Analytic properties and mean values of several double zeta-functions

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§1. Zeta-functions and mean value theorems

For complex variables $s_j = \sigma_j + it_j$ $(j = 1, 2)$, the Euler-Zagier double zeta-function is defined by

$$\zeta_{EZ,2}(s_1, s_2) = \sum_{m,n=1}^{\infty} \frac{1}{m^{s_1}(m+n)^{s_2}}.$$

This series is convergent absolutely in the region when $\sigma_2 > 1$ and $\sigma_1+\sigma_2 > 2$. Zhao [8] and Akiyama, Egami and Tanigawa [1] independently proved that $\zeta_{EZ,2}(s_1, s_2)$ can be continued meromorphically to $\mathbb{C}^2$, and has singularities on $s_2 = 1$ and $s_1 + s_2 = 2, 1, 0, -2, -4, \ldots$ ([1]).

This function is a generalization of the Riemann zeta-function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. The analytic behavior of the Riemann zeta-function in the critical strip is a long-standing problem. Here, we recall the well-known mean square formulas for the Riemann zeta-function:

Theorem 0.1 (see [6, Theorem 7.2 and 7.3]). *For $T \geq 2$, we have*

$$\int_2^T |\zeta(\sigma + it)|^2 dt \sim \begin{cases} \zeta(2\sigma)T & (1/2 < \sigma < 1) \\ T \log T & (\sigma = \frac{1}{2}). \end{cases}$$

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In 2015, Matsumoto and Tsumura [4] considered a double analog of Theorem 0.1. They showed that for a fixed complex number s_1,

$$\int_2^T |\zeta_{EZ,2}(s_1,s_2)|^2 dt_2 = \sum_{k=2}^{\infty} \left| \sum_{m=1}^{k-1} \frac{1}{m^{s_1}} \right|^2 \frac{1}{k^{2\sigma_2}} T + o(T) \tag{1}$$

holds in some regions of $\mathbb{C}^2$-space. The implicit constant depends on s_1 and σ_2. The series

$$\sum_{k=2}^{\infty} \left| \sum_{m=1}^{k-1} \frac{1}{m^{s_1}} \right|^2 \frac{1}{k^{2\sigma_2}}$$

converges absolutely in the region $\sigma_2 > 1/2$ and $\sigma_1 + \sigma_2 > 3/2$.

Remark 0.2. *They derived the concrete order of error terms in (1), but $o(T)$. It is complicated to state them precisely, so we omit details here. Also, we know the concrete order of error terms in the following (3), Theorem 0.5 and Theorem 0.8, but we omit them in the same reason.*

Remark 0.3. *The asymptotic formula (1) corresponds to the case $1/2 \leq \sigma < 1$ in Theorem 0.1. Later, Ikeda, Matsuoka and Nagata [2] obtained an asymptotic formula that corresponds to the case $\sigma = 1/2$.*

As for other multiple zeta-functions, Okamoto and Onozuka [5] treated the following double sum, named the Mordell-Tornheim double zeta-function:

$$\zeta_{MT,2}(s_1,s_2,s_3) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^{s_1} n^{s_2} (m+n)^{s_3}} \tag{2}$$

for $s_1, s_2, s_3 \in \mathbb{C}$. This is absolutely convergent when $\sigma_1 + \sigma_3 > 1, \sigma_2 + \sigma_3 > 1$ and $\sigma_1 + \sigma_2 + \sigma_3 > 2$ ([5, Theorem 2.2]). They showed that for fixed complex numbers s_1 and s_2,

$$\int_2^T |\zeta_{MT,2}(s_1,s_2,s_3)|^2 dt_3 = \sum_{k=2}^{\infty} \left| \sum_{m=1}^{k-1} \frac{1}{m^{s_1}(k-m)^{s_2}} \right|^2 \frac{1}{k^{2\sigma_3}} T + o(T) \tag{3}$$

holds in the domain $\mathcal{D}$, where

$$\begin{aligned}\mathcal{D} = &\{(s_1,s_2,s_3) | \sigma_1 + \sigma_3 > 1, \sigma_2 + \sigma_3 > 1 \text{ and } \sigma_1 + \sigma_2 + \sigma_3 > 2\} \\ &\cup \{(s_1,s_2,s_3) | \sigma_1 > 1, \sigma_2 \geq 0, \sigma_3 > 0, t_2 \geq 0, 1/2 < \sigma_2 + \sigma_3 \leq 1, \\ &\quad \sigma_1 + \sigma_2 + \sigma_3 > 2, s_2 + s_3 \neq 1 \text{ and } 2 \leq t_3 \leq T\} \\ &\cup \{(s_1,s_2,s_3) | 1/2 < \sigma_1 < 3/2, \sigma_2 \geq 0, \sigma_3 > 0, t_2 \geq 0, \sigma_1 + \sigma_3 > 1,\end{aligned}$$

$1/2 < \sigma_2 + \sigma_3 \leq 1, 3/2 < \sigma_1 + \sigma_2 + \sigma_3 \leq 2, s_2 + s_3 \neq 1,$
$s_1 + s_2 + s_3 \neq 2$ and $2 \leq t_3 \leq T\}$.

