# GEOMETRICAL CONSTANTS OF 2-DIMENSIONAL SPACES WITH EXTREME NORMS IN $AN_2$

Naoto Komuro (Hokkaido University of Education)
Ken-Ichi Mitani (Okayama Prefectural University)
Tomoyuki Okazaki (Asahikawa Kita Highschool)

**Abstract**

$AN_2$ (the set of all absolute normalized norms on $\mathbb{R}^2$) has convex structure, and some geometric constants have weak convex properties. We focus on the set of all extreme points of $AN_2$ (denoted by $E(AN_2)$) and report some results obtained by the calculation of modified von Neumann-Jordan constant of the norms in $E(AN_2)$. Also we summarize the previous results on convex property of von Neumann-Jordan constant and James constant.

## 1. Introduction and Preliminaries

A norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be absolute if $\|(x,y)\| = \|(|x|,|y|)\|$ for all $(x,y) \in \mathbb{R}^2$, and normalized if $\|(1,0)\| = \|(0,1)\| = 1$. The set of all absolute normalized norms on $\mathbb{R}^2$ is denoted by $AN_2$. Let $\Psi_2$ be the set of all convex functions $\psi$ on $[0,1]$ satisfying $\max\{1-t,t\} \leq \psi(t) \leq 1$ for $t \in [0,1]$. $\Psi_2$ and $AN_2$ can be identified by a one to one correspondence $\psi \to \|\cdot\|_\psi$ with the relation $\psi(t) = \|(1-t,t)\|_\psi$ for $t \in [0,1]$.

For $1 \leq p \leq \infty$, we denote

$$\psi_p(t) = \begin{cases} \{(1-t)^p + t^p\}^{\frac{1}{p}} & (1 \leq p < \infty) \\ \max\{1-t,t\} & (p=\infty). \end{cases}$$

$\psi_p \in \Psi_2$, and they correspond to the $l_p$-norms $\|\cdot\|_p$ on $\mathbb{R}^2$ defined by

$$\|(x_1,x_2)\|_p = \begin{cases} (|x_1|^p + |x_2|^p)^{\frac{1}{p}} & (1 \leq p < \infty) \\ \max\{|x_1|,|x_2|\} & (p=\infty). \end{cases}$$

A subset S of $N_2$ (the set of all norms on $\mathbb{R}^2$) is said to have convex structure (or to be convex) if

$$\|\cdot\|,\ \|\cdot\|' \in S \Longrightarrow (1-\lambda)\|\cdot\| + \lambda\|\cdot\|' \in S \quad (\lambda \in [0,1]).$$

---

2010 *Mathematics Subject Classification.* 46B20.

*Key words and phrases.* Absolute normalized norm, von Neumann-Jordan constant, modified von Neumann-Jordan constant, extreme norm.

The second author was supported in part by Grants-in-Aid for Scientific Research (No.21K03275), Japan Society for the Promotion of Science.

It is clear that $AN_2$ has convex structure, and the correspondence $\psi \to \|\cdot\|_\psi$ preserves the operation to take convex combination, that is,

$$(1-\lambda)\|\cdot\|_\psi + \lambda\|\cdot\|_{\psi'} = \|\cdot\|_{(1-\lambda)\psi+\lambda\psi'}$$

for $\psi,\ \psi' \in \Psi_2$ and $\lambda \in [0,1]$. It is known that neither James constant nor von Neumann-Jordan constant is convex as functions on $AN_2$. ([7, 8]) In Section 2, we will give some convex subsets of $N_2$ on which they are convex function. We also summarize the important facts which hold in those subsets.

In section 3, we will focus our consideration on the norms which are extreme points of $AN_2$, and report some results obtained by the calculation of modified von Neumann-Jordan constant of those extreme norms.

## 2. Convex properties of von Neumann-Jordan constant and James constant

For a normed space $(X, \|\cdot\|)$, von Neumann-Jordan constant (NJ constant) and James constant are defined by

$$C_{NJ}((X,\|\cdot\|)) = \sup\left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2+\|y\|^2)} \mid (x,y) \neq (0,0) \right\},$$
$$J((X,\|\cdot\|)) = \sup\{\min\{\|x+y\|, \|x-y\|\} \mid x, y \in S_X\},$$

where $S_X = \{x \in X \mid \|x\| = 1\}$.

For the basic properties and applications, we refer to ([3],[4],[5],[12],[16],[17], [20]). In this note we consider only the case that $\dim X = 2$. The following proposition suggests that the two-dimensional case is fundamental to investigate these constants.

