ソボレフ流と二重非線形放物型方程式について(*)

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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \geq 3)$ with smooth boundary $\partial\Omega$. We consider the following doubly nonlinear degenerate and singular parabolic equation, called $p-$Sobolev flow,

$$(1.1)\qquad \begin{cases} \partial_t u^q - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \lambda(t) u^q & \text{in } \Omega_\infty := \Omega \times (0,\infty) \\ \|u(t)\|_{q+1} = 1 & \text{for } t \geq 0 \\ u = u_0 & \text{on } \partial_p\Omega_\infty := \partial\Omega \times (0,\infty) \end{cases}$$

where $p > 1$, $p \leq q+1 \leq p^*$ with $p^* := \frac{np}{n-p}$ if $1 < p < n$ and any finite positive number if $p \geq n$, $u = u(x,\,t)$ is a nonnegative function defined for $(x,\,t) \in \Omega_\infty$, $\nabla_\alpha = \partial/\partial x_\alpha$, $\alpha = 1,\dots,m$, $\nabla u = (\nabla_\alpha u)$ is the spatial gradient of a function u, $|\nabla u|^2 = \sum_{\alpha=1}^m (\nabla_\alpha u)^2$ and $\partial_t u$ is the derivative on time t. The initial and boundary data $u_0 = u_0(x)$ is in the Sobolev space $\mathrm{W}_0^{1,p}(\Omega)$, nonnegative and $\|u_0\|_{q+1} = 1$. By multiplying the equation by u and integration by parts on space,

$$\frac{d}{dt}\frac{q}{q+1}\|u(t)\|_{q+1}^{q+1} + \|\nabla u(t)\|_p^p = \lambda(t)\|u(t)\|_{q+1}^{q+1} \Longrightarrow \lambda(t) = \|\nabla u(t)\|_p^p,$$

where $\|f\|_p$ is the $L^p(\Omega)-$norm of a function f, $E(u) := \|\nabla u\|_p^p/p$ is the $p-$energy of a function u. The system above describes the negative directed gradient flow in the constrained extremal problem for the $p-$energy. The corresponding Euler-Lagrange equation is given as the $p-$Laplace type equation, which has only trivial solution if the domain Ω is star-shaped with the origin. It is proved by a Pohožaev type identity and Hopf's maximum principle, which are available for regularized $p-$Laplace equation (see [M. Guedda, L. Veron: Nonlinear Anal. **13** (1989) 879-902]). Thus, a solution of the evolution equation may have concentration points of volume, local $(q+1)-$powered integral, at infinite time, by the volume conservation $\|u(t)\|_{q+1} = 1$. Our main purpose is to study such asymptotic behavior of a solution to the evolution equation above. In this paper we shall report on some results recently obtained.

The first result is the global existence a weak solution of (1.1) and its regularity (see [T. Kuusi, M. Misawa, K. Nakamura: J. Goem. Anal. **30** (2020) 1918-1964; J. Differential Equation **279** (2021) 245-281]).

Theorem 1 (A global existence and regularity) *Let $p > 1$ and $p \leq q+1 \leq p^*$. Suppose that u_0 belongs to $\mathrm{W}_0^{1,p}(\Omega)$, is nonnegative, bounded, $\|u_0\|_\infty < \infty$, and $\|u_0\|_{q+1} = 1$. Then, there exists a global weak solution $u \in C\left([0,\,\infty); L^{q+1}(\Omega)\right) \cap L^\infty(0,\,\infty; W_0^{1,p}(\Omega))$ of the Cauchy-Dirichlet problem (1.1), satisfying the energy inequalities*

$$(1.2)\qquad \|u(t)\|_{q+1} = 1, \qquad \forall t \geq 0,$$

$$(1.3)\qquad \|\partial_t u^{\frac{q+1}{2}}\|_{\mathrm{L}^2(\Omega_\infty)}^2 + \sup_{0<t<\infty} E(u(t)) \leq E(u_0).$$

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Moreover, the solution u is positive and bounded, $0 < u(t,x) \leq e^{pE(u_0)t/q}\|u_0\|_\infty$ for any $(t,x) \in \Omega_\infty$, and u and its spatial gradient ∇u are locally in time-space continuous in Ω_∞.

The boundedness and non-negativity of a solution is proved by a comparison type argument (see [Propositions 3.4, 3.5 and Lemma 3.6, J. Goem. Anal. **30** (2020)]). An expansion of positivity of a solution is derived by some De Giorgi's truncated local energy estimates and De Giorgi's Sobolev type inequalities with the truncation (see [Sect. 4, J. Goem. Anal. **30** (2020)], [Theorem 5.2, p. 258, Propositions 5.4 and 5.4, p.259, Appendix A, J. Differential Equation **279** (2021)]), those are reminiscent of the weak type estimate in L^p estimates. These local estimates are performed under a scaling setting intrinsic to the doubly nonlinear parabolic operator which is the principal term of our equation. The existence of a weak solution is based on that of the doubly nonlinear equation only with principal part (see [J. Geom. Anal. **33** (2023), no. 33]) and a special scaling transformation (see [Proposition 4.1, p. 255, J. Differential Equation **279** (2021)] and refer to [M. Misawa, N. Nakamura: Adv. Calc. Var. (2021)]), yielding a solution satisfying the conservation of volume $\|u(t)\|_{q+1} = 1$, $t \geq 0$.

We are interested in studying the asymptotic behavior around infinite time of the global solution to (1.1) obtained in Theorem 1. There may exist two possibilities about the behavior: The global solution of (1.1) strongly converges to a limit function in $W^{1,p}_{\mathrm{loc}}(\Omega)$ along a time-sequence increasingly tending to ∞. Otherwise, the global solution only weakly does so. The limit function is naturally a weak solution of the stationary equation corresponding to $(1.1)_1$. In the case of the weak convergence there may appear the so-called energy and volume gap at infinite time, which leads to energy and volume concentration. We study a concentration phenomenon of the weakly convergent solutions to (1.1).
We shall state the following results, the asymptotic behavior of the global solution of (1.1), obtained in Theroem 1. The first one characterizes concentration at infinity time, controls the global solution of (1.1) for large time and leads to deriving the asymptotic convergence to a limit function which is a stationary solution corresponding to (1.1).

