ON BEILINSON'S l-ADIC EISENSTEIN CLASSES

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Abstract. In this paper, we will reconstruct Beilinson's l-adic Eisenstein classes of the universal elliptic curve. Moreover, we will give an idea to construct some elements of étale cohomology group from the residues of differential forms.

1. Motivation

This is a summary of our talk at "Algebraic Number Theory and related topics 2022" held at RIMS. This is a continuation of our work talking at "Algebraic Number Theory and related topics 2011" held at RIMS. Our work originated in the work of Takako Fukaya, Kazuya Kato, and Nobushige Kurokawa in [7]. They constructed some collection of differential forms of the product of two modular curves, which satisfies some norm relations, by restricting some collection of Siegel Eisenstein series of Sp(4) to the diagonal.

In the author's master's thesis [15], we have constructed some differential Euler systems associated to the symmetric squares of modular forms by modifying their differential forms.

In [16] (our RIMS talk in 2011), we have made a conjecture that there exist Euler systems of the Milnor K-group $K_3^{(M)}$ of the function fields of the product of two modular curves such that their images by the dlog map are the differential Euler systems mentioned above.

Our strategy to prove this conjecture is to construct elements of $K_3^{(M)}$ of the function fields of Siegel modular varieties of $Sp(4)$ such that their images by the dlog map are the Siegel Eisenstein series (not restriction to diagonal) appeared in [16].

One of final goals of our research is to construct Euler systems in K-groups (or étale cohomology) from some differential Euler systems constructed from Eisenstein series of some Shimura varieties.

To do this, in this paper, we will give an idea to construct some elements of étale cohomology from the residues of differential forms. It is based on the analogy between residues of étale cohomology and residues of differential forms. This work is incomplete but we can reconstruct Beilinson's l-adic Eisenstein classes of the universal elliptic curve using it. This reconstruction is the main result of this paper. We also prove the Manin-Drinfeld theorem by the same idea.

The author would like to thank Tomokazu Kashio for his hospitality at the symposium. The author also would like to thank Masataka Chida for teaching me the references [4], [14], [10] and [9]. Finally, I would like to thank Yoichi Mieda for a comment to our talk (see Section 12 for the details of the comment).

2. Introduction

In this section, we give a brief summary of this paper. Throughout this paper, for $N \geq 3$, let $Y(N)$ be modular curve over $\mathbb{Q}$ of level N without cusps, and $X(N)$ be the smooth compactification of $Y(N)$. In this paper, we only use étale cohomology unless otherwise specified. At first, we recall the Manin-Drinfeld theorem.

Theorem 2.1. *(Theorem 4.1.)(Manin-Drinfeld theorem)*
Let P be a cusp of $X(N)$, O be the point at infinity of $X(N)$. There exists a positive integer C such that $C(P - O)$ is a principal divisor.

We will write summaries of the classical proof and our proof in Section 4. Next, we will reconstruct the l-adic realization of Beilinson's Eisenstein symbol, which we call Beilinson's l-adic Eisenstein

This work was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

classes, in Section 6.2. Note that Beilinson constructed the Eisenstein symbol in the motivic cohomology $H^2_{\mathcal{M}}(E, \mathbb{Z}(2))$ using some elements of $\Gamma(E - E[m], O_E^\times)$, where E is the universal elliptic curve over $Y(N)$, $E[m]$ is the divisor consisting of all m-torsion sections of E, and m is some positive integer.

The idea of our reconstruction is as follows. Let $X = X_{1,univ}(N)$, $Z = D_1 \cup D_2$, and $U = X - Z$. Here $X_{1,univ}(N)$, D_1 and D_2 will be defined in Section 5.2 (Definition 8, Definition 9). Roughly speaking $X_{1,\mathrm{univ}}(N)$ is a "proper smooth universal elliptic curve over $X(N)$" and Z is the fiber of $i\infty \in X(N)$. Consider the following residue exact sequence

$$(1) \quad \cdots \to H^2(X, \mathbb{Z}/l^n\mathbb{Z}(2)) \to H^2(U, \mathbb{Z}/l^n\mathbb{Z}(2)) \overset{\mathrm{Res}}{\to} H^3_Z(X, \mathbb{Z}/l^n\mathbb{Z}(2)) \overset{Gys_3}{\to} H^3(X, \mathbb{Z}/l^n\mathbb{Z}(2)) \to \cdots .$$

At first, we will construct the element $\mathrm{Eis}_{N,n}$ of $H^3_Z(X, \mathbb{Z}/l^n\mathbb{Z}(2))$ corresponding to some weight 3 Eisenstein series and show that $\mathrm{Gys}_3(C_N \mathrm{Eis}_{N,n}) = 0$ for some non-zero integer C_N using the Eichler-Shimura relation. The precise statement of our reconstruction is as follows. This is the main result of this paper.

Theorem 2.2. *(Theorem 6.1.) There exist a non-zero integer C (not depending on n) and $Z_{\mathrm{Eis}_{N,n}} \in H^2(U, \mathbb{Z}/l^n\mathbb{Z}(2))$ for each positive integer n satisfying the following property.*

- *The equality $\mathrm{Res}(Z_{\mathrm{Eis}_{N,n}}) = C\mathrm{Eis}_{N,n}$ is satisfied for each positive integer n, where Res is the residue map from $H^2(U, \mathbb{Z}/l^n\mathbb{Z}(2))$ to $H^3_Z(X, \mathbb{Z}/l^n\mathbb{Z}(2))$.*

Moreover $Z_{\mathrm{Eis}_{N,n}}$ satisfies the following properties.

- *Let p be a prime such that $p \equiv 1 \mod N$, $p \neq l$ and $p \in S_{X,good}$, where $S_{X,good}$ is defined in Section 3. (Note that there exist inifinitely many such prime numbers p.) There exists (a Hecke operator) $T \in \mathbb{Z}[T_{z,p}, T_{w,p}]$ such that $TZ_{\mathrm{Eis}_{N,n}}$ does not depend on the choice of $Z_{\mathrm{Eis}_{N,n}}$ for each positive integer n and $TZ_{\mathrm{Eis}_N} := (TZ_{\mathrm{Eis}_{N,n}})_{n \in \mathbb{Z}_{\geq 1}} \in H^2(U, \mathbb{Z}_l(2))$ is non-zero.*
- *Let m be an integer and q be a prime number congruent to 1 modulo N. The equalities $T_{z,q}(Z_{\mathrm{eig},\mathrm{Eis}_N}) = (q^2+1)Z_{\mathrm{eig},\mathrm{Eis}_N}$ and $T_{w,m}(Z_{\mathrm{eig},\mathrm{Eis}_N}) = mZ_{\mathrm{eig},\mathrm{Eis}_N}$ are satisfied, where we put $Z_{\mathrm{eig},\mathrm{Eis}_N} = TZ_{\mathrm{Eis}_N}$. Note that these eigenvalues are the same as those of weight 3 Eisenstein series.*

Note that the etale cohomology class $Z_{\mathrm{eig},\mathrm{Eis}}$ is the same as Beilinson's l-adic Eisenstein classes.

One of (conjectural) generalizations of our main theorem is to construct étale cohomology classes corresponding to Siegel Eisenstein series of Sp(4). In Section 7, we mention the l-adic Siegel Eisenstein classes without exact mathematical definitions.

As mentioned in Section 1, one of final goals of our work is to construct some Euler systems. In Section 8, we describe our two attempts to construct Euler systems of $K_3^{(M)}(\mathrm{Func}(X(N) \times X(N)))$. One attempt is to use the elements of $H^3_D(X(N) \times X(N))$, where D denotes the diagonal divisor. We also use the elements of $Z^1(X(N) \times X(N), \mathrm{Gerst}_3)$, where Z^1 denotes the 1-cocycle and Gerst denotes the Gersten complex. It is not related to the construction of étale cohomology classes from differential forms. But it is the origin of the work of this paper (see Section 1 and Section 8).

