prim}} \\ \mathbf{x} \equiv u\mathbf{c} \pmod q \\ \mathbf{x} \in \mathscr{C}_{n+1}(\xi_\infty, \sigma_\infty)}} \sum_{1 \leq \nu \leq N_{d,n}-2} \frac{A^\nu}{\lambda_1(\Lambda_{\nu_{d,n}(\mathbf{x})}) \cdots \lambda_\nu(\Lambda_{\nu_{d,n}(\mathbf{x})})}.$$

For the main term T, after carrying out partial summation, as in Lemma 3.5 of [3], we can use Theorem 1.3 of [1] since the congruence condition can be translated to a lattice and the archimedean condition $\mathbf{x} \in \mathscr{C}_{n+1}(\xi_\infty, \sigma_\infty)$ and the function $\|\nu_{d,n}(\mathbf{x})\|$ are semi-algebraic. In order to get rid of the effect of the error term $O(1)$ piled up over the reduced residues (mod q), we use the following lemma, which can be proven by induction over r:

Lemma 1 (Matsuzawa–S.). *Consider*

- *A lattice* $\Lambda \subseteq \mathbb{R}^N$ *of rank* $r \geq 1$,
- *A vector* $\mathbf{c} \in \Lambda$ *and* $q \in \mathbb{N}$ *such that*

$$d \mid q \text{ and } \mathbf{c} \in d\Lambda \implies d = 1.$$

- *A ball $\mathbf{a}+\mathcal{B}_N(T)$ centered at $\mathbf{a}\in\mathbb{R}^N$ with radius $T\geq 1$.*

If $T\leq Cq\lambda_1(\Lambda)$ with some $C\geq 1$, then we have

$$\sideset{}{^*}\sum_{u \pmod q} \#((\mathbf{a}+\mathcal{B}_N(T))\cap(u\mathbf{c}+\Lambda)) \ll \frac{T}{\lambda_1(\Lambda)}+1,$$

where the implicit constant depends only on r, C.

By using Lemma 1 to take an advantage of the average over the reduced residues (mod q), we can argue similarly as the proof of Theorem 3 of [7] to obtain

$$T=\widetilde{C}_{d,n}(\overline{\boldsymbol{\xi}},\boldsymbol{\sigma})A^{N_{d,n}-1}B^{n+1-d}\Big(1+O((\mathfrak{q}B^{-1}+\mathfrak{q}^nB^{-(n+1-d)})(\mathfrak{q}B)^{\varepsilon})\Big).$$

For the error term E, we shall use the following bound for the largest successive minima of $\Lambda_{\nu_{d,n}(\mathbf{x})}$ due to le Boudec, Browning and Sawin [3]. Let us define

$$\mathfrak{d}_r(\mathbf{x}) \coloneqq \min\{\det(\Lambda) \mid \Lambda\subseteq\mathbb{R}^{n+1} : \text{a lattice of rank } r \text{ such that } \mathbf{x}\in\Lambda\}$$

Lemma 2 (le Boudec–Browning–Sawin [3, Lemma 3.15]). *For $n,d\geq 1$ and $\mathbf{x}\in\mathbb{Z}^{n+1}_{\mathrm{prim}}$,*

$$\lambda_{N_{d,n}-1}(\Lambda_{\nu_{d,n}(\mathbf{x})}) \leq \mu(\mathbf{x}) \coloneqq n\frac{\|\mathbf{x}\|}{\mathfrak{d}_2(\mathbf{x})}.$$

We then dissect the error term E as

$$E=\sum_{\mu(\mathbf{x})\leq A}+\sum_{\mu(\mathbf{x})>A}=E_1+E_2.$$

In E_1, by the second Minkowski theorem, we can use the bound

$$\sum_{1\leq\nu\leq N_{d,n}-2}\frac{A^\nu}{\lambda_1(\Lambda_{\nu_{d,n}(\mathbf{x})})\cdots\lambda_\nu(\Lambda_{\nu_{d,n}(\mathbf{x})})} \ll A^{N_{d,n}-2}\frac{\mu(\mathbf{x})}{\|\mathbf{x}\|^d}.$$

In E_2, we use the idea of Lemma 3.6 of [3] to get

$$\sum_{1\leq\nu\leq N_{d,n}-2}\frac{A^\nu}{\lambda_1(\Lambda_{\nu_{d,n}(\mathbf{x})})\cdots\lambda_\nu(\Lambda_{\nu_{d,n}(\mathbf{x})})} \ll A^{N_{d,n}-1}\left(\frac{1}{\|\mathbf{x}\|^d}\left(\frac{\mu(\mathbf{x})}{A}\right)^{n-1}+\left(\frac{1}{A}\right)^n\right)+A.$$