Theorem 2 ($\varepsilon-$strong compactness) *Let $\frac{2n}{n+2} < p < n$ and $q+1 = p^*$. Let $\{t_k\}$, $t_k \nearrow \infty$ as $k \to \infty$. There exist subsequence $\{t_k\}$ (non-relabelled), a positive number $\epsilon_0 > 0$ and at most finite points $\{x_1, \dots, x_N\} \subset \Omega$, $0 < N < \infty$, such that there holds for any positive number $r \leq 1$ and all $i = 1, \dots, N$,*

$$\liminf_{k \nearrow \infty} \frac{1}{r^p} \int_{t_k - r^p}^{t_k} \|u(t)\|_{L^{q+1}(B(x_i, r))}\, dt \geq \varepsilon_0 \tag{1.4}$$

and the sequence $\{u(t_k)\}$ is strongly convergent in the Sobolev space $W^{1,p}$ on any compact subset of $\Omega \setminus \{x_1, \dots, x_N\}$.

The following result yields the asymptotic profile at a concetration point of the global soluton of (1.1).

Theorem 3 (Volume and energy concentration) *Let $\frac{2n}{n+2} < p < n$ and $q+1 = p^*$. Let $\{t_k\}$, $t_k \nearrow \infty$ and $\{r_k\}$, $r_k \searrow 0$ as $k \to \infty$. There exist subsequences $\{t_k\}$, $\{r_k\}$ (non-relabelled), a sequence $L_k \nearrow \infty$ as $k \to \infty$ and, a positive number $\epsilon_0 > 0$ and at most finite points $\{x_1, \dots, x_N\} \subset \Omega$, $0 < N < \infty$, such that the followings hold for each $x' :=$ each x_i, $i = 1, \dots, N$,*

$$\liminf_{k \nearrow \infty} \|u(t_k)\|_{L^{q+1}(B(x', r_k))} \geq \varepsilon_0;$$

$$w_k(x) := L_k^{-1} u\left(x' + L_k^{\frac{p-(q+1)}{p}} x,\ t_k\right)$$
$$\longrightarrow w \quad \text{strongly and locally in } W^{1,p} \cap L^{q+1}(\mathbb{R}^n)\,(k \to \infty),$$

where w is a positive and bounded weak solution of $-div(|\nabla w|^{p-2}\nabla w) = \lambda_\infty w^q$ in $\mathbb{R}^n$ with a positive constant λ_∞, and w and its gradient ∇w are locally continuous in $\mathbb{R}^n$.

To explain the meaning of Theorems 2 and 3 we recall some results on the asymptotic convergence of the Palais-Smale like sequence. In the Laplacian case (p=2) we have the global compactness result established by Struwe ([M. Struwe: Math. Z **187** (1984) 511-517). The result was extended to the $p-$Laplacian case for $1 < p < n$ (see [C. Mercuri, M. Willem: Discrete Contin. Dyn. Syst. A **28** (2010) 469-493; N. Santier: Calc. Var. **25** (2006) 299-331; A. Farina, C. Mercuri, M. Willem: Calc. Var. **58:153** (2019)]). Based on Theorems 2 and 3, we can establish the so-called energy and volume equalities, which completely characterize the asymptotic behavior as infinity-time of the nonnegative solutions to (1.1). See [Proposition 2.1, p. 513, Math. Z **187** (1984)], [Lemma 3.4, p.72, H. Schwetlick, M. Struwe, J. reine angew. Math. **563** (2003)] for the case $p = 2$, [Theorem 1.2, pp. 471-472, Discrete Contin. Dyn. Syst. A **28** (2010)] for the case1 $< p < n$. We shall present the result elsewhere in the near future.
The limit function w at time-infinity in Theorem 3 is given as the extremal function attaining the best constant in the Sobolev inequality, called Talenti function. Refer to [G. Talenti: Ann. Mat. Pura Appl. (4) **110** (1976) 353-372] for the Laplacian case, [B. Sciunzi: Adv. Math. **291** (2016) 12 -23; L. Damascelli, S. Merchán, L. Montoro, B. Sciumzi: Adv. Math. **256** (2014) 313-335; J. Vétois: J. Differential Equations **260** (1) (2016) 149-161] for the $p-$Laplacian case. In Theorem 2, any boundary concentration does not appear because the limit function is a positive stationary solution with zero Dirichlet boundary condition in the half space $\mathbb{R}^n_+$, which, however should be trivial (see [Theorem 1.1, pp. 470-471, Discrete Contin. Dyn. Syst. A **28** (2010)]), where the strong maximum principle is used (refer to [J. L. Vazquez: Appl. Math. Optim. **12** (1984) 191-202). Thus, the concentration points are in the interior of the domain Ω.

2 Lemmata

We present the local boundedness available for a nonnegative weak solution to (1.1), obtained in Theorem 1. This is the key estimation for showing the volume concentration at the limit as time tends to ∞ of a solution of (1.1).