The other attempt is to use the (conjectural) l-adic Siegel Eisenstein classes. As mentioned in Section 1, our strategy to construct Euler systems is to construct elements of $H^3(V_2, \mathbb{Z}/l^n\mathbb{Z}(3))$ corresponding to the Siegel Eisenstein series (not restriction to diagonal) appeared in [16], where V_2 denotes some Shimura variety of $Sp(4)$. We have not constructed these elements yet. But, in Section 9, generalizing our idea to construct Beilinson's l-adic Eisenstein classes, we give the idea to construct etale cohomology classes from the residues of differential forms. (Note that we can regard weight 3 Siegel Eisenstein series of $Sp(4)$ as differential forms of degree 3 (see Section 7).) It is as follows. Let X be a i-dimensional scheme, Z be its divisor. Put $U = X - Z$. Consider the following residue exact sequence.

$$(2) \quad \cdots \to H^i(X, \mathbb{Z}/l^n\mathbb{Z}(i)) \to H^i(U, \mathbb{Z}/l^n\mathbb{Z}(i)) \overset{\mathrm{Res}}{\to} H^{i+1}_Z(X, \mathbb{Z}/l^n\mathbb{Z}(i)) \overset{\mathrm{Gys}_{i+1}}{\to} H^{i+1}(X, \mathbb{Z}/l^n\mathbb{Z}(i)) \to \cdots .$$

At first, we construct an element $Z_{i+1,\mathrm{supp}}$ of $H^{i+1}_Z(X, \mathbb{Z}/l^n\mathbb{Z}(i))$ and show that $\mathrm{Gys}_{i+1}(CZ_{i+1,\mathrm{supp}}) = 0$ for some non-zero integer C. We hope we can construct some elements of étale cohomology corresponding to differential forms using this method inductively. When X is a Shimura variety, main tools to show $\mathrm{Gys}_{i+1}(Z) = 0$ are Proposition 6 and the Eichler-Shimura relation.

The rest of this introduction, we mention some results related to our work.

- Faltings considered the Arithmetic Eisenstein classes on the Siegel space (see [4] for more details).
- The construction of the Eisenstein symbol using the polylogarithm (see [9] for more details).

Our method to construct elements of étale cohomology is related to that of Harder's.

- Harder constructed some elements of étale cohomology of some Shimura varieties using the theory of Eisenstein cohomology (see [6] for more details).

3. Definitions, Notations, and Phrases

In this paper, we always use the following notations unless otherwise stated.

Notation 1.

- Let l be a prime number.
- Let p be a prime number different from l.
- Let $\mathbb{Z}$ denote the set of all integers, and $\mathbb{Z}_{\geq 1}$ denote the set of all positive integers.
- For a field L, let G_L denote the absolute Galois group $\mathrm{Gal}(\bar{L}/L)$.
- For a scheme X, we will write the function field of X as $\mathrm{Func}(X)$.
- Let K be a number field.
- Let O_K be a discrete valuation ring such that $\mathrm{Frac}(O_K) = K$, where Frac means the fractional field.
- Let k be the residue field of the ring O_K, and $\bar{k}$ be its algebraic closure.
- Define $\bar{X} = X \otimes_{\mathrm{Spec}(K)} \mathrm{Spec}(\bar{K})$ for a scheme X over $\mathrm{Spec}(K)$.
- Let $\mathcal{X}$ be a scheme over $\mathrm{Spec}(O_K)$ such that $\mathcal{X} \otimes_{\mathrm{Spec}(O_K)} \mathrm{Spec}(K) = X$.
- Define $\mathcal{X}_{\bar{k}} = \mathcal{X} \otimes_{\mathrm{Spec}(O_K)} \bar{k}$ for a scheme $\mathcal{X}$ over $\mathrm{Spec}(O_K)$. We also use the notation $(\mathcal{X})_{\bar{k}}$ instead of $\mathcal{X}_{\bar{k}}$.
- Let Fr be the Frobenius endomorphism defined in Section 13.
- For a scheme X over K, let $S_{X,good}$ be the set of all prime numbers p such that there exist a discrete valuation ring O_K with mixed characteristic $(0, p)$ and a proper smooth scheme $\mathcal{X}$ over $\mathrm{Spec}(O_K)$ such that $\mathrm{Frac}(O_K) = K$ and $\mathcal{X} \otimes \mathrm{Spec}(\mathbb{Q}) = X$.
- Define $\Gamma_1(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid a \equiv d \equiv 1 \pmod{N}, c \equiv 0 \pmod{N}\}$.
- For $N \geq 3$, let $Y(N)$ be modular curve over $\mathbb{Q}$ of level N without cusps, and $X(N)$ be the smooth compactification of $Y(N)$.
- For $N \geq 3$, let $Y_1(N)$ be modular curve over $\mathbb{Q}$ of level $\Gamma_1(N)$ without cusps, and $X_1(N)$ be the smooth compactification of $Y_1(N)$.
- Let N be an integer greater than or equal to 3.
- For a positive integer n and an integer m, let $T_{z,n}$ and $T_{w,m}$ denote the Hecke operators defined in Section 12.2.

Phrase 1. In this paper, every phrase like "There exists a non-zero integer C such that (a statement like $\mathrm{Gys}(Z) = 0$)" always means "There exists a non-zero integer C (not depending on n) such that (a statement like $\mathrm{Gys}(Z) = 0$) for any integer n" unless otherwise stated.

4. Proofs of the Manin-Drinfeld theorem

In this section, we will write summaries of the classical proof and our proof of the Manin-Drinfeld theorem. At first, we recall the Manin-Drinfeld theorem.

Theorem 4.1. *(Manin-Drinfeld theorem)*
Let P be a cusp of $X(N)$, O be the point at infinity of $X(N)$. There exists a positive integer C such that $C(P - O)$ is a principal divisor.

Proof . At first, we will give a summary of the classical proof in [3]. The Hecke operator T_p acts on $(P - O) \in \mathrm{Pic}(X(N))$ by $T_p(P - O) = (p+1)(P - O)$, where p is a prime number congruent to 1 modulo N. By the Eichler-Shimura relation and the Weil conjecture, we see the all "eigenvalues" of the action of T_p on $\mathrm{Pic}_0(X(N)) \cong H^1(X(N), O_X)/H^1(X, \mathbb{Z})$ are all different from $p + 1$, where we write Pic_0 for the degree 0 part of the Picard group. From these two facts, we conclude $C(P - O) = 0 \in \mathrm{Pic}(X(N))$ for some non-zero integer C.

We will write a summary of our proof. Let $X = X(N)$, $U = Y(N)$ and $Z = X(N) - Y(N)$. We will consider the following Gysin sequence.

$$H^1(X, \mathbb{Z}/l^n\mathbb{Z}(1)) \to H^1(U, \mathbb{Z}/l^n\mathbb{Z}(1)) \stackrel{\mathrm{Res}}{\to} H^2_Z(X, \mathbb{Z}/l^n\mathbb{Z}(1)) \stackrel{\mathrm{Gys}_2}{\to} H^2(X, \mathbb{Z}/l^n\mathbb{Z}(1)) \tag{3}$$

We regard $P - O$ as the element of $H^2_Z(X, \mathbb{Z}/l^n\mathbb{Z}(1))$. The key of our proof is the following proposition.

Proposition 1. *There exists a positive integer C such that* $\mathrm{Gys}_2(C(P - O)) = 0$.

Proof . By definition, $\mathrm{Gys}_2(C(P-O)) \in H^2(X(N), \mathbb{Z}/l^n\mathbb{Z}(1))$. At first, we will decompose $H^2(X(N), \mathbb{Z}/l^n\mathbb{Z}(1))$ using the following Hochschild-Serre spectral sequence.

$$E_2^{p',q'} = H^{p'}(G_{\mathbb{Q}(\zeta_N)}, H^{q'}(\overline{X(N)}, \mathbb{Z}/l^n\mathbb{Z}(1))) \Rightarrow H^{n'}(X(N), \mathbb{Z}/l^n\mathbb{Z}(1)) \tag{4}$$

We use this spectral sequence in the case $n' = 2$. By the definition of spectral sequence, $E^2 := H^2(X(N), \mathbb{Z}/l^n\mathbb{Z}(1))$ has the filtration $E^2 = E^2_0 \supset E^2_1 \supset E^2_2 \supset E^2_3 = 0$ such that $E^2_{p'}/E^2_{p'+1} (= E_\infty^{p',2-p'})$ is the subquotient of $H^{p'}(G_{\mathbb{Q}(\zeta_N)}, H^{2-p'}(\overline{X(N)}, \mathbb{Z}/l^n\mathbb{Z}(1)))$ for each $p' = 0, 1, 2$. For simplicity, we write Fil^i instead of E^2_i for each $i = 0, 1, 2, 3$.