**Proposition 2.1.** *For every Banach space $X$, we have*

$$C_{NJ}((X,\|\cdot\|)) = \sup\{C_{NJ}((E,\|\cdot\|)) \mid E : \textit{subspace of } X,\ \dim E = 2\}$$
$$J((X,\|\cdot\|)) = \sup\{\ J((E,\|\cdot\|))\ \mid E : \textit{subspace of } X,\ \dim E = 2\}.$$

Let

$$\Psi_2^+ = \{\psi \in \Psi_2 \mid \psi(t) \geq \psi_2(t)\ (t \in [0,1])\},$$
$$\Psi_2^- = \{\psi \in \Psi_2 \mid \psi(t) \leq \psi_2(t)\ (t \in [0,1])\},$$

and let

$$AN_2^+ = \{\|\cdot\| \in AN_2 \mid \|x\| \geq \|x\|_2\ (x \in \mathbb{R}^2)\},$$
$$AN_2^- = \{\|\cdot\| \in AN_2 \mid \|x\| \leq \|x\|_2\ (x \in \mathbb{R}^2)\}.$$

Then $\|\cdot\|_\psi \in AN_2^+ \Leftrightarrow \psi \in \Psi_2^+$, and $\|\cdot\|_\psi \in AN_2^- \Leftrightarrow \psi \in \Psi_2^-$. It is easy to see that $AN_2^+$ and $AN_2^-$ are both convex subsets of $AN_2$.

**Theorem 2.1.** ([20]) $\max\{M_1^2, M_2^2\} \le C_{NJ}((\mathbb{R}^2, \|\cdot\|_\psi)) \le M_1^2 M_2^2 \quad (\psi \in \Psi_2)$ *where*

$$M_1 = \max_{t\in[0,1]} \frac{\psi(t)}{\psi_2(t)} = \max_{x\neq 0} \frac{\|x\|_\psi}{\|x\|_2}, \quad M_2 = \max_{t\in[0,1]} \frac{\psi_2(t)}{\psi(t)} = \max_{x\neq 0} \frac{\|x\|_2}{\|x\|_\psi}.$$

**Corollary 2.1.** *If $\psi \in \Psi_2^+$, then $C_{NJ}((\mathbb{R}^2, \|\cdot\|_\psi)) = M_1^2$, and if $\psi \in \Psi_2^-$, then $C_{NJ}((\mathbb{R}^2, \|\cdot\|_\psi)) = M_2^2$.*

**Theorem 2.2.** ([8, 19]) *von Neumann-Jordan constant $C_{NJ}((\mathbb{R}^2, \|\cdot\|))$ is a convex function on each of $AN_2^+$ and $AN_2^-$.*

Let $\Psi_2^s = \{\psi \in \Psi_2 \mid \psi(t) = \psi(1-t) \ (t \in [0,1])\}$, and $AN_2^s = \{\|\cdot\|_\psi \in AN_2 \mid \psi \in \Psi_2^s\}$. It is easy to see that $\|\cdot\| \in AN_2^s$ if and only if $\|(x,y)\| = \|(y,x)\|$ holds for all $(x,y) \in \mathbb{R}^2$. Regarding the fact that NJ constant is not convex on $AN_2$, the following proposition gives a stronger result.

**Proposition 2.2.** ([8, 19]) *$AN_2^s$ is a convex subset of $AN_2$, and $C_{NJ}((\mathbb{R}^2, \|\cdot\|)$ is not a convex function on $AN_2^s$.*

For $\theta \in (0, \pi]$, let $M(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$,
and let

$$N_2(\theta) = \{\|\cdot\| \in N_2 \mid \|M(\theta)x\| = \|x\| \ (x \in \mathbb{R}^2)\}.$$

If $\|\cdot\| \in N_2(\theta)$, $S_{(\mathbb{R}^2, \|\cdot\|)}$ is invariant under the rotation of angle $\theta$. It is known that if $\|\cdot\| \in N_2(\pi/4)$, then $J((\mathbb{R}^2, \|\cdot\|)) = \sqrt{2}$. It is easy to see that $N_2(\pi/4) \subset N_2(\pi/2) \subset N_2(\pi) = N_2$, and all $\ell_p$ norms $\|\cdot\|_p$ on $\mathbb{R}^2$ belong to $N_2(\pi/2)$. Moreover we have