Lemma 4 (Local boundedness) *Let $\frac{2n}{n+2} < p < n$ and $q + 1 = p^*$. Suppose that u a nonnegative weak solution to (1.1), obtained in Theorem 1. Let r_0 be a positive number satisfying $r_0 E(u_0) \le 1$ and $Q(r_0) \equiv B(x_0, r_0) \times (t_0 - (r_0)^p, t_0) \subset \Omega_\infty$. Put $\gamma = \frac{p(n+2)}{n}$. There exists a positive constant $\hat{\delta}_0 = \hat{\delta}_0(n,p,q) \le 1$ such that the following holds true: For any positive number $\delta_0 \le \hat{\delta}_0$, there exist a positive number k_0 such that*

$$(2.1) \qquad k_0 \ge \frac{1}{r_0^{n+p}\delta_0^\gamma}, \qquad 1 = \frac{1}{\delta_0}\left(\frac{1}{r_0^{n+p}k_0} + \frac{1}{|\hat{Q}|}\int_{\hat{Q}} \frac{u^{q+1}}{k_0^{q+1}}\,dx\,dt\right)^{1/\gamma},$$

where $\hat{Q}(k_0, r_0) \equiv B\left(x_0,\, k_0^{(p-(q+1))/p} r_0\right) \times (t_0 - r_0^p, t_0)$, and there holds

$$(2.2) \qquad u(x_0, t_0) \le 4k_0.$$

The proof of Lemma 4 is based on De Giorgi's type local energy estimates for truncated solutions, of which the detail will be appeared in a fothcoming paper. Here we shall show how to determine the local boundedness constant, of which the way is intrinsic to a solution and may be of its own interest. We emphasize that the equation (2.1) corresponds to (2.3) in the following proposition.

Proposition 5 (Intrinsic local boundedness) *Let $r_0 > 0$ and $\delta_0 \in (0,1)$. Let $Q(r_0) = B(x_0, r_0) \times (t_0 - (r_0)^p, t_0) \subset \Omega_\infty$. Put $\beta = \frac{p(q-1)}{n}$ and $\gamma = \frac{p(n+2)}{n}$ (so that $\beta+\gamma = q+1 = p^*$). Then there is a unique positive real number k_0 such that if $u \in L^{q+1}(Q(r_0))$ and $u \geq 0$, then there is a unique solution k_0, $k_0 \geq r_0^{-n-p}\delta_0^{-\gamma}$, to the equation*

$$k_0 = \frac{1}{\delta_0}\left(\frac{k_0^{-1+\gamma}}{r_0^{n+p}} + \fint_{\hat{Q}(k_0,r_0)} \frac{u^\beta}{k_0^\beta} u^\gamma \, dx\, dt\right)^{1/\gamma}, \tag{2.3}$$

where $\hat{Q}(k_0, r_0) = B\left(x_0, k_0^{(p-(q+1))/p} r_0\right) \times (t_0 - (r_0)^p, t_0)$. Moreover, the root satisfies $k_0 \equiv k(u, r_0, \delta_0) \nearrow \infty$ as $r_0 \searrow 0$ or $\delta_0 \searrow 0$.

Proof. Since

$$\fint_{\hat{Q}(k_0,r_0)} \frac{u^\beta}{k_0^\beta} u^\gamma \, dx\, dt = \frac{k_0^{-\beta+\frac{n}{p}(q+1-p)}}{r_0^{n+p}|B(0,1)|} \int_{\hat{Q}(k_0,r_0)} u^{\beta+\gamma} \, dx\, dt$$

and

$$-\beta + \frac{n}{p}(q+1-p) = \gamma - q - 1 + \frac{n}{p}(q+1) - n = \gamma + \frac{n-p}{p}(q+1) - n = \gamma\,,$$

we have that

$$\left(\frac{k_0^{-1+\gamma}}{r_0^{n+p}} + \fint_{\hat{Q}(k_0,r_0)} \frac{u^\beta}{k_0^\beta} u^\gamma \, dx\, dt\right)^{1/\gamma}$$
$$= k_0 \left(\frac{1}{r_0^{n+p}}\left[\frac{1}{k_0} + \frac{1}{|B(0,1)|}\int_{\hat{Q}(k_0,r_0)} u^{q+1} \, dx\, dt\right]\right)^{1/\gamma}. \tag{2.4}$$

The function

$$k_0 \mapsto h(k_0)\,, \qquad h(k_0) = \frac{1}{r_0^{n+p}}\left[\frac{1}{k_0} + \frac{1}{|B(0,1)|}\int_{\hat{Q}(k_0,r_0)} u^{q+1} \, dx\, dt\right],$$

is continuous and strictly decreasing function of k_0 and $h(k_0) \downarrow 0$ as $k_0 \uparrow \infty$ for any given $r_0 > 0$. Moreover $h(r_0^{-n-p}) \geq 1$. Therefore there must be a unique $k_0^* > r_0^{-n-p}$ such that

$$h(k_0^*) = \delta_0^\gamma\,.$$

It is easy to see that this root converges to infinity as r_0 or δ_0 tends to zero. This proves the claim. □

By the use of Lemma 4 we show the uniform local boundedness for solutions of (1.1).

Lemma 6 (Uniform boundedness) *Let $\frac{2n}{n+2} < p < n$ and $q+1 = p^*$. Suppose that u a nonnegative weak solution to (1.1), obtained in Theorem 1. Suppose that, for some positive numbrs ε_0, $r_0 \le 1$ and $Q(r_0) = B(x_0, r_0) \times (t_0 - r_0^p, t_0) \subset \Omega_\infty$,*

$$\frac{1}{r_0^p} \int_{t_0 - r_0^p}^{t_0} \|u\|_{L^{q+1}(B(x_0, r_0))} \le \varepsilon_0. \tag{2.5}$$