Let p_i be the quotient homomorphism from Fil^i to $\mathrm{Fil}^i/\mathrm{Fil}^{i+1}$ for each $i = 0, 1, 2$. We see $p_0(\mathrm{Gys}_2(C(P-O))) = 0 \in H^0(G_{\mathbb{Q}(\zeta_N)}, H^2(\overline{X(N)}, \mathbb{Z}/l^n\mathbb{Z}(1)))$ because $\deg(n(P - O)) = 0$. We see $p_1(\mathrm{Gys}_2(C(P - O))) = 0$ for some non-zero integer C because, by the Weil conjecture, all eigenvalues of the action of the Hecke operator T_p on $H^1(G_{\mathbb{Q}(\zeta_N)}, H^1(\overline{X(N)}, \mathbb{Z}/l^n\mathbb{Z}(1)))$ are all distinct from $p + 1$. The equality $p_2(\mathrm{Gys}_2(C(P-O))) = 0$ is proved by restricting the preceding Gysin sequence to the cusp of $X(N)$ other than O and P. (The details are left to the reader.) This completes the proof of the proposition.

From the preceding proposition, we see $C(P - O) \in \mathrm{Im}(H^1(U, \mathbb{Z}/l^n\mathbb{Z}(1)) \to H^2_Z(X, \mathbb{Z}/l^n\mathbb{Z}(1)))$. This implies $C(P - O) \in l^n Pic(X)$. Hence we see $C(P - O) = 0 \in \cap_{n=1}^\infty l^n \mathrm{Pic}(X) = 0$ by the Mordell-Weil theorem as desired.

5. Explicit descriptions of the universal elliptic curve

In this section, we will give the definition of the universal elliptic curve. Our definition is slightly different from the classical one. In the former part of this section, we will define an analytic version of the universal elliptic curve. In the latter part, we will define an algebraic structure of the analytic universal elliptic curve.

5.1. Analytic expression of the Universal Elliptic curve.

Definition 1. We define $\mathfrak{h}$ and $Y_{1,\mathrm{univ},analytic}(N)$ as follows.

- $\mathfrak{h} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$.
- $Y_{1,\mathrm{univ},analytic}(N) := \mathbb{Z}^2 \rtimes \Gamma_1(N) \backslash (\mathfrak{h} \times \mathbb{C})$.

Here the actions of $\mathbb{Z}^2$ and $SL_2(\mathbb{Z})$ on $\mathfrak{h} \times \mathbb{C}$ are given as follows.

- $(t_1, t_2)((z, w)) = (z, w - t_1 z - t_2)$ for $(t_1, t_2) \in \mathbb{Z}^2, (z, w) \in \mathfrak{h} \times \mathbb{C}$.
- $\begin{pmatrix} a & b \\ c & d \end{pmatrix}((z, w)) = ((az + b)/(cz + d), w/(cz + d))$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), (z, w) \in \mathfrak{h} \times \mathbb{C}$.

Definition 2. Let L be a lattice in $\mathbb{C}$. We will define $\wp_L(z)$ and $\wp'_L(z)$ as follows.

$$\wp_L(z) := \frac{1}{z^2} + \sum_{0 \neq l \in L} \frac{1}{(z - l)^2} - \frac{1}{l^2} \tag{5}$$

The sums converge absolutely and uniformally on any compact subset of $\mathbb{C} - L$.

$$\wp'_L(z) := (\partial/\partial z)\wp(z) \tag{6}$$

Note that, in many papers, the notations $\wp$ and $\wp'$ are used instead of $\wp_L$ and $\wp'_L$ respectively. For $z \in \mathfrak{h}$, define L_z as follows.

$$L_z := \mathbb{Z} + z\mathbb{Z}. \tag{7}$$

Lemma 1. *The following relation between the $\wp$-function and $\wp'$-function holds.*

$$(8)\quad \begin{aligned}\wp'_{L_z}(w)^2 &= 4(\wp_{L_z}(w)-\wp_{L_z}(z/2))(\wp_{L_z}(w)-\wp_{L_z}((z+1)/2))(\wp_{L_z}(w)-\wp_{L_z}(1/2))\\ &= 4\wp_{L_z}(w)^3-\frac{4}{3}\pi^4E_4(z)\wp_{L_z}(w)-\frac{8}{27}\pi^6E_6(z)\end{aligned}$$

Here E_4 and E_6 denote the normalized Eisenstein series of weight 4 and 6 respectively.

Definition 3. Put $\lambda(z)=\wp'_{L_z}(1/N)/\wp_{L_z}(1/N)$. It is well-known that λ is a modular function of weight 1.

Put $x(z,w)=\lambda(z)^2\wp_{L_z}(w)$, $y(z,w)=\lambda(z)^3\wp'_{L_z}(w)$, $c_4=\frac{4}{3}\pi^4$, and $c_6=\frac{8}{27}\pi^6$. For simplicity of notation, we write x (resp. y) instead of $x(z,w)$ (resp. $y(z,w)$). Then the equation (8) becomes

$$(9)\quad y^2=4x^3-(c_4E_4(z)/\lambda(z)^4)x-c_6E_6(z)/\lambda(z)^6.$$

The N-torsion point (corresponding to $w=1/N$) of the elliptic curve defined by the above equation is calculated as follows.

$$(10)\quad (x(z,1/N),y(z,1/N))=(\wp_{L_z}(1/N)^3/\wp'_{L_z}(1/N)^2,\wp_{L_z}(1/N)^3/\wp'_{L_z}(1/N)^2).$$

Definition 4. We will denote by $F(N)$, $F_1(N)$ the function fields of $X(N)$, $X_1(N)\otimes_{\mathrm{Spec}(\mathbb{Q})}\mathrm{Spec}(\mathbb{Q}(\zeta_N))$ respectively. Namely, we put

$$(11)\quad F(N)=\mathbb{Q}(\{j(Az)\}_{A\in SL_2(\mathbb{Z}),\det(A)|N})$$

$$(12)\quad F_1(N)=F(N)^{\Gamma_1(N)}$$

Definition 5. Let b_4 and b_6 be given by $b_4=c_4E_4(z)/\lambda(z)^4$ and $b_6=c_6E_6(z)/\lambda(z)^6$.

Lemma 2. *The functions $b_4,b_6,x(z,1/N)=y(z,1/N)$ are in $F_1(N)$. The functions $x(z,z/N)$ and $y(z,z/N)$ are in $F(N)$.*

Proof . Obviously, the functions $b_4,b_6,x(z,1/N)=y(z,1/N),x(z,z/N)$ and $y(z,z/N)$ are modular functions of level N. They are in $F(N)$, because their q-expansions are in $\mathbb{Q}(\zeta_N)[[q]]$,

5.2. Algebraic expression of the Universal Elliptic curve.

Now, we give the structure of the algebraic variety to $Y_{1,\mathrm{univ},analytic}(N)$. Let $Y_1(N)$ be the modular curve over $\mathbb{Q}$ of level $\Gamma_1(N)$ and $X_1(N)$ be the smooth compactification of $Y_1(N)$. Put $Y_1(N)_{\mathbb{Q}(\zeta_N)}=Y_1(N)\otimes\mathrm{Spec}(\mathbb{Q}(\zeta_N))$ and $X_1(N)_{\mathbb{Q}(\zeta_N)}=X_1(N)\otimes\mathrm{Spec}(\mathbb{Q}(\zeta_N))$. We choose an affine open covering $\{U_i\}_{1\le i\le m}$ of $X_1(N)_{\mathbb{Q}(\zeta_N)}$ so that there exist $\lambda_1,\cdots,\lambda_m\in\mathrm{Func}(X_1(N)_{\mathbb{Q}(\zeta_N)})$ such that $\lambda\lambda_i(z)\neq 0$ for all $i=1,\cdots,m$ and all $z\in p_a^{-1}(U_i(\mathbb{C}))$, where p_a is the quotient morphism from $\mathfrak{h}$ to $X_1(N)(\mathbb{C})$. Here, for a scheme Y, we denote by $Y(\mathbb{C})$ the set of all $\mathbb{C}$-valued point of Y (regarded as the analytic space).