Then, for both of E_1, E_2, we dissect the sum dyadically according to the size of $\|\mathbf{x}\|$ and $\mu(\mathbf{x})$. We then have to estimate the distribution of $\mu(\mathbf{x})$, i.e. we need to estimate

$$L_{r,n}(X,\Delta;\mathbf{c},q;\xi,\sigma) \coloneqq \#\left\{\mathbf{x}\in\left(\sideset{}{^*}\bigcup_{u \pmod q}(u\mathbf{c}+q\mathbb{Z}^{n+1})\cap\mathscr{C}_{n+1}(\xi,\sigma)\cap\mathcal{B}_{n+1}(X)\right) \,\middle|\, \mathfrak{d}_r(\mathbf{x})\leq\Delta\right\}.$$

Such an upper bound is given by le Boudec, Browning and Sawin [3] as

Lemma 3 (le Boudec–Browning–Sawin [3, Lemma 3.20]). *For $X,\Delta\geq 1$, we have*

$$\#\{\mathbf{x}\in\mathbb{Z}^{n+1}\cap\mathcal{B}_{n+1}(X) \mid \mathfrak{d}_r(\mathbf{x})\leq\Delta\} \ll X^r\Delta^n\log 2\Delta$$

provided $n\geq 2$ and $r\in\{2,\ldots,r+1\}$.

However, if we use Lemma 3, the effect of the weak approximation

$$\mathbf{x} \in \bigcup_{u \pmod q}^{*} (u\mathbf{c} + q\mathbb{Z}^{n+1}) \cap \mathscr{C}_{n+1}(\xi, \sigma) \tag{5}$$

will be discarded, which gives a very weak uniformity over $(\overline{\boldsymbol{\xi}}, \boldsymbol{\sigma})$. We thus need to get an extension of Lemma 3 with keeping the effect of (5). Such an estimate is given by the next lemma:

Lemma 4 (Matsuzawa–S.). *Consider*

- *An integral vector* $\mathbf{c} \in \mathbb{Z}^{n+1}$ *and* $q \geq 1$ *with* $\gcd(\mathbf{c}, q) = 1$ *and* $n \geq 3$.
- *A vector* $\xi \in \mathbb{R}^{n+1} \setminus \{0\}$ *and a real number* $0 < \sigma \leq 1$.

For $X, \Delta \geq 1$, *we then have*

$$L_{2,n}(X, \Delta; \mathbf{c}, q; \xi, \sigma) \ll \left(\left(\left(\frac{\sigma}{q}\right)^{n} + \left(\frac{\sigma}{q}\right)^{n-1} \Delta^{\frac{1}{2}} + \frac{\sigma}{q}\frac{1}{\Delta} \right) \Delta^n X^2 + \Delta^n X \right) (\mathsf{q}\Delta)^{\varepsilon},$$

where the implicit constant depends only on n, ε.

The proof of Lemma 4 goes similarly to the proof of Lemma 3.20 of [3]. However, in order to keep the effect of (5), we need to modify Lemma 3.19 of [3] as well. Lemma 3.19 of [3] counts the number of primitive lattices with given sizes of successive minima using the method of Schmidt [10], i.e. we recursively add a new vector to the lattice and count the possibility of the newly added vectors. In our case, we need to keep a trace of how (5) is weakened in this recursive process and use the theorem of Barroero–Widmer [1, Theorem 1.3] with the aid of Lemma 1 to take the advantage of the average over $u \pmod q$.

By using Lemma 4, we can bound E_1, E_2 to obtain Main Theorem A.

5.2. **Proof of Main Theorem B.** The proof of Main Theorem B follows the line of the proof of Poonen and Voloch [9, Theorem 3.6]. However, we need to make the estimates more quantitative and again we should keep the uniformity over $(\overline{\boldsymbol{\xi}}, \boldsymbol{\sigma})$. We translate the p-adic solubility to some other conditions suitable for counting.

For $p = +\infty$, we can just use the theorem of Barroero and Widmer [1, Theorem 1.3] to count the hypersurfaces since the real solubility is just a semi-algebraic condition.

For large primes p (so that not taking part in the weak approximation), we use the Lang–Weil estimate [6] to find a smooth $\mathbb{F}_p$-point on a given $V \in \mathbb{V}_{d,n}(A)$ and apply Hensel's lemma to get a $\mathbb{Q}_p$-point. To this end, we need to discard hypersurfaces being reducible (mod p). The space of (mod p)-reducible hypersurfaces has codimension 2 if $n \geq d \geq 2$ and $(n, d) \neq (2, 2)$ and so we can use Ekedahl's sieve [4] (or its quantitative version) to discard the contribution of hypersurfaces reducible (mod p) for some large prime p.

For small primes p, we just cover the space of hypersurfaces $\mathbb{P}^{N_{d,n}-1}(\mathbb{Q})$ by p-adic balls of a suitably chosen radius and we count the hypersurfaces contained in the p-adic balls totally consists of $\mathbb{Q}_p$-soluble hypersurfaces. (Note that the condition "being contained in a p-adic ball" can be interpreted as just some congruence.) This may loss hypersurfaces contained in the p-adic balls consisting of a mixture of $\mathbb{Q}_p$-soluble and $\mathbb{Q}_p$-insoluble hypersurfaces. However, by using the argument similar to Hensel's lemma, we can bound the number of such exceptional ball so that they are negligible if we choose the radius of p-adic balls suitably.

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