Then there holds, for positive numbers $r_0' = r_0'(r_0, \varepsilon_0)$, $C = C(r_0, \varepsilon_0)$ and $Q(r_0') = B(x_0, r_0') \times (t_0 - (r_0')^p, t_0)$,

$$\sup_{(x,\, t) \in Q(r_0')} |u| \le C. \tag{2.6}$$

Proof. Let $(x_0', t_0') \in Q(r_0/2)$ be an arbitrarily taken and fixed. We shall employ Lemma 4 for the proof, where r_0 and (x_0, t_0) are replaced by $r_0/2$ and (x_0', t_0'), respectively. Clearly, $Q'(r_0/2) \equiv B(x_0', r_0/2) \times (t_0' - (r_0/2)^p, t_0')$ is contained in $Q(r_0)$. Let k_0 be chosen as in (2.1) of Lemma 4. From $\dfrac{n(p-(q+1))}{p} = -\dfrac{np}{n-p} = -(q+1)$ and (2.5), it follows that, letting $\hat{Q}'(k_0, r_0/2) \equiv B\left(x_0', k_0^{(p-(q+1))/p}(r_0/2)\right) \times (t_0' - (r_0/2)^p, t_0')$,

$$\frac{1}{|\hat{Q}'(k_0, r_0/2)|} \int_{\hat{Q}'(k_0, r_0/2)} \frac{u^{q+1}}{k_0^{q+1}}\, dx\, dt = \frac{2^{n+p}}{r_0^{n+p}|B(1)|} \int_{Q(r_0)} u^{q+1}\, dx\, dt \le \frac{2^{n+p}\varepsilon_0}{r_0^n |B(1)|}, \tag{2.7}$$

where we note that $k_0 \ge 1$ by $\delta_0 \le 1$ and $r_0 \le 1$. Choosing $k_0' \ge k_0$ so large that

$$1 \le \frac{1}{\delta_0^\gamma} \left(\frac{1}{r_0^{n+p} k_0'} + \frac{2^{n+p}\varepsilon_0}{r_0^n |B(1)|} \right) \le \frac{2^{n+p+1}\varepsilon_0}{\delta_0^\gamma r_0^n |2B(1)|}, \tag{2.8}$$

we obtain from (2.2) in Lemma 4 that

$$u(x_0', t_0') \le 4k_0' \quad \text{for any } (x_0', t_0') \in Q(r_0/2). \tag{2.9}$$

Here we notice the dependence of k_0', $k_0' = k_0'(r_0, \delta_0, \varepsilon_0, n, p)$, and thus, the assertion (2.6) follows from (2.9), letting $r_0' = (k_0')^{\frac{p-(q+1)}{p}} (r_0/2) (\le r_0/2)$. □

Proof of Theorem 2. First we notice that the conditions, $\frac{2n}{n+2} < p < n$ and $q+1 = p^*$, imply that $q \ge 1$. Let u be nonnegative weak solution to (1.1), obtained in Theorem 1. We shall show the following: There exists a sequence of times $\{\tau_k\}$, $\tau_k \nearrow \infty$ as $k \to \infty$ such that the sequence of solutions $\{u(\tau_k)\}$ converges to a weak solution of the stationary equation corresponding to (1.1).
First we take a subsequence $\{t_k''\}$ of $\{t_k'\}$ such that $t_{k+1}'' - t_k'' \ge 1$ for all $k = 1, \ldots$ and $t_k'' \nearrow \infty$ as $k \to \infty$. Write as $I(k) = \left(t_k'', t_{k+1}''\right)$, $k = 1, \ldots$. Now we prove that there exists a sequence $\{\tau_k\}$ such that $\tau_k \in I_k$, $k = 1, \ldots$, $\tau_k \nearrow \infty$ as $k \to \infty$ and

$$\lim_{k \to \infty} \int_\Omega |\partial_t u^q(\tau_k)|\, dx = 0. \tag{2.10}$$

Indeed, by (1.3) in Theorem 1 there holds

$$\sum_{k=1}^\infty \frac{1}{|I_k|} \int_{I_k} \int_\Omega \left|\partial_t u^{\frac{q+1}{2}}\right|^2 dx\, dt \le \int_0^\infty \int_\Omega \left|\partial_t u^{\frac{q+1}{2}}\right|^2 dx\, dt < \infty,$$

where we use that the length $|I_k|$ of I_k is larger than 1 by the choice of t''_k. From the mean-value theorem, for each $k = 1, \ldots$ there exists a number $\tau_k \in I_k$ such that, as $k \to \infty$,

$$\int_\Omega \left|\partial_t u^{\frac{q+1}{2}}(\tau_k)\right|^2 \, dx \le \frac{1}{|I_k|} \int_{I_k} \int_\Omega \left|\partial_t u^{\frac{q+1}{2}}\right|^2 \, dx\, dt \to 0.$$

For $q \geq 1$ the chain rule of weak differential enables us to compute as

$$\partial_t u^q = \frac{2q}{q+1} u^{\frac{q-1}{2}} \partial_t u^{\frac{q+1}{2}}, \tag{2.11}$$

since the function $z^{\frac{2q}{q+1}}$ is locally Lipschitz on $z \in [0, \infty)$. The fact above and the Hölder inequality yield the estimation

$$\begin{aligned}
\int_\Omega |\partial_t u^q(\tau_k)| \, dx &\le \frac{2q}{q+1} \|u^{\frac{q-1}{2}}(\tau_k)\|_2 \|\partial_t u^{\frac{q+1}{2}}(\tau_k)\|_2 \\
&\le \frac{2q}{q+1} |\Omega|^{\frac{1}{q+1}} \sup_{t \in (0,\infty)} \|u(t)\|_{q+1}^{\frac{q-1}{2}} \|\partial_t u^{\frac{q+1}{2}}(\tau_k)\|_2 \\
&\le \|u_0\|_{q+1}^{\frac{q-1}{2}} \|\partial_t u^{\frac{q+1}{2}}(\tau_k)\|_2 \longrightarrow 0
\end{aligned}$$

as $k \to \infty$, which gives (2.10).

Next we claim that the integral equation

$$\int_\Omega \left(\partial_t u^q(\tau_k)\phi + |\nabla u|^{p-2} \nabla u(\tau_k) \cdot \nabla \phi - (\lambda u^q)(\tau_k)\phi\right) \, dx = 0 \tag{2.12}$$

holds true for any $\phi \in C_0^\infty(\Omega)$.