**Lemma 2.1.** *$N_2(\pi/2) \cap AN_2 = AN_2^s$ holds.*

A vector $x \in \mathbb{R}^2$ is said to be isoceles orthogonal to a vector $y \in \mathbb{R}^2$ ($x \perp_I y$) if $\|x + y\| = \|x - y\|$. ([1]) It is known that if $x \in S_{(\mathbb{R}^2, \|\cdot\|)}$, there is only two vectors $y, -y \in S_{(\mathbb{R}^2, \|\cdot\|)}$ such that $x \perp_I y$ ([3]). The following is the key result for the assertions of this section. We write $x \perp y$ if the angle between $x$ and $y$ is $\pi/2$. In this note we write

$$\overline{x} = M(\pi/2)x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x$$

for $x \in \mathbb{R}^2$.

**Theorem 2.3.** ([7]) *Let $\|\cdot\| \in N_2$, and let $x, y \in S_{(\mathbb{R}^2, \|\cdot\|)}$. Then the following conditions are equivalent.*

(1) $\|\cdot\| \in N_2(\pi/2)$.

(2) $x \perp_I y \Longleftrightarrow x \perp y$.

From these facts, we can see that the definition of James constant has some other expressions as follows.

**Corollary 2.2.** ([6]) *$J((\mathbb{R}^2, \|\cdot\|)) = \sup\{\|x+y\| \mid x, y \in S_X,\ x \perp_I y\}$ holds, and if $\|\cdot\| \in N_2(\pi/2)$, then*

$$J((\mathbb{R}^2, \|\cdot\|)) = \sup\{\|x+\bar{x}\| \mid x \in S_X\} \tag{2.1}$$

$$= \sqrt{2}\|M(\pi/4)\|. \tag{2.2}$$

*where $\|M(\pi/4)\|$ is the operator norm of $M(\pi/4)$.*

It follows from (2.2) that $J((\mathbb{R}^2, \|\cdot\|)) = \sqrt{2} \;\Leftrightarrow\; \|\cdot\| \in N_2(\pi/4)$ holds for $\|\cdot\| \in N_2(\pi/2)$.([6]) Using (2.1), we obtain the following formula which is useful in calculation of James constant.

**Theorem 2.4.** *If $\|\cdot\|_\psi \in N_2(\pi/2)$, then*

$$J((\mathbb{R}^2, \|\cdot\|_\psi)) = \max_{0\le t\le \frac{1}{2}} \frac{2-2t}{\psi(t)}\psi(\frac{1}{2-2t}). \tag{2.3}$$

**Remark.** In this theorem, the definition of $\psi$ is extended to the norms in $N_2(\pi/2)$ by the same formula $\psi(t) = \|(1-t, t)\|_\psi$.

The formula (2.3) was first proved in ([12]) in the case that $\|\cdot\|_\psi \in AN_2^s (= AN_2 \cap N_2(\pi/2))$. We have another version of this formula as follows.

**Theorem 2.5.** ([7]) *Let $\|\cdot\| \in N_2(\pi/2)$, and let $\varphi(t) = \|(t,1)\|$ $(t \in \mathbb{R})$. Then $\varphi$ is a convex function on $\mathbb{R}$, and the James constant is given by*

$$J((\mathbb{R}^2, \|\cdot\|)) = \sup_{t\ge 0} \frac{t+1}{\varphi(t)}\varphi(\frac{t-1}{t+1}).$$

A function $f$ on a convex set $C$ is said to be quasiconvex if for any $\alpha \in \mathbb{R}$, $\{x \in C \mid f(x) \le \alpha\}$ is a convex subset of $C$, equivalently, $f((1-\lambda)x + \lambda x') \le \max\{f(x), f(x')\}$ for $x, x' \in C$ and $\lambda \in [0,1]$. Convex functions are always quasiconvex, and every quasiconvex function on bouded closed convex set $C$ takes its maximum at an extreme point of $C$. The following result is obtained by Corollary 2.2.

**Theorem 2.6.** ([7]) *James constant $J((\mathbb{R}^2, \|\cdot\|))$ is quasiconvex on $N_2(\pi/2)$.*

For the dual space $X^*$ of $X$, the formula $J(X^*) = J(X)$ is not always valid while $C_{NJ}(X^*) = C_{NJ}(X)$ holds in general. In ([18]), it was proved that $J((\mathbb{R}^2, \|\cdot\|^*)) = J((\mathbb{R}^2, \|\cdot\|))$ holds when $\|\cdot\| \in AN_2^s$. This result can be generalized by the following theorem which is obtained by the last formula in Corollary 2.2.