We find that the series in the main term

$$\sum_{k=2}^{\infty} \left| \sum_{m=1}^{k-1} \frac{1}{m^{s_1}(k-m)^{s_2}} \right|^2 \frac{1}{k^{s_3}}$$

is absolutely convergent when $\sigma_1 + \sigma_3 > 1/2, \sigma_2 + \sigma_3 > 1/2$ and $\sigma_1 + \sigma_2 + \sigma_3 > 3/2$ ([5, Theorem 2.2]).

Remark 0.4. *Since $\zeta_{MT,2}(s_1, 0, s_3) = \zeta_{EZ,2}(s_1, s_2)$, the result of [5] contains (1) as a special case.*

§2. Main result

In this article, we apply the method of Okamoto and Onozuka for another double zeta-function, named the Apostol-Vu double zeta-function. It is defined by

$$\zeta_{AV,2}(s_1, s_2, s_3) = \sum_{m=1}^{\infty} \sum_{n<m} \frac{1}{m^{s_1} n^{s_2} (m+n)^{s_3}} \tag{4}$$

in the region $\sigma_j > 1$ $(j = 1, 2, 3)$. It can be continued meromorphically to the whole $\mathbb{C}^3$-space, and its singularities are $s_1 + s_3 = 1 - \ell$ or $s_1 + s_2 + s_3 = 2 - \ell$ for $\ell \in \mathbb{N}_{\geq 0}$ ([3, Theorem 2]). Moreover, it has the following relation between the Mordell-Tornheim double zeta-function ([3, (5.6)]):

$$\zeta_{MT,2}(s_1, s_2, s_3) = 2^{-s_3}\zeta(s_1 + s_2 + s_3) + \zeta_{AV,2}(s_1, s_2, s_3) + \zeta_{AV,2}(s_2, s_1, s_3). \tag{5}$$

By applying the method due to [5], we obtain the following result.

Theorem 0.5 ([7])**.** *For $s_j = \sigma_j + it_j \in \mathbb{C}$ $(j = 1, 2, 3)$, assume that when t_3 moves from 2 to T, the point (s_1, s_2, s_3) does not encounter the hyperplane $s_1 + s_3 = 1$ and $s_1 + s_2 + s_3 = 2$. Then for fixed complex numbers s_1, s_2,*

$$\int_2^T |\zeta_{AV,2}(s_1, s_2, s_3)|^2 dt_3 = \sum_{k=2}^{\infty} \left| \sum_{k/2 < m \leq k-1} \frac{1}{m^{s_1}(k-m)^{s_2}} \right|^2 \frac{1}{k^{2\sigma_3}} T + o(T)$$

holds in the domain $\mathcal{D}_{AV}$, where

$$\begin{aligned}\mathcal{D}_{AV} := &\{(s_1, s_2, s_3) | \sigma_1 + \sigma_3 > 1 \text{ and } \sigma_1 + \sigma_2 + \sigma_3 > 2\} \\ &\cup \{(s_1, s_2, s_3) | \sigma_1 \geq 0, t_1 \geq 0, \sigma_3 > 0, 2 \leq t_3, \sigma_1 + \sigma_3 > 1/2, \\ &\quad \sigma_1 + \sigma_2 + \sigma_3 > 3/2, s_1 + s_3 \neq 1 \text{ and } s_1 + s_2 + s_3 \neq 2\},\end{aligned}$$

and the implicit constant depends on s_1, s_2 and σ_3.

In the region $\sigma_1+\sigma_3>1, \sigma_1+\sigma_2+\sigma_3>2$, we can easily obtain the mean square formula, since (4) is absolute convergent. For the remainded region of $\mathcal{D}_{AV}$, we need some approximation formulas for $\zeta_{AV,2}(s_1,s_2,s_3)$.

For the remainded region of $\mathcal{D}_{AV}$, we derived the following approximation formula.