Let $0 < \varepsilon, h \searrow 0$ and define a cut-off function on time $\eta_h = \eta_h(t)$ such that η_h is Lipschitz on $\mathbb{R}$, $\eta_h = 1$ in $[\tau_k - \varepsilon + h,\, \tau_k + \varepsilon - h]$, $\eta_h = 0$ in $\mathbb{R} \setminus (\tau_k - \varepsilon,\, \tau_k + \varepsilon)$ and $|\partial_t \eta_h| \le 1/h$ in $\mathbb{R}$. Then, we use the test function $\phi\eta_h$ in the weak form of $(1.1)_1$. Noting the integrability of each term appearing in the resulting equality, by the Lebesgue convergence theorem we pass to the limit as $h \searrow 0$ and have

$$\int_{\tau_k - \varepsilon}^{\tau_k + \varepsilon} \left\{ \int_\Omega \left(\partial_t u^q \phi + |\nabla u|^{p-2} \nabla u \cdot \nabla \phi - (\lambda u^q)\phi\right) \, dx \right\} dt = 0$$

and then, dividing the both side of the above equation by 2ε, from the Lebesgue's differenrtial theorem available for integrable functions we can take the limit as $\varepsilon \searrow 0$ in the resulting equation to obtain the claim (2.12).

From (1.2) and (1.3) in Theorem 1 we see that the sequence $\{u(\tau_k)\}$ is bounded in $W^{1,p}(\Omega)$ and thus, by the compactness of Sobolev empbedding into the Lebesgue space we have a (non-relabeled) subsequence $\{\tau_k\}$, the limit function $w \in W_0^{1,p}(\Omega)$ and a finite numberλ_∞ such that, as $k \to \infty$,

$$\begin{aligned}
&u(\tau_k) \longrightarrow w && \text{weakly in } W^{1,p}(\Omega), \\
&u(\tau_k) \longrightarrow w && \text{strongly in } L^r(\Omega), \quad \forall r \in [1, p^*), \text{ and almost everywhere } \Omega, \\
&\lambda(\tau_k) \longrightarrow \lambda_\infty, &&
\end{aligned} \tag{2.13}$$

where we use Mazur's theorem verifying that the closed subspace $W_0^{1,p}(\Omega)$ of $W^{1,p}(\Omega)$ is weakly closed in $W^{1,p}(\Omega)$.

We also have the following strong convergence of gradients: There exists a (non-relabeled) subsequence $\{u(\tau_k)$ such that

$$Du(\tau_k) \longrightarrow Dw \qquad \text{strongly in } L^r(\Omega), \quad \forall r \in [1, p), \tag{2.14}$$

of which the proof is referred in [Lemma 5.3, p. 19, Appendix E, p. 43, M. Misawa, K. Nakamura, Md Abu Hanif Sarkar: Nonlinear Differ. Eqn. Appl. (2023) **30** :43].
By means of the convergences (2.10), (2.13) and (2.14) we have the identity holding true for any $\phi \in C^\infty(\Omega)$

$$\int_\Omega \left(|\nabla w|^{p-2} \nabla w \cdot \nabla \phi - \lambda_\infty w^q \phi \right) \, dx = 0. \tag{2.15}$$

Further we can verify that the sequence $\{u(\tau_k)\}$ strongly converges to the limit w in $W^{1,p}(\Omega \setminus \mathcal{N})$ for some set of finitely many points $\mathcal{N} = \{x_1, \ldots, x_N\}$. In fact we shall demonstrate the convergence

$$Du(\tau_k) \longrightarrow Dw \qquad \text{strongly in } L^p_{\text{loc}}(\Omega \setminus \mathcal{N}). \tag{2.16}$$

For the proof we shall employ the local boundedness of the solution to (1.1).
Fix $x_0 \in \Omega$ and assume that for some positive $r_0 \leq 1$ there holds

$$\liminf_{k \nearrow \infty} \frac{1}{r_0^p} \int_{t_k - r_0^p}^{t_k} \|u(t)\|_{L^{q+1}(B(x_0, r_0))} < \varepsilon_0.$$

Then we choose a subsequence $\{t'_k\}$ of $\{t_k\}$ such that

$$\frac{1}{r_0^p} \int_{t'_k - r_0^p}^{t'_k} \|u(t)\|_{L^{q+1}(B(x_0, r_0))} \, dt \leq \varepsilon_0. \tag{2.17}$$

Applying Lemma 6 with (2.17), we have positive numbers $r'_0 = r'_0(r_0, \varepsilon_0)$, $C = C(r_0, \varepsilon_0)$ and $Q(r_0; x_0, t'_k) \equiv B(x_0, r'_0) \times (t'_k - (r'_0)^p, t'_k)$,

$$\sup_{Q(r_0; x_0, t'_k)} |u| \leq C, \tag{2.18}$$

yields the uniform on t'_k boundedness of the solutions $\{u \,\text{in}\, Q(r_0; x_0, t'_k)\}$, $k = 1, \ldots$.
Next we shall show the validity of the following convergences as $k \to \infty$:

$$\int_\Omega \partial_t u(\tau_k) \, (u(\tau_k) - w) \, dx \longrightarrow 0, \tag{2.19}$$

$$\int_{B(x_0, r'_0/2)} (u^q(\tau_k) - w^q) \, (u(\tau_k) - w) \, dx \longrightarrow 0. \tag{2.20}$$

Noting (2.11) again we estimate as

$$\begin{aligned} \int_\Omega |\partial_t u(\tau_k)| |u(\tau_k) - w| \, dx &\leq \frac{2q}{q+1} \int_\Omega |\partial_t u^{\frac{q+1}{2}}(\tau_k)| \left(u^{\frac{q+1}{2}}(\tau_k) + u^{\frac{q-1}{2}}(\tau_k)|w| \right) dx \\ &\leq \frac{2q}{q+1} \|\partial_t u^{\frac{q+1}{2}}(\tau_k)\|_2 \left(\|u(\tau_k)\|_{q+1}^{\frac{q+1}{2}} + \|u(\tau_k)\|_{q+1}^{\frac{q-1}{2}} \|w\|_{q+1} \right), \end{aligned} \tag{2.21}$$

where the Hölder inequality is used in the second inequality with $q \geq 1$. Thus, the convergence (2.19) folows from (2.10).