**Theorem 2.7.** ([6]) *$J((\mathbb{R}^2, \|\cdot\|^*)) = J((\mathbb{R}^2, \|\cdot\|))$ holds for $\|\cdot\| \in N_2(\pi/2)$.*

## 3. Modified von Neumann-Jordan constant of extreme norms of $AN_2$

For $\alpha,\ \beta$ with $0 \le \alpha \le \frac{1}{2} < \beta \le 1$, we define

$$\psi_{\alpha,\beta}(t) = \begin{cases} 1-t & (t \in [0,\alpha]) \\ \dfrac{\alpha+\beta-1}{\beta-\alpha}t + \dfrac{\beta-2\alpha\beta}{\beta-\alpha} & (t \in [\alpha,\beta]) \\ t & (t \in [\beta,1]), \end{cases}$$

and when $0 \le \alpha \le \frac{1}{2} = \beta$, we put $\psi_{\alpha,\beta} = \psi_\infty$. The corresponding norm is

$$\|(x_1,x_2)\|_{\psi_{\alpha,\beta}} = \begin{cases} |x_1| & (|x_2| \le \dfrac{\alpha}{1-\alpha}|x_1|) \\ f(x_1,x_2) & (\dfrac{\alpha}{1-\alpha}|x_1| \le |x_2|,\ \dfrac{1-\beta}{\beta}|x_2| \le |x_1|) \\ |x_2| & (|x_1| \le \dfrac{1-\beta}{\beta}|x_2|), \end{cases}$$

where $f(x_1,x_2) = \dfrac{(1-2\alpha)\beta}{\beta-\alpha}|x_1| + \dfrac{(2\beta-1)(1-\alpha)}{\beta-\alpha}|x_2|$, for $(x_1,x_2) \in \mathbb{R}^2$.

We denote $E = \{\psi_{\alpha,\beta} \mid 0 \le \alpha \le \frac{1}{2} \le \beta \le 1\}$. Then it is easy to see that $E \subset \Psi_2$. We have

**Theorem 3.1.** ([11]) *The following conditions are equivalent.*

(1) $\|\cdot\|_\psi$ *is an extreme point of* $AN_2$.
(2) $\psi$ *is an extreme point of* $\Psi_2$.
(3) $\psi \in E$.

**Theorem 3.2.** ([9])

$$J((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})) = \max\{\frac{1}{\psi_{\alpha,\beta}(1/2)},\ 1+\frac{1}{2\psi_{\alpha,\beta}(1/2)+\gamma},\ 2\psi_{\alpha,\beta}(1/2)\},$$

*where*

$$\gamma = \gamma(\alpha,\beta) = \begin{cases} \dfrac{2\beta-1}{\beta-\alpha} & (\alpha+\beta \le 1) \\ \dfrac{1-2\alpha}{\beta-\alpha} & (\alpha+\beta \ge 1). \end{cases}$$

For a fixed $\psi_{\alpha,\beta} \in E$, we denote $E_{\alpha,\beta} = \{\psi \in \Psi_2 \mid \psi_\infty \le \psi \le \psi_{\alpha,\beta}\}$. It is easy to see that $E_{\alpha,\beta}$ is a convex subset of $\Psi_2$. James constant has another convex property as follows.

**Theorem 3.3.** ([10]) *$E$ has infinitely many elements $\psi_{\alpha,\beta}$ satisfying* $\max\{\beta-\alpha,\ 2\beta-1\} \le \alpha\beta$. *For such an element $\psi_{\alpha,\beta}$, we have*

(1) $J((\mathbb{R}^2,\|\cdot\|_\psi)) = 1/\psi(1/2)$ *holds for* $\psi \in E_{\alpha,\beta}$.
(2) *The function* $E_{\alpha,\beta} \ni \psi \longrightarrow J((\mathbb{R}^2,\|\cdot\|_\psi))$ *is convex on* $E_{\alpha,\beta}$.

**Definition.** For a normed space $(X, \|\cdot\|)$,

$$C'_{NJ}((X, \|\cdot\|)) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{4} \mid x, y \in S_X\right\}.$$

is called modified von Neumann-Jordan constant (mod. NJ constant).

It was introduced by J. Gao in ([3]) and Y. Takahashi in ([21]). The following proposition is a summary of the basic properties of mod. NJ constant.