Define $X'_{1,univ}(U_i)=\mathrm{Proj}(\oplus_{n=0}^{\infty}I_i^n)$. Here we put

- $A_i=\Gamma(U_i,O_{X_1(N)_{\mathbb{Q}(\zeta_N)}})[x_i,y_i,z_i]/(y_i^2z_i-4x_i^3-(b_4/\lambda_i^4)x_iz_i^2-(b_6/\lambda_i^6)z_i^3)$.
- $I_i^n=(x_i,y_i,z_i)^nA_i$.

The natural inclusion $\Gamma(U_i,O_{X_1(N)_{\mathbb{Q}(\zeta_N)}})\to A_i$ induces the morphism $f_i:X'_{1,univ}(U_i)\to U_i$ for each $i=1,\cdots,m$.

Definition 6.

- Define $X_{1,\mathrm{univ}}(N)'$ by $X'_{1,\mathrm{univ}}(N)=\cup_{i=1}^mX'_{1,\mathrm{univ}}(U_i)$, where the open subscheme $f_i^{-1}(U_i\cap U_j)$ of $X'_{1,\mathrm{univ}}(U_i)$ and the open subscheme $f_j^{-1}(U_i\cap U_j)$ of $X'_{1,\mathrm{univ}}(U_j)$ are identified by the change of variables $x_j=(\lambda_j/\lambda_i)^2x_i$, $y_j=(\lambda_j/\lambda_i)^3y_i$, and $z_j=z_i$.
- Gluing f_i $(i=1,\cdots,m)$ together, we obtain the morphism $X'_{1,univ}(N)\to X_1(N)_{\mathbb{Q}(\zeta_N)}$. We will denote by $p_{e,X_1(N)}$ it.

Definition 7. Define the point ∞ and the curve C_∞ as follows.

- Let ∞ be the point on $X_1(N)_{\mathbb{Q}(\zeta_N)}$ corresponding to $z=i\infty$.
- Put $C_\infty=p_{e,X_1(N)}^{-1}(\infty)$, where $p_{e,X_1(N)}:X_{1,\mathrm{univ}}(N)'\to X_1(N)_{\mathbb{Q}(\zeta_N)}$ is defined in Definition 6.
- Put $U_{1,\mathrm{univ}}(N)=p_{e,X_1(N)}^{-1}(Y_1(N)_{\mathbb{Q}(\zeta_N)})$.

Remark that it is easy to see that $U_{1,\mathrm{univ}}(N)(\mathbb{C})$ is isomorphic to $Y_{1,\mathrm{univ},analytic}(N)$ as an analytic manifold.

Lemma 3. *There exists a following "canonical" isomorphism of scheme over* $\mathrm{Spec}(\mathbb{Q}(\zeta_N))$.

$$C_\infty \overset{\cong}{\to} \{(x:y:z) \in \mathbb{P}^2_{\mathbb{Q}(\zeta_N)} \mid y^2 z = x^2(x+1/4z)\}. \tag{13}$$

Proof . By definition, there exists a canonical isomorphism

$$C_\infty \overset{\cong}{\to} \{(x:y:z) \in \mathbb{P}^2_{\mathbb{Q}(\zeta_N)} \mid y^2 z = 4x^3 - b_4(i\infty)xz^2 - b_6(i\infty)z^3\}. \tag{14}$$

Note that
(15)

$$\begin{aligned}4x^3 - b_4(i\infty)xz^2 - b_6(i\infty)z^3 &= 4(x-(4\zeta(2)/\lambda(i\infty)^2)z)(x+(2\zeta(2)/\lambda(i\infty)^2)z)(x+(2\zeta(2)/\lambda(i\infty)^2)z) \in \mathbb{Q}[x,z] \\ &= 4x'^2(x'-(6\zeta(2)/\lambda(i\infty)^2)z) \in \mathbb{Q}[x',z]\end{aligned}$$

Here we put $x' = x + (2\zeta(2)/\lambda(i\infty)^2)z$. By the above computation and the fact $\lambda(i\infty) \in 2\pi i\mathbb{Q}(\zeta_N)$, we see the Lemma.

Definition 8.

- Let $X_{1,\mathrm{univ}}(N)$ denote the blow-up of $X'_{1,\mathrm{univ}}(N)$ at the point p_{excep}. Here p_{excep} is the singular point of C_∞ corresponding to $(x:y:z) = (1:0:0)$ by the isomorphism in the preceding lemma.
- Let $p_b : X_{1,univ}(N) \to X'_{1,\mathrm{univ}}(N)$ be the blow-up morphism at the point p_{excep}, which is the isomorphism on $X'_{1,univ}(N)\backslash p_{\mathrm{excep}}$.

Proposition 2. *The scheme $X_{1,univ}(N)$ is a proper smooth scheme over* $\mathrm{Spec}(\mathbb{Q}(\zeta_N))$.

Proof . This propositon easily follows from the fact that $v_\infty(b'_4) = v_\infty(b'_6) = 1$. Here we define the three elements b'_2, b'_4, b'_6 of $F_1(N)$ by $x^3 + b'_2x^2 + b'_4x + b'_6 = (4(x+t)^3 - b_4(x+t) - b_6)/4$ with $t = -2\zeta(2)/\lambda(i\infty)^2$, and v_∞ denotes the normalized discrete valuation corresponding to the point ∞ of $X_1(N)_{\mathbb{Q}(\zeta_N)}$.

We use the following definitions in this section and the next section.

Definition 9. Define D_1 and D_2 as follows.

- $D_1 :=$ *the Zariski closure of* $p_b^{-1}(C_\infty - p_{\mathrm{excep}})$ *in* $X_{1,\mathrm{univ}}(N)$, where p_b is the blow-up morphism in Definition 8.
- $D_2 := p_b^{-1}(p_{\mathrm{excep}}) \cong \mathbb{P}^1_{\mathbb{Q}(\zeta_N)}$.

We easily see the following Lemma.

Lemma 4. *Let $D = p^{-1}_{e,X_1(N)}(\infty)$. Here $p_{e,X_1(N)}$ is defined in Definition 6. Then $D = D_1 \cup D_2$ is satisfied. Moreover, $D_1 \cup D_2$ is a simple normal crossing divisor and $D_1 \cap D_2$ consists of two points.*

Definition 10. We call D (in the preceding lemma) the boundary of the universal elliptic curve.

6. Construction of Beilinson's l-adic Eisenstein classes of the universal elliptic curve

In this section, we will reconstruct Beilinson's l-adic Eisenstein classes. At first, we describe a general theory. Let X be a scheme, Z be its divisor, and F be a sheaf on X. Put $U = X - Z$. The following exact sequence exists (see [13] for the details).

$$\cdots \to H^i(X,F) \to H^i(U,F) \overset{\mathrm{Res}}{\to} H^{i+1}_Z(X,F) \overset{\mathrm{Gys}_{i+1}}{\to} H^{i+1}(X,F) \to H^{i+1}(U,F) \to \cdots \tag{16}$$

We call this long exact sequence the Gysin sequence or the residue exact sequence in this paper.

Our method is to construct some elements $Z_{i,\mathrm{supp}}$ of $H^{i+1}_Z(X,F)$ at first, and prove $\mathrm{Gys}_{i+1}(Z_{i,\mathrm{supp}}) = 0$. This implies that there exists an element Z_i of $H^i(U,F)$ such that $\mathrm{Res}(Z_i) = Z_{i,\mathrm{supp}}$. The details will be explained in Section 9.

In universal elliptic curve's case, we use the preceding residue exact sequence in the case $X = X_{1,\mathrm{univ}}(N)$ and $Z = D_1 \cup D_2$, where $X_{1,univ}(N)$, D_1, and D_2 are defined in Section 5.2 (Definition 8 and Definition 9). We construct the element Eis_N of $H^3_{D_1\cup D_2}(X_{1,\mathrm{univ}}(N), \mathbb{Z}/l^n\mathbb{Z}(2))$ at first and prove $\mathrm{Gys}_3(C\mathrm{Eis}_N) = 0$ for some non-zero integer C, where Eis_N will be defined in Section 6.1. This implies that there exists $Z_{\mathrm{Eis}_N} \in H^2(U, \mathbb{Z}/l^n\mathbb{Z}(2))$ such that $\mathrm{Res}(Z_{\mathrm{Eis}_N}) = C\mathrm{Eis}_N$ for some non-zero integer C. The explicit statement is mentioned in Theorem 6.1. This is the main result of this paper.