For the proof of (2.20) we note the strong integral convergence as

$$u(\tau_k) \longrightarrow w \qquad \text{strongly in } L^\gamma(B(x_0, r_0')) \text{ for any finite } \gamma \geq 1, \tag{2.22}$$

wichi is verified by the use of the convergence (2.13) and the uniform boundedness (2.18). By the Hölder inequality we simply estimate and take the limit as $k \to \infty$ in the resulting inequality as

$$\begin{aligned} \int\limits_{B(x_0,r_0')} |u^q(\tau_k) - w^q| \, |u(\tau_k) - w| \, dx &\leq \int\limits_{B(x_0,r_0')} (|u(\tau_k)|^q + |w|^q) \, |u(\tau_k) - w| \, dx \\ &\leq \left(\|u(\tau_k)\|_{q+1}^q + \|w\|_{q+1}^q \right) \|u(\tau_k) - w\|_{L^{q+1}(B(x_0,r_0'))} \longrightarrow 0, \end{aligned}$$

where the convergence (2.22) is used in the last line. The validity of (2.20) is shown.

Here we recall the algebraic inequalities as follows: There holds for any vectors $P, Q \in \mathbb{R}^n$ that

$$\begin{aligned} &\left(|P|^{p-2}P - |Q|^{p-2}Q \right) \cdot (P - Q) \geq C_1 \left(|P| + |Q| \right)^{p-2} |P - Q|^2 , \\ &\left| |P|^{p-2}P - |Q|^{p-2}Q \right| \leq C_2 \left(|P| + |Q| \right)^{p-2} |P - Q| . \end{aligned} \tag{2.23}$$

Now we subtract (2.15) from (2.12) and use the test function $\eta^2 \left(u(\tau_k) - w \right)$ in the resulting equation, where the function $\eta = \eta(x)$ is Lipschitz on $\mathbb{R}^n$ such that $\eta = 1$ in $B(x_0, r_0'/2)$, $\eta = 0$ outside $B(x_0, r_0')$ and $|\nabla \eta| \leq 2/r_0'$. By the use of (2.23) we have, if $p > 2$,

$$\begin{aligned} &C' \int\limits_{B(x_0,r_0')} \eta^2 \, |\nabla u(\tau_k) - \nabla w|^p \\ &\leq C \int\limits_{B(x_0,r_0')} |\nabla \eta|^2 \left(|\nabla u(\tau_k)| + |\nabla w| \right)^{p-2} |u(\tau_k) - w|^2 \, dx \\ &\quad - \int\limits_{B(x_0,r_0')} \eta^2 \partial_t u(\tau_k) \left(u(\tau_k) - w \right) dx + \lambda_\infty \int\limits_{B(x_0,r_0')} \eta^2 \left(u^q(\tau_k) - w^q \right) \left(u(\tau_k) - w \right) dx \\ &\qquad + \left(\lambda(\tau_k) - \lambda_\infty \right) \int\limits_{B(x_0,r_0')} \eta^2 u^q(\tau_k) \left(u(\tau_k) - w \right) dx \\ &\longrightarrow 0 \qquad \text{as } k \to \infty, \end{aligned} \tag{2.24}$$

where we use the convergences (2.19), (2.20) and $(2.13)_3$.
If $1 < p < 2$ we use (2.23) to have the inequality

$$\begin{aligned} &\int\limits_{B(x_0,r_0')} \eta^2 \, |\nabla u(\tau_k) - \nabla w|^p \\ &\leq \left(\int\limits_{B(x_0,r_0')} \eta^2 \left(|\nabla u(\tau_k)| + |\nabla w| \right)^{p-2} |\nabla u(\tau_k) - \nabla w|^2 \, dx \right)^{\frac{p}{2}} \left(\int\limits_{B(x_0,r_0')} \eta^2 \left(|\nabla u(\tau_k)| + |\nabla w| \right)^p dx \right)^{\frac{2-p}{2}} \\ &\leq C \left(\int\limits_{B(x_0,r_0')} \eta^2 \left(|\nabla u(\tau_k)|^{p-2} \nabla u(\tau_k) - |\nabla w|^{p-2} \nabla w \right) \cdot \left(\nabla u(\tau_k) - \nabla w \right) dx \right)^{\frac{p}{2}} \\ &\times \left(\int\limits_{B(x_0,r_0')} \eta^2 \left(|\nabla u(\tau_k)| + |\nabla w| \right)^p dx \right)^{\frac{2-p}{2}} . \end{aligned}$$

At this stage we evaluate the last term in the above inequality. The integral term in the 1st brace is equal to the same as 3rd one in (2.24) and the integral term in the 2nd brace is bounded by (1.3) in Theorem 1. Thus, from the same resoning as (2.24) this last term converges to 0 as $k \to \infty$.
Therefore we have that $\nabla u(\tau_k)$ converges to ∇w strongly in $L^p(B(x_0, r_0'/2))$ as $k \to \infty$. The convergence (2.16) follows from by a usual covering argument with the strong convergence of gradients above.