**Proposition 3.1.** *For any normed space $X$, we have the following.*

(1) $1 \le C'_{NJ}(X) \le C_{NJ}(X) \le 2.$

(2) $C_{NJ}(X) = 1 \Leftrightarrow C'_{NJ}(X) = 1 \Leftrightarrow X$ *is an inner product space.* ([2])

(3) $C_{NJ}(X) = 2 \Leftrightarrow C'_{NJ}(X) = 2 \Leftrightarrow X$ *is not uniformly nonsquare.*

(4) $C'_{NJ}(X^*) = C'_{NJ}(X)$ *is not true* ([21]) *while* $C_{NJ}(X^*) = C_{NJ}(X)$ *holds.*

(5) $\max\{M_1^2, M_2^2\} \le C'_{NJ}(X)$ *is not true* ([15]) *while* $\max\{M_1^2, M_2^2\} \le C_{NJ}(X) \le M_1^2 M_2^2$ *holds.*

(6) *If* $\psi \in \Psi_2^-$, *then* $C'_{NJ}((\mathbb{R}^2, \|\cdot\|_\psi)) = C_{NJ}((\mathbb{R}^2, \|\cdot\|_\psi)) = M_2^2$. ([15])

(7) $C'_{NJ}((\mathbb{R}^2, \|\cdot\|_p)) = C_{NJ}((\mathbb{R}^2, \|\cdot\|_p)) = \max\{2^{\frac{2}{p}-1},\ 2^{1-\frac{2}{p}}\}$. ([15])

To state the result of calculation of mod. NJ constant, we adopt the new parameter $(s, t)$ defined by

$$s = \frac{\alpha}{1-\alpha}, \quad t = \frac{1-\beta}{\beta} \quad (0 \le \alpha \le \tfrac{1}{2} \le \beta \le 1).$$

It is easy to see that $0 \le s, t \le 1$, and the unit sphere of $\|\cdot\|_{\psi_{\alpha,\beta}}$ is an octagon with vertices $(1, s)$, $(t, 1)$ in the first quadrant when $0 < s, t < 1$.

**Theorem 3.4.** *If* $\|\cdot\|_{\psi_{\alpha,\beta}} \in E(AN_2)$, *and* $\psi_{\alpha,\beta} \in \Psi_2^-$, *then*

$$\begin{aligned} C'_{NJ}((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})) &= C_{NJ}((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})) \\ &= M_2^2 \\ &= \begin{cases} 1 + s^2 & (0 \le t \le s \le 1) \\ 1 + t^2 & (0 \le s \le t \le 1). \end{cases} \end{aligned}$$

*The main result of this note is the following. Since* $C'_{NJ}((\mathbb{R}^2, \|\cdot\|_\infty)) = C_{NJ}((\mathbb{R}^2, \|\cdot\|_\infty)) = 2$ *is known, we consider the case when* $\alpha,\ \beta \ne \frac{1}{2}$ *(equivalently,* $s,\ t \ne 1$*).*

**Theorem 3.5.** *Suppose that* $\|\cdot\|_{\psi_{\alpha,\beta}} \in E(AN_2)$ *and* $\alpha,\ \beta \neq \frac{1}{2}$, *then*

$$C'_{NJ}((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})) = \max\left\{1+s^2,\ 1+t^2,\ 1+\left(\frac{(1-s)(1-t)}{1-st}\right)^2\right\}.$$

In the most cases that $\psi_{\alpha,\beta} \in E \setminus \Psi_2^-$, $C_{NJ}((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}}))$ is still unknown. However, if $0 \leq s, t < 1$, $s \neq t$, and $\max\{s,t\} < \dfrac{(1-s)(1-t)}{1-st}$, then $\psi_{\alpha,\beta} \in E \setminus \Psi_2^-$, and we can assert that $C_{NJ}((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}}))$ does not coincide with $C'_{NJ}((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}}))$.

**Theorem 3.6.** *Suppose that* $\psi_{\alpha,\beta} \in E$. *We have*

(1) *If* $0 \leq s, t < 1$, $s \neq t$, *and* $\max\{s,t\} < \dfrac{(1-s)(1-t)}{1-st}$, *then* $C'_{NJ}((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})) < C_{NJ}((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}}))$.

(2) $\max\{M_1^2, M_2^2\} \leq C'_{NJ}((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}}))$ *holds for every* $\psi_{\alpha,\beta} \in E$.