Proposition 0.6. *For $(s_1,s_2,s_3)\in\mathbb{C}^3$ with s_1,s_2 being fixed, we assume that $\sigma_1\geq 0, t_1\geq 0$ and $\sigma_3>\max\{0,\frac{1}{2}-\sigma_1,\frac{3}{2}-\sigma_1-\sigma_2\}, t_3\geq 2$. Then for $s_1+s_3\neq 1$, $s_1+s_2+s_3\neq 2$, we have*

$$\zeta_{AV,2}(s_1,s_2,s_3)=\sum_{m\leq at_3}\sum_{n<m}\frac{1}{m^{s_1}n^{s_2}(m+n)^{s_3}}+\begin{cases} O(t_3^{-\sigma_1-\sigma_3}) & (\sigma_2>\frac{3}{2}) \\ O(t_3^{-\sigma_1-\sigma_3}\log t_3) & (\sigma_2=\frac{3}{2}) \\ O(t_3^{\frac{3}{2}-\sigma_1-\sigma_2-\sigma_3}) & (\sigma_2<\frac{3}{2}), \end{cases}$$

where $a=\max\{1,|t_1|\}$, and implicit constants depend on s_1 and σ_3.

§3. Application

Theorem 0.5 corresponds to (3). Indeed we apply the method of Okamoto and Onozuka to (4) directly. However, by (5) and the information about the Apostol-Vu double zeta-function, we can deduce the mean square formula for the Mordell-Tornheim double zeta-function in a region where that was not available due to [5].

We obtain the following approximation formula for (2) by combining (5) and Proposition 0.6 with replacing a by $b=\max\{1,|t_1|,|t_2|\}$.

Proposition 0.7. *For $(s_1,s_2,s_3)\in\mathbb{C}^3$ with s_1,s_2 being fixed, we assume that $\sigma_1\geq 0, t_1\geq 0, \sigma_2\geq 0$, $t_2\geq 0$. Moreover, Let $\sigma_3>\max\{0,\frac{1}{2}-\sigma_1,\frac{1}{2}-\sigma_2,\frac{3}{2}-\sigma_1-\sigma_2\}, t_3\geq 2$. Then for $s_1+s_3\neq 1, s_2+s_3\neq 1$, $s_1+s_2+s_3\neq 2$, we have*

$$\zeta_{MT,2}(s_1,s_2,s_3)=\sum_{m\leq bt_3}\sum_{n\leq bt_3}\frac{1}{m^{s_1}n^{s_2}(m+n)^{s_3}} \\ +\begin{cases} O(t_3^{-\min\{\sigma_1+\sigma_3,\sigma_2+\sigma_3\}}) & (\max\{\sigma_1,\sigma_2\}>\frac{3}{2}) \\ O(t_3^{-\min\{\sigma_1+\sigma_3,\sigma_2+\sigma_3\}}\log t_3) & (\max\{\sigma_1,\sigma_2\}=\frac{3}{2}) \\ O(t_3^{\frac{3}{2}-\sigma_1-\sigma_2-\sigma_3}) & (\sigma_1,\sigma_2<\frac{3}{2}), \end{cases}$$

where $b=\max\{1,|t_1|,|t_2|\}$, and implicit constants depend on s_1,s_2 and σ_3.

By the above approximation formula, we find that the same asymptotic formula (3) holds in the $\mathcal{D}'$, where

$$\mathcal{D}'=\{(s_1,s_2,s_3)|\sigma_1\geq 0,\sigma_2\geq 0,\sigma_3>0,t_1\geq 0,t_2\geq 0,t_3\geq 2,$$

$$\sigma_1+\sigma_3 \le 1, \sigma_2+\sigma_3 \le 1, \sigma_1+\sigma_2+\sigma_3 > 3/2,$$
$$s_1+s_3 \ne 1, s_2+s_3 \ne 1, s_1+s_2+s_3 \ne 2\} .$$

More precisely, we prove the following:

Theorem 0.8 ([7]). *Let $(s_1, s_2, s_3) \in \mathcal{D}'$. We assume that when t_3 moves from 2 to T, the point (s_1, s_2, s_3) does not encounter the hyperplane $s_1+s_3 = 1, s_2+s_3 = 1$ and $s_1+s_2+s_3 = 2$. Then for fixed complex number s_1, s_2, we have*

$$\int_2^T |\zeta_{MT,2}(s_1, s_2, s_3)|^2 \, dt_3 = \sum_{k=2}^{\infty} \left| \sum_{m=1}^{k-1} \frac{1}{m^{s_1}(k-m)^{s_2}} \right|^2 \frac{1}{k^{2\sigma_3}} T + o(T),$$

where the implicit constant depends on s_1, s_2 and σ_3.

Remark 0.9. *By using the Propositon 0.7, we can deduce the mean square formula for (2) not only in $\mathcal{D}'$ but also in a subregion of $\mathcal{D}$. However, the order of the error term is worse than the order of it in (3).*

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