The finiteness of concetration points $\mathcal{N}$ in (1.4) is verfied as follows: We compute as

$$
\begin{aligned}
\sum_{i=1}^{N} \frac{1}{r_0^p} \int_{t_k'-r_0^p}^{t_k'} \|u(t)\|_{L^{q+1}(B(x_i,r_0))}\,dt &= \frac{1}{r_0^p} \int_{t_k'-r_0^p}^{t_k'} \sum_{i=1}^{N} \|u(t)\|_{L^{q+1}(B(x_i,r_0))}\,dt \\
&= \frac{1}{r_0^p} \int_{t_k'-r_0^p}^{t_k'} \|u(t)\|_{L^{q+1}\left(\cup_{i=1}^{N} B(x_i,r_0)\right)}\,dt \\
&\le \frac{1}{r_0^p} \int_{t_k'-r_0^p}^{t_k'} \|u(t)\|_{L^{q+1}(\Omega)}\,dt = 1,
\end{aligned}
$$

where we use (1.2) in the last line. Taking the limitinf on $k \nearrow \infty$ in both side in the above inequality yields the estimation

$$
\begin{aligned}
N\varepsilon_0 &\le \sum_{i=1}^{N} \liminf_{k\nearrow\infty} \frac{1}{r_0^p} \int_{t_k'-r_0^p}^{t_k'} \|u(t)\|_{L^{q+1}(B(x_i,r_0))}\,dt \\
&\le \liminf_{k\nearrow\infty} \sum_{i=1}^{N} \frac{1}{r_0^p} \int_{t_k'-r_0^p}^{t_k'} \|u(t)\|_{L^{q+1}(B(x_i,r_0))}\,dt \le 1
\end{aligned}
$$

and thus,

$$
N \le \frac{1}{\varepsilon_0}. \tag{2.25}
$$

The proof of Theorem 2 is completed. □

Proof of Theorem 3. We shall show the validity of Theorem 3. Let $x_0 = x_i$, $i = 1, \ldots, N$ be any point where (1.4) holds true for any positive $r \le 1$. Let $\{t_k\}$, $t_k \nearrow \infty$ as $k \to \infty$. Let $\{r_l\}$ be a sequence of postive numbers $r_l \searrow 0$ as $l \to \infty$. Then, by (1.4) we have, for any r_l, $l = 1, \ldots,$

$$
\liminf_{k\nearrow\infty} \frac{1}{(r_l)^p} \int_{t_k-(r_l)^p}^{t_k} \|u(t)\|_{L^{q+1}(B(x_0,r_l))}\,dt \ge \varepsilon_0 \tag{2.26}
$$

and from the mean-value theorem there exists a number t_{kl}, $t_k - (r_l)^p < t_{kl} < t_k$ for each $k, l = 1, \ldots$ such that

$$
\|u(t_{kl})\|_{q+1} = \frac{1}{(r_l)^p} \int_{t_k-(r_l)^p}^{t_k} \|u(t)\|_{L^{q+1}(B(x_0,r_l))}\,dt. \tag{2.27}
$$

By Cantor's diagonal argument, (2.26) and (2.27) we can take subsequences $\{r'_k\}$ of $\{r_l\}$ and $\{t'_{kk}\}$ of $\{t_{kl}\}$ such that $t_k - (r'_k)^p < t'_{kk} < t_k$ and

$$\begin{aligned} & t'_{kk} \nearrow \infty, \qquad r'_k \searrow 0 \qquad \text{as } k \to \infty, \\ & \liminf_{k \nearrow \infty} \|u(t'_{kk})\|_{L^{q+1}(B(x_0, r'_k))} \geq \frac{\varepsilon_0}{2}. \end{aligned} \tag{2.28}$$

Let us write $\{t'_{kk}\}$ as $\{t_k\}$ and $\{r'_k\}$ as $\{r_k\}$.

Hereafter we shall fix $k = 1, \ldots$ and write as $t_0 = t_k$ and $r_0 = r_k$. Let $Q(r_0) = B(x_0, r_0) \times (t_0 - (r_0)^p, t_0)$. Let $(x'_0, t'_0) \in Q(r_0)$ be arbitrarily taken and be fixed. Make a local prabolic cylinder $Q'(r_0) = B(x'_0, r_0) \times (t'_0 - (r_0)^p, t'_0)$ with vertex at (x'_0, t'_0). We now employ Lemma 4 in $Q'(r_0)$. Thus, we have positive numbers $\delta_0 \leq 1$ and L' such that

$$\begin{aligned} & L' \geq (r_0)^{-n-p} \delta_0^{-\gamma}, \\ & \hat{Q}'(L', r_0) = B\left(x'_0, (L')^{(p-(q+1))/p} r_0\right) \times (t'_0 - (r_0)^p, t'_0), \\ & 1 = \frac{1}{\delta_0} \left(\frac{1}{(r_0)^{n+p} L'} + \frac{1}{|\hat{Q}'(L', r_0)|} \int_{\hat{Q}'(L', r_0)} \frac{u^{q+1}}{(L')^{q+1}} \, dx \, dt \right)^{1/\gamma}, \end{aligned} \tag{2.29}$$

$$u(x'_0, t'_0) \leq 4L'. \tag{2.30}$$

Here, in $(2.29)_3$ and (2.30), the positive number L' may depend on (x'_0, t'_0). Now we claim that the positive numbers L' is bounded uniformly on (x'_0, t'_0). Indeed there exists a positive $L > L'$ such that

$$\begin{aligned} & L \geq (2r_0)^{-n-p} \delta_0^{-\gamma}, \\ & \frac{1}{(\delta_0)^{\gamma}} \left(\frac{1}{(r_0)^{n+p} L} + \frac{1}{|Q(r_0)|} \int_{Q(2r_0)} u^{q+1} \, dx \, dt \right) < \frac{2}{(\delta_0)^{\gamma} |Q(r_0)|} \int_{Q(2r_0)} u^{q+1} \, dx \, dt \end{aligned} \tag{2.31}$$