## References


[1] J. Alonso, H. Martini and S. Wu, *On Birkhoff orthogonality and isoceles orthogonality in normed linear spaces*, Aequat. Math., 83 (2012) pp.153-189.

[2] Mahlon M. Day, *Some characterizations of inner product spaces*, Trans. A.M.S., 62 (1947) pp.320-337.

[3] J. Gao, *A Pythagorian approach in Banach spaces*, J. Inequal. Appl. Art. ID 94982 (2006) pp.1-11.

[4] M. Kato and L. Maligranda, *On the James and Jordan-von Neumann constants of Lorentz sequence spaces*, J. Math. Anal. Appl., 258 (2001) pp.457-465.

[5] M. Kato, L. Maligranda and Y. Takahashi, *On the James and Jordan-von Neumann constants and normal structure coefficient of Banach spaces*, Studia Math., 144 (2001) pp.275-295.

[6] N. Komuro, K. -S. Saito and R. Tanaka, *A sufficient condition that $J(X^*) = J(X)$ holds for a Banach space $X$*, Tokyo J. Math., 41 (2018) pp.219-223.

[7] N. Komuro, K. -S. Saito and K.-I. Mitani, *Convex property of James and von Neumann-Jordan constant of absolute norms on $\mathbb{R}^2$*, Proceedings of the 8th International Conference on NACA, (2015) pp.301-308.

[8] N. Komuro, K. -S. Saito, K.-I. Mitani and R. Tanaka, *On the convexity of von Neumann-Jordan constant*, J. Nonlinear Convex Anal., 16 no.11(2015) pp.2263-2268.

[9] N. Komuro, K.-S. Saito, and K.-I. Mitani, *Extremal structure of absolute normalized norms on $\mathbb{R}^2$ and the James constant*, Appl. Math. Comput., 217 (2011), pp.10035-10048.

[10] N. Komuro, K.-S. Saito, and K.-I. Mitani, *Extremal structure of Absolute Normalized Norms on $\mathbb{R}^2$ III*, Proc. of 3rd International Conference on ISBFS, 2009 (2011) pp.271-279.

[11] N. Komuro, K.-S. Saito, and K.-I. Mitani, *Extremal structure of the set of absolute norms on $\mathbb{R}^2$ and the von Neumann-Jordan constant*, J. Math. Anal. and Appl., 370 (2010), pp.101-106.

[12] K.-I. Mitani and K.-S. Saito, *Dual of two dimensional Lorentz sequence spaces*, Nonlinear Analysis, 71 (2009), pp.5238-5247.

[13] K.-I. Mitani, K.-S. Saito and T. Suzuki, *On the calculation of the James constant of Lorentz sequence spaces*, J. Math. Anal. Appl., 343 (2008) pp.310-314.

[14] K.-I. Mitani and K.-S. Saito, *The James constant of absolute norms on* $\mathbb{R}^2$, J. Nonlinear Convex Anal., 4 (2003) pp.399-410.

[15] Hiroyasu Mizuguchi and K.-S. Saito, *Some geometric constants of absolute normalized norms on* $\mathbb{R}^2$, Ann. Funct. Anal., 2(2011) no.2 pp.22-33.

[16] W. Nilsrakoo and S. Saejung, *The James constant of normalized norms on* $\mathbb{R}^2$, J. Inequal. Appl., (2006) Art. ID 26265, 12pp.

[17] S. Saejung, *On James and von Neumann-Jordan constants and sufficient conditions for the fixed point property*, J. Math. Anal. Appl., 323 (2006) pp.1018-1024.

[18] K.-S. Saito, M. Sato and R.Tanaka *When does the equality* $J(X^*) = J(X)$ *hold for a two-dimensional Banach space* $X$, Acta Math. Sin. Engl. Ser., 31(2015), no.8 pp1303-1314.

[19] K.-S. Saito, *How to calculate geometric constants of Banach Spaces*, Proc. of 7th International Conference on NACA2011, (2013). pp.193-203.

[20] K.-S. Saito, M.Kato and Y. Takahashi, *von Neumann-Jordan constant of absolute normalized norms on* $\mathbb{C}^2$, J. Math. Anal. Appl., 244(2000) no.2 pp.515-532.

[21] Y. Takahashi, *Some geometric constants of Banach spaces. A unified approach.*, Proc. of 2nd International Conference on ISBFS, (2006) pp.191-220.