6.1. **Construction of some elements of etale cohomology of the universal elliptic curve with support on the boundary.**

In this subsection, we will construct the element Eis_N of $H^3_{D_1\cup D_2}(X_{1,\mathrm{univ}}(N),\mathbb{Z}/l^n\mathbb{Z}(2))$ corresponding to some weight 3 Eisenstein series. Throughout this section, we use the notations D_1 and D_2 in Section 5.2 (Definition 9).

Definition 11.

- Let Q_1 be the point at infinity of the elliptic curve C_∞ defined in Section 5.2 (Definition 7).
- Let Q_2 be the intersection point of $X_{1,univ}[2]$ and D_2, where $X_{1,univ}(N)[2]$ is the divisor consisting of all 2-torsion sections of $X_{1,univ}(N)$. Note that the point Q_2 is determined uniquely.

Definition 12.

- $f_1 := (y/x-1/2)/(y/x+1/2)\in \mathrm{Func}(D_1)$.

Here we use the symbols x, y in Lemma 3. Note that $f_1(Q_1)=1$.

Lemma 5. *There exists a function f_2 in $\mathrm{Func}(D_2)$ such that $\mathrm{ord}_P(f_1)=-\mathrm{ord}_P(f_2)$ for any point P in $D_1\cap D_2$ and $f_2(Q_2)=1$.*

Proof . It is obvious from the fact $D_2\cong \mathbb{P}^1_{\mathbb{Q}(\zeta_N)}$.

To construct some elements of étale cohomology of the universal elliptic curve with support on the boundary from f_1 and f_2, we use the following homomorphisms.

$$
\begin{aligned}
&H^3_{D_1\cup D_2}(X_{1,\mathrm{univ}}(N),\mathbb{Z}/l^n\mathbb{Z}(2))\\
&\overset{\mathrm{res}}{\to} H^3_{D_1-D_1\cap D_2}(X-D_2,\mathbb{Z}/l^n\mathbb{Z}(2))\oplus H^3_{D_2-D_1\cap D_2}(X-D_1,\mathbb{Z}/l^n\mathbb{Z}(2))\\
&\overset{\cong}{\to} H^1(D_1-D_1\cap D_2,\mathbb{Z}/l^n\mathbb{Z}(1))\oplus H^1(D_2-D_1\cap D_2,\mathbb{Z}/l^n\mathbb{Z}(1))
\end{aligned}
\tag{17}
$$

We will denote by $r_{D_1\cup D_2}$ the composite of the above two homomorphisms. Here the first homomorphism sends $e\in H^3_{D_1\cup D_2}(X_{1,\mathrm{univ}}(N),\mathbb{Z}/l^n\mathbb{Z}(2))$ to $\mathrm{res}(e):=(\mathrm{res}_{X-D_2}(e),\mathrm{res}_{X-D_1}(e))\in H^3_{D_1-D_1\cap D_2}(X-D_2,\mathbb{Z}/l^n\mathbb{Z}(2))\oplus H^3_{D_2-D_1\cap D_2}(X-D_1,\mathbb{Z}/l^n\mathbb{Z}(2))$ and res_{X-D_i} is the restriction homomorphism from X to $X-D_i$ for each $i=1,2$, and the second isomorphism is induced by the purity theorem.

Using the homomorphism $r_{D_1\cup D_2}$, we can determine the group $H^3_{D_1\cup D_2}(X_{1,univ}(N),\mathbb{Z}/l^n\mathbb{Z}(2))$ explicitly (see the lemma below).

Lemma 6. *The homomorphism $r_{D_1\cup D_2}$ is injective, and the image of $r_{D_1\cup D_2}$ is determined as follows.*

$$
\begin{aligned}
\mathrm{Im}(r_{D_1\cup D_2}) = \{&(c_1,c_2)\in H^1(D_1-D_1\cap D_2,\mathbb{Z}/l^n\mathbb{Z}(1))\oplus H^1(D_2-D_1\cap D_2,\mathbb{Z}/l^n\mathbb{Z}(1))\\
&\mid \mathrm{Res}_P(c_1)=-\mathrm{Res}_P(c_2)\ (\forall P\in D_1\cap D_2)\},
\end{aligned}
\tag{18}
$$

where Res_P is the residue map from $H^1(D_1-D_1\cap D_2,\mathbb{Z}/l^n\mathbb{Z}(1))$ (or $H^1(D_2-D_1\cap D_2,\mathbb{Z}/l^n\mathbb{Z}(1))$) to $H^0(P,\mathbb{Z}/l^n\mathbb{Z})$.

We write $(D_1-D_1\cap D_2,c_1)+(D_2-D_1\cap D_2,c_2)$ for the element of $H^3_{D_1\cup D_2}(X_{1,univ}(N),\mathbb{Z}/l^n\mathbb{Z}(2))$ corresponding to (c_1,c_2) in the above equality if (c_1,c_2) belongs to the right hand side. Namely, we put $(D_1-D_1\cap D_2,c_1)+(D_2-D_1\cap D_2,c_2)=r^{-1}_{D_1\cup D_2}((c_1,c_2))$ if (c_1,c_2) belongs to the right hand side.

Proof . Well-known.

By definition, f_1 and f_2 have the opposite orders of zero at the points of $D_1\cap D_2$. Hence we can apply the above lemma to construct the element $\mathrm{Eis}_{N,n}$ (defined below) of $H^3_{D_1\cup D_2}(X_{1,\mathrm{univ}}(N),\mathbb{Z}/l^n\mathbb{Z}(2))$ as follows.

Definition 13. Define $Eis_{N,n}=(D_1-D_1\cap D_2,\mathrm{kum}(f_1))+(D_2-D_1\cap D_2,\mathrm{kum}(f_2))\in H^3_{D_1\cup D_2}(X_{1,\mathrm{univ}}(N),\mathbb{Z}/l^n\mathbb{Z}(2))$ for each positive integer n, where $\mathrm{kum}: H^0(X_{1,\mathrm{univ}}(N),\mathbb{G}_m)\to H^1(X_{1,\mathrm{univ}}(N),\mathbb{Z}/l^n\mathbb{Z}(1))$ is the Kummer map. We will write $\mathrm{Eis}_{N,n}$ simply Eis_N when no confusion can arise.

Proposition 3. *Let q be a prime number congruent to 1 modulo N and m be an integer. The equalities $T_{z,q}Eis_N=(q^2+1)Eis_N$ and $T_{w,m}Eis_N=mEis_N$ are satisfied, where $T_{z,q}$ and $T_{w,m}$ are the Hecke operators defined in Section 12.2.*

Proof . This proposition follows from the direct computation. The details are left to the reader.

6.2. **Reconstruction of Beilinson's l-adic Eisenstein classes.**

In this subsection, we will reconstruct the Beilinson's l-adic Eisenstein classes. Throughout this section, we use the following notations unless otherwise stated.

Notation 2.

- Let $X = X_{1,univ}(N)$, $Z = D_1 \cup D_2$, and $U = X - Z$. where $X_{1,univ}(N)$, D_1 and D_2 are defined in Section 5.2 (Definition 8 and Definition 9).
- Let p be a prime number such that $p \equiv 1 \mod N$, $p \neq l$, and $p \in S_{X,good}$, where $S_{X,good}$ is defined in Section 3. (There are infinitely many p satisfying these properties.) Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}[\zeta_N]$ above p.
- Let $K = \mathbb{Q}(\zeta_N)$ and $O_K = \mathbb{Z}[\zeta_N]_{\mathfrak{p}}$, where $\mathbb{Z}[\zeta_N]_{\mathfrak{p}}$ is the local ring of $\mathbb{Z}[\zeta_N]$ at $\mathfrak{p}$. Let $k = \mathbb{Z}[\zeta_N]/\mathfrak{p}$ and $\bar{k}$ be an algebraic closure of k.
- Let $\mathcal{X}$ be a proper smooth scheme over $\operatorname{Spec}(O_K)$ such that $X = \mathcal{X} \otimes \operatorname{Spec}(\mathbb{Q})$. Note that there exists such $\mathcal{X}$ because $p \in S_{X,good}$.

Proposition 4. *Let K be a field, O_K be a discrete valuation ring such that $\operatorname{Frac}(O_K) = K$ and k be the residue field of O_K. Assume that the characteristic of k is not equal to l. For any proper smooth scheme $\mathcal{X}$ over $\operatorname{Spec}(O_K)$, there exists a canonical isomorphism $H^i(\bar{X}, \mathbb{Z}/l^n\mathbb{Z}) \overset{\cong}{\to} H^i(\mathcal{X}_{\bar{k}}, \mathbb{Z}/l^n\mathbb{Z})$, where we put $\bar{X} = \mathcal{X} \otimes \operatorname{Spec}(\bar{\mathbb{Q}})$ and $\mathcal{X}_{\bar{k}} = \mathcal{X} \otimes_{\operatorname{Spec}(O_K)} \operatorname{Spec}(\bar{k})$. Moreover this isomorphism is functorial and commutative to any correspondence.*

Proof . The former statement follows from the proper base change theorem and the smooth base change theorem, and the latter statement follows from these two theorems and results in Section 12.

Definition 14. We give the action of Fr on $H^i(\bar{X}, \mathbb{Z}/l^n\mathbb{Z})$ using the isomorphism in the preceding proposition.

We use the following residue exact sequence to reconstruct Beilinson's l-adic Eisenstein classes.

$$(19) \qquad \cdots \to H^2(X, \mathbb{Z}/l^n\mathbb{Z}(2)) \to H^2(U, \mathbb{Z}/l^n\mathbb{Z}(2)) \overset{\text{Res}}{\to} H^3_Z(X, \mathbb{Z}/l^n\mathbb{Z}(2)) \overset{Gys_3}{\to} H^3(X, \mathbb{Z}/l^n\mathbb{Z}(2)) \to \cdots .$$

Proposition 5. *There exists a non-zero integer C_N such that* $\operatorname{Gys}_3(C_N \operatorname{Eis}_N) = 0$.

Proof . By definition, $\operatorname{Gys}_3(C_N \operatorname{Eis}_N) \in H^3(X, \mathbb{Z}/l^n\mathbb{Z}(2))$. At first, we will decompose $H^3(X, \mathbb{Z}/l^n\mathbb{Z}(2))$ using the following Hochschild-Serre spectral sequences.

$$(20) \qquad H^{p'}(G_K, H^{q'}(\bar{X}, \mathbb{Z}/l^n\mathbb{Z}(2))) \Rightarrow H^{n'}(X, \mathbb{Z}/l^n\mathbb{Z}(2))$$

- $E^{n'} := H^{n'}(X, \mathbb{Z}/l^n\mathbb{Z}(2))$ has a decreasing filtration $E^{n'} = E^{n'}_0 \supset E^{n'}_1 \supset \cdots \supset E^{n'}_{n'} \supset E^{n'}_{n'+1} = 0$. We often write $\operatorname{Fil}^i(X)$ instead of $E^{n'}_i$ for $i = 0, 1, \cdots, n'+1$.
- $\operatorname{Fil}^{p'}(X)/\operatorname{Fil}^{p'+1}(X) (= E^{\infty}_{p',n'-p'})$ is isomorphic to the subquotient of $H^{p'}(G_K, H^{n'-p'}(\bar{X}, \mathbb{Z}/l^n\mathbb{Z}(2)))$ determined by the spectral sequence for $p' = 0, 1, \cdots, n'$.

We treat the case $n' = 3$. To prove the theorem, it is sufficient to prove, for each $i = 0, 1, 2, 3$, that there exists a positive integer C' (depending only on N and i) such that $\operatorname{Gys}_3(C' \operatorname{Eis}_N) \in \operatorname{Fil}^{i+1}(X)$ under the assumption that $\operatorname{Gys}_3(C \operatorname{Eis}_N) \in \operatorname{Fil}^i(X)$ for some non-zero integer C (not depending on n).

(1) We may assume $i \geq 1$, because, by the Weil conjecture, there exists a constant C not depending on n such that $H^0(G_{\mathbb{Q}(\zeta_N)}, H^3(\bar{X}, \mathbb{Z}/l^n\mathbb{Z}(2))$, which includes $\operatorname{Fil}^0(X)/\operatorname{Fil}^1(X)$, is C-torsion.

(2) In the case $i = 1$ (i.e. $\operatorname{Gys}_3(C \operatorname{Eis}_N) \in \operatorname{Fil}^1(X)$).

We put $V_X = \operatorname{Fil}^1(X)/\operatorname{Fil}^2(X) \subset H^1(G_K, H^2(\bar{X}, \mathbb{Z}/l^n\mathbb{Z}(2)))$. Recall that the Eichler-Shimura relation is the equality

$$(21) \qquad T_{z,p} = \operatorname{Fr} + T_{w,p} \operatorname{Fr}^*$$

as a correspondence on $\mathcal{X} \otimes \operatorname{Spec}(\mathbb{F}_p)$ (see Section 13). This implies $(\operatorname{Fr} + pT_{w,p}\operatorname{Fr}^{-1} - T_{z,p})(V_X) = 0$. Hence $(\operatorname{Fr}^2 - T_{z,p}\operatorname{Fr} + pT_{w,p})(V_X) = 0$. Especially $g_{\text{Eis}}(\operatorname{Fr})(\operatorname{Gys}_3(C_N \operatorname{Eis}_N)) = (\operatorname{Fr}^2 - (1+p^2)\operatorname{Fr} + p^2)(\operatorname{Gys}_3(C_N \operatorname{Eis}_N)) = 0$. Here we put $g_{\text{Eis}}(x) = (x-1)(x-p^2)$.

Moreover, by the Weil conjecture, the absolute value of every eigenvalue of the action of Fr on $H^2(\bar{X}, \mathbb{Z}_l(2))$ is p. Hence there exists a positive integer C' such that $\operatorname{Gys}_3(C' \operatorname{Eis}_N) \in \operatorname{Fil}^2(X)$.

(3) In the case $i=2$. (i.e. $\mathrm{Gys}_3(C\mathrm{Eis}_N) \in \mathrm{Fil}^2(X)$).
We can prove $\mathrm{Gys}_3(C'\mathrm{Eis}_N) \in \mathrm{Fil}^3(X)$ for some non-zero integer C' completely the same as above using the fact that the absolute value of every eigenvalue of the action of Fr on $H^1(\bar{X}, \mathbb{Z}_l(2))$ is $p^{1/2}$.

(4) In the case $i=3$. (i.e. $\mathrm{Gys}_3(C\mathrm{Eis}_N) \in \mathrm{Fil}^3(X)$).
Using the fact K is totally imaginary, we see that the q-cohomological dimension of G_K is 2 for all prime numbers q. This implies $H^3(G_K, H^0(\bar{X}, \mathbb{Z}/l^n\mathbb{Z})(2)) = 0$. Therefore $\mathrm{Gys}_3(C\mathrm{Eis}_N) \in \mathrm{Fil}^4(X) = 0$. This completes the proof of the proposition.

Theorem 6.1. *There exist a non-zero integer C (not depending on n) and $Z_{\mathrm{Eis}_{N,n}} \in H^2(U, \mathbb{Z}/l^n\mathbb{Z}(2))$ for each positive integer n satisfying the following property.*

- *The equality $\mathrm{Res}(Z_{\mathrm{Eis}_{N,n}}) = C\mathrm{Eis}_{N,n}$ is satisfied for each positive integer n, where Res is the residue map from $H^2(U, \mathbb{Z}/l^n\mathbb{Z}(2))$ to $H^3_Z(X, \mathbb{Z}/l^n\mathbb{Z}(2))$.*

Moreover $Z_{\mathrm{Eis}_{N,n}}$ satisfies the following properties.

- *Let p be a prime such that $p \equiv 1 \mod N$, $p \neq l$ and $p \in S_{X,good}$, where $S_{X,good}$ is defined in Section 3. (Note that there exist inifinitely many such prime numbers p.) There exists (a Hecke operator) $T \in \mathbb{Z}[T_{z,p}, T_{w,p}]$ such that $TZ_{\mathrm{Eis}_{N,n}}$ does not depend on the choice of $Z_{\mathrm{Eis}_{N,n}}$ for each positive integer n and $TZ_{\mathrm{Eis}_N} := (TZ_{\mathrm{Eis}_{N,n}})_{n\in\mathbb{Z}_{\geq 1}} \in H^2(U, \mathbb{Z}_l(2))$ is non-zero.*
- *Let m be an integer and q be a prime number congruent to 1 modulo N. The equalities $T_{z,q}(Z_{\mathrm{eig},\mathrm{Eis}_N}) = (q^2+1)Z_{\mathrm{eig},\mathrm{Eis}_N}$ and $T_{w,m}(Z_{\mathrm{eig},\mathrm{Eis}_N}) = mZ_{\mathrm{eig},\mathrm{Eis}_N}$ are satisfied, where we put $Z_{\mathrm{eig},\mathrm{Eis}_N} = TZ_{\mathrm{Eis}_N}$. Note that these eigenvalues are the same as those of weight 3 Eisenstein series.*

Proof . By the preceding proposition, there exist a non-zero integer C (not depending on n) and $Z_{\mathrm{Eis}_{N,n}} \in H^2(U, \mathbb{Z}/l^n\mathbb{Z}(2))$ (for each positive integer n) satisfying $\mathrm{Res}(Z_{\mathrm{Eis}_{N,n}}) = C\mathrm{Eis}_{N,n}$ for each positive integer n. This is the former part of the theorem. To show the latter part, it is sufficient to show that there exists (a Hecke operator) $T \in \mathbb{Z}[T_{z,p}, T_{w,p}]$ such that $T(V_{X,U}) = 0$ for all positive integers n and $TZ_{\mathrm{Eis}_N} \in H^2(U, \mathbb{Z}_l(2))$ is non-zero. Here we put $V_{X,U} = \mathrm{Im}(H^2(X, \mathbb{Z}/l^n\mathbb{Z}(2)) \to H^2(U, \mathbb{Z}/l^n\mathbb{Z}(2)))$. This follows from the following three facts.

- By the Weil conjecture, we see $H^0(G_K, H^2(\bar{X}, \mathbb{Z}/l^n\mathbb{Z}(2))$ is C-torsion for some non-zero integer C not depending on n.
- The absolute value of every eigenvalue of the action of Fr on $H^1(G_K, H^1(\bar{X}, \mathbb{Z}_l(2))$ is $p^{1/2}$.
- The Hecke operator $T_{w,p}$ acts trivially on $H^2(G_K, H^0(\bar{X}, \mathbb{Z}/l^n\mathbb{Z}(2))$.

In fact, by these three facts, we see that there exists (a Hecke operator) $T \in \mathbb{Z}[T_{z,p}, T_{w,p}]$ such that all eigenvalues of the action of T on $V_{X,U}$ are all distinct from that of $\mathrm{Eis}_{N,n}$ for some $n \in \mathbb{Z}_{\geq 1}$. This completes the proof of the theorem.

Remark 1. The element $Z_{\mathrm{eig},\mathrm{Eis}_N} \in H^2(U, \mathbb{Z}_l(2))$ is the same as Beilinson's l-adic Eisenstein classes.

7. The l-adic Siegel Eisenstein classes (without exact mathematical definitions)

In this section, we introduce l-adic Siegel Eisenstein classes without exact mathematical definitions. Throughout this section, we use the following notations.

Notation 3.

- Let $\mathbb{H}_2$ be the Siegel upper half-space of degree 2. Namely
$$\mathbb{H}_2 := \{\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \in M_2(\mathbb{C}) \mid \Im(z) \text{ is positive definite } \}$$
- Define $Sp_4(\mathbb{Z}) = \{A \in M_2(\mathbb{Z}) \mid {}^tAJ_2A = J_2\}$, where $J_2 := \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}$ with $1_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
- Define $\Gamma_2(N) = \{\gamma \in Sp_4(\mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} 1_2 & 0 \\ 0 & 1_2 \end{pmatrix} \mod N\}$.
- Let V_2 be the Shimura variety corresponding to the analytic space $\mathbb{H}_2/\Gamma_2(N)$.
- Let $\tilde{V_2}$ be a smooth compactification of V_2 obtained from the toroidal compactification of V_2 by blow-ups. See Faltings-Chai [5] for the toroidal compactification.
- Let D be the divisor of $\tilde{V_2}$ defined by "$\{q_1 = 0\}$", where we put $q_1 = e^{2\pi i z_1}$.

We choose $\tilde{V_2}$ so that $\tilde{V_2}\backslash V_2$ is a simple normal crossing divisor. We may assume $\tilde{V_2}\backslash V_2 = \cup_{1\leq i\leq m} D_i$, where $D_1 = D$ and D_i is a divisor of $\tilde{V_2}$ for each $i = 2, \cdots, m$. It is well-known that there is a following canonical isomorphism.

$$D \overset{\cong}{\to} X_{univ}(N) \tag{22}$$

Notation 4.

- Let $E_{\mathrm{Sp}(4),3}$ be a Siegel Eisenstein series of degree 2, weight 3, level N (see [20] for more details).
- Put $\omega_0 = E_{Sp(4),3} dz_1 \wedge dz_2 \wedge dz_3$.

The differential form ω_0 is the algebraic differential form on V_2 (see Lemma 5.1 in [14]), and the residue of ω_0 on D is "corresponding to" an Eisenstein series of weight 3, level N.

Roughly speaking, some Siegel Eisenstein series of weight 3, degree 2 corresponds to $Z_{\mathrm{Eis}} \in H^2(D - D\cap(\cup_{2\leq i\leq m} D_i), \mathbb{Z}/l^n\mathbb{Z}(2)) \cong H^4_D(\tilde{V_2} - \cup_{2\leq i\leq m} D_i, \mathbb{Z}/l^n\mathbb{Z}(3))$, where Z_{Eis} is some l-adic Eisenstein class of the universal elliptic curve, and a l-adic Siegel Eisenstein class is its (Hecke eigen) lift to $H^3(V_2, \mathbb{Z}/l^n\mathbb{Z}(3))$. We write a picture about l-adic Siegel Eisenstein classes of degree 2 without exact mathematical definitions.

$$\text{some Siegel Eisenstein series of weight 3} \overset{residue}{\Rightarrow} \text{some Eisenstein series of weight 3}$$

$$\exists ? \text{Siegel Eisenstein class} \in H^3(V_2, \mathbb{Z}/l^n\mathbb{Z}(3)) \overset{residue}{\Rightarrow} C_N Z_{\mathrm{Eis}} \in H^4_D(\tilde{V_2} - \cup_{2\leq i\leq m} D_i, \mathbb{Z}/l^n\mathbb{Z}(3)).$$

To construct them, we have to show $\mathrm{Gys}_4(C_N Z_{\mathrm{Eis}}) = 0$ for some positive integer C_N, where we regard Z_{Eis} as the element of $H^4_D(\tilde{V_2} - \cup_{2\leq i\leq m} D_i, \mathbb{Z}/l^n\mathbb{Z}(3))$. But we have not proved this yet.

8. Euler systems

In this section, we will describe our attempts to construct Euler systems without proofs. We haven't constructed "True" Euler systems related to the Galois representation of the symmetric squares of modular forms yet, but we hope our attempts is useful to construct "True" Euler systems.

We describe two attempts to construct Euler systems. The first attempt is to use some elements of $H^1(Y(N) \times Y(N), \mathrm{Gerst}_3)$ or $H^3_Z(Y(N) \times Y(N), \mathbb{Z}/l^n\mathbb{Z}(3))$, where Z run through some divisors of $Y(N) \times Y(N)$. It originated in the discussion of the seminar of author's master's thesis with Kazuya Kato in 2007.

Second attempt is to use conjectural l-adic Siegel Eisenstein classes. This is the main theme of the paper.

There are several works related to the first attempt. For example, Lei, Loeffler and Zerbes constructed Euler systems of $H^1(Y(N) \times Y(N), K_2)$ associated to the tensor products of modular forms (see [11] for more details). Loeffler, Skinner and Zerbes constructed Euler systems for $GSp(4)$ (see [12] for more details).

There are several results related to the second attempt. For example, Harder constructed some elements of étale cohomology of Shimura varieties using the theory of the Eisenstein cohomology (see [6] for more details).

8.1. Construction of some elements of $H^1(X, \mathrm{Gerst}_3(X))$.

We recall the definition of the Gersten complex.

Definition 15. Let F be a field. Suppose that X is a smooth variety of finite type over a field F. For a non-negative ingeger i, define the Gersten complex $\mathrm{Gerst}_i(X)$ as follows.

$$\mathrm{Gerst}_i(X) : 0 \to \oplus_{x\in X^0} K_i(\kappa(x)) \to \oplus_{x\in X^1} K_{i-1}(\kappa(x)) \to \cdots \to \oplus_{x\in X^i} K_0(\kappa(X))) \to 0. \tag{23}$$

Here, for any integer j, X^j denotes the set of all codimension j points of X, K_j denotes the j-th Quillen (or Milnor) K-group, and $\kappa(x)$ denotes the residue field of the local ring $O_{X,x}$.

Throughout this section, we use the following definitions unless otherwise stated.

Definition 16.

- Write $X = Y(N) \otimes_{\mathrm{Spec}(\mathbb{Q}(\zeta_N))} Y(N)$.
- Let D be the diagonal divisor of X. Namely we put $D = \{(z, z) \in X \mid z \in Y(N)\}$.
- Let Z_N be the element of $H^1(X, \mathrm{Gerst}_3(X))$ corresponding to the element $\{{}^cSiegel_{1/N,0}(z), {}^cSiegel_{0,1/N}(z)\}$ of $K_2(\kappa(D))$ $(\subset \oplus_{x\in X^1} K_2(\kappa(x)))$, where ${}^cSiegel$ is the Siegel unit defined in [8]. By the same way, we can define the element of $Z^1(X, \mathrm{Gerst}_3(X))$, where Z^1 means the 1-cocycle. By abuse of notation, we use the same symbol Z_N for this element.

The elements Z_N ($N \in \mathbb{Z}_{\geq 1}$) constructed above satisfy the following norm relations.

Theorem 8.1. *Let p be a prime number. Then, the following norm relations hold for all positive integers n.*

$$\mathrm{Norm}_{X(Np^{n+1})\times X(Np^{n+1})/X(Np^n)\times X(Np^n)} Z_{Np^{n+1}} = Z_{Np^n} \tag{24}$$

Note that these norm relations also hold as elements of $Z^1(X, \mathrm{Gerst}_3)$.

We can also construct an Euler system by generalizing the element Z_N (see [17] for more details).

8.2. **Construction of some elements of $H^4_D(X, \mathbb{Z}/l^n\mathbb{Z}(3))$.**
In this subsection, we will define $Z'_N \in H^4_D(X, \mathbb{Z}/l^n\mathbb{Z}(3))$ (an analogy of Z_N), and state theorems on norm relations without proofs (for the proofs, we refer the reader to [17]).
The following lemma is well-known (use the purity theorem).

Lemma 7. *Let X be a scheme over a field of characteristic 0. There exists a canonical isomorphism $H^2(Z, \mathbb{Z}/l^n\mathbb{Z}(2)) \overset{\cong}{\to} H^4_Z(X, \mathbb{Z}/l^n\mathbb{Z}(3))$ for each smooth divisor Z of X.*

Definition 17. Define $Z'_N = \{{}^cSiegel_{1/N,0}(z), {}^cSiegel_{0,1/N}(z)\} \in H^2(D, \mathbb{Z}/l^n\mathbb{Z}(2)) \cong H^4_D(X, \mathbb{Z}/l^n\mathbb{Z}(3))$

Theorem 8.2. *Let p be a prime number relatively prime to N. Then, the following norm relations hold for all positive integers n.*

$$\mathrm{Norm}_{X(Np^{n+1})\times X(Np^{n+1})/X(Np^n)\times X(Np^n)} Z'_{Np^{n+1}} = Z'_{Np^n} \tag{25}$$

Moreover we can construct the elements $Z'_{1,Np} \in H^4_{D_p}(X \otimes \mathrm{Spec}(\mathbb{Q}(\zeta_p)), \mathbb{Z}/l^n\mathbb{Z}(3))$ (See [17] for more details.) satisfying the first step of Euler system relations (see the following theorem). Here we put $D_p = D \cup <p> T_{p^2}$, where $<\cdot>$ means the diamond operator and T_{p^2} is the Hecke operator (regarded as the divisor of X).

Theorem 8.3. *Let p be a prime number relatively prime to N. The following norm relation holds.*

$$\mathrm{Norm}_{X\otimes \mathrm{Spec}(\mathbb{Q}(\zeta_p))/X} Z'_{1,Np} = (1-\alpha_p\otimes\alpha_p)(1-\alpha_p\otimes\beta_p)(1-\beta_p\otimes\alpha_p)(1-\beta_p\otimes\beta_p)Z'_N \tag{26}$$

Here we put "$\alpha_p + \beta_p = T_p$, $\alpha_p\beta_p = p<p>$" (see [16] for the meaning of the right hand side).

We consider the following residue exact sequence to construct Euler systems.

$$\begin{aligned} \cdots \to H^3(X, \mathbb{Z}/l^n\mathbb{Z}(3)) \to H^3(U, \mathbb{Z}/l^n\mathbb{Z}(3)) \to H^4_Z(X, \mathbb{Z}/l^n\mathbb{Z}(3)) \\ \to H^4(X, \mathbb{Z}/l^n\mathbb{Z}(3)) \to H^4(U, \mathbb{Z}/l^n\mathbb{Z}(3)) \to \cdots \end{aligned} \tag{27}$$

Here Z is some union of Hecke operators (regarded as the divisor of X) of $Y(N)$, and $U = X - Z$.
Let T be an element of the Hecke ring of X such that $T(H^3(X, \mathbb{Z}/l^n\mathbb{Z}(3))) = 0$ and $T(H^4(X, \mathbb{Z}/l^n\mathbb{Z}(3))) = 0$. We can construct an Euler system of $K^{(M)}_3(\mathrm{Func}(X))/(l^n)$ from $TZ'_{1,Np}$ and so on (see [17] for more details).

8.3. **Dream.** We hope that there exists an element corresponding to "$\{D, {}^cSiegel_{1/N,0}(z_1), {}^cSiegel_{0,1/N}(z_2)\}$" in the Milnor K-group $K^{(M)}_3(\mathrm{Func}(X))$. Note that D is not a principal divisor. Can we use the function $j(z_1) - j(z_2) \in \mathrm{Func}(X)$?, where j is the j-invariant and $(z_1, z_2) \in \mathfrak{h} \times \mathfrak{h}$.

8.4. **Dream to construct Euler systems using (conjectural) Siegel Eisenstein classes.** If we can construct the "l-adic Siegel Eisenstein classes of degree 2" (see Section 7) corresponding to the Siegel Eisenstein series in [16], it may happen we get an Euler system of the Galois module $\mathrm{Sym}^2(H^1(\overline{X(N)}, \mathbb{Z}_l))(3)$ by restricting the preceding Siegel Eisenstein classes to the diagonal.
Probably, we can construct an element $Z_{3,\mathrm{supp}}$ of $H^4_{\overline{V_2}-V_2}(V_2, \mathbb{Z}/l^n\mathbb{Z}(3))$ corresponding to the Siegel Eisenstein series in [16] by using l-adic Eisenstein classes constructed from the polylogarithms. But, to construct the l-adic Siegel Eisenstein class corresponding to it, we have to show $\mathrm{Gys}_4(Z_{3,\mathrm{supp}}) = 0$. We haven't proved it yet.

9. Construction of some elements of étale cohomology from the residues of differential forms

In this section, we describe a general strategy to construct some elements of étale cohomology from (the residues of) differential forms.

9.1. **General Strategies.**
Throughout this section, we will use the following notations unless otherwise stated.

Notation 5.

- Let i be a non-negative integer.
- Let X be an i-dimensional proper smooth scheme over a number field K.
- Let ω_0 be a differential form of degree i on an open dense subscheme of X.
- Put $S_{\mathrm{res}} = \{D \mid D \in \mathrm{Div}(X),\ \textit{the residue of } \omega_0 \textit{ on } D \textit{ is not identically zero}\}$, where $\mathrm{Div}(X)$ denotes the set of all irreducible divisors of X.
- Put $S^j_{\mathrm{res}} = \{D''_1 \cap \cdots \cap D''_j \mid D''_k \in S_{\mathrm{res}} (1 \leq \forall k \leq j), D''_{k_1} \neq D''_{k_2} (1 \leq \forall k_1 < \forall k_2