Because the positive constant L in (2.31) does not depend on any $(x'_0, t'_0) \in Q$. In fact, for any positive $l \geq 1$ and any point $(x'_0, t'_0) \in Q(r_0)$, $\hat{Q}'(l, r_0)$ is contained in $Q(2r_0)$ and thus, we have,

$$\begin{aligned} & \frac{1}{\delta_0} \left(\frac{1}{(r_0)^{n+p} l} + \frac{1}{|\hat{Q}'(l)|} \int_{\hat{Q}'(l, r_0)} \frac{u^{q+1}}{l^{q+1}} \, dx \, dt \right)^{1/\gamma} \\ & = \frac{1}{\delta_0} \left(\frac{1}{(r_0)^{n+p} l} + \frac{1}{|B(1)| r_0^{n+p}} \int_{\hat{Q}'(l, r_0)} u^{q+1} \, dx \, dt \right)^{1/\gamma} \\ & < \frac{1}{\delta_0} \left(\frac{1}{(r_0)^{n+p} l} + \frac{1}{|Q(r_0)|} \int_{Q(2r_0)} u^{q+1} \, dx \, dt \right)^{1/\gamma}. \end{aligned} \tag{2.32}$$

L' is a root of the equation (2.29) and L is that of (2.31) and thus, L doed not dpend on $(x'_0, t'_0) \in Q$ and $L' < L$.

Therefore from (2.30) and the observation above it follows that

$$u(x'_0, t'_0) \leq 4L \qquad \text{for any } (x'_0, t'_0) \in Q. \tag{2.33}$$

We write L as L_k to indicate the dependence of L on t_k and r_k. Now we introduce the scaled solution defined as

$$(2.34)\qquad v_k(x,t) = \frac{u\left(x_0 + L_k^{\frac{p-(q+1)}{p}} x,\ t_k + t\right)}{L_k},$$

$$(x,t) \in Q'(k) = B(k) \times J(k), \qquad B(k) = B\left(0,\ r_k L_k^{\frac{q+1-p}{p}}\right), \quad J_k = (-(r_k)^p,\ 0)\,.$$

From $(2.31)_1$, the space-width of $Q'(k)$ is computed as

$$(2.35)\qquad r_k L_k^{\frac{q+1-p}{p}} \geq (\delta_0)^{-\frac{p\gamma}{n-p}} (r_k)^{\frac{-p(1+n+p)-n}{n-p}} \nearrow \infty \quad \text{as } k \to \infty,$$

because $\frac{-p(1+n+p)-n}{n-p} < 0$ and δ_0 is a fixed positive number, and the time-length $(r_k)^p \searrow 0$ as $k \to \infty$, and thus, the sequence of sets $\{Q'(k)\}$ converges to all of space $\mathbb{R}^n$. By (2.33), we have the boundedness

$$(2.36)\qquad \sup_{Q'(k)} v_k \leq 4$$

and compute the integral quantities of v_k for any $t \in (-(r_k)^p,\ 0)\,,$ as

$$(2.37)\qquad \|v_k(t)\|_{L^{q+1}(B(k))} = \|u(t_k + t)\|_{L^{q+1}(B(x_0,\, r_k)},$$

$$(2.38)\qquad \|\nabla v_k(t)\|_{L^p(B(k))} = \|\nabla u(t_k + t)\|_{L^p(B(x_0, r_k)}$$

$$(2.39)\qquad \int_{-(r_k)^p}^{0} \|\partial_t v_k(t)\|^2_{L^2(B(k))}\, dt = \int_{t_k-(r_k)^p}^{t_k} \|\partial_t u(t)\|^2_{L^2(B(x_0, r_k/2))}\, dt.$$

By virtue of the boundedness (1.2) and (1.3) we can argue similarly as (2.10)-(2.15) to have subsequences $\{t'_k\}$, $\{r'_k\}$ (non-relabelled), a sequence $\{\tau_k\}$ and the limit $w \in W^{1,p} \cap L^{q+1}(\mathbb{R}^n)$ such that

$$t'_{k+1} - t'_k \geq 1, \qquad \tau_k \in (t'_k - (r'_k)^p,\ t'_k) \qquad \text{for all } k = 1, \ldots,$$

$$\int_{B(k)} |\partial_t v_k(\tau_k)|^2\, dx \longrightarrow 0 \qquad \text{as } k \to \infty,$$

$$\int_{\mathbb{R}^n} \left(\partial_t v_k(\tau'_k)\phi + |\nabla v_k(\tau_k)|^{p-2} \nabla v_k(\tau_k) \cdot \nabla\phi - (\lambda v_k^q)(\tau_k)\right) dx = 0 \qquad \text{for all } \phi \in C_0^\infty(\mathbb{R}^n),$$

$$v_k(\tau_k) \longrightarrow w \qquad \text{weakly in } W^{1,p}(\mathbb{R}^n),$$

$$v_k(\tau_k) \longrightarrow w \qquad \text{strongly in } L^r(\mathbb{R}^n), \quad \forall r \in [1, p^*), \text{ and almost everywhere } \mathbb{R}^n,$$

$$\lambda(\tau_k) \longrightarrow \lambda_\infty,$$

$$Dv_k(\tau_k) \longrightarrow Dw \qquad \text{strongly in } L^r(\mathbb{R}^n), \quad \forall r \in [1, p),$$

$$\int_{\mathbb{R}^n} \left(|\nabla w|^{p-2} \nabla w \cdot \nabla\phi - \lambda_\infty w^q \phi\right) dx = 0 \qquad \text{for all } \phi \in C_0^\infty(\mathbb{R}^n).$$

Further we have the strong convergence of gradients

$$(2.40)\qquad Dv_k(\tau_k) \longrightarrow Dw \qquad \text{strongly in } L^p(\mathbb{R}^n),$$

of which the proof is performed similarly as in (2.19)-(2.24) by the use of (2.36). The proof of Theorem 3 is completed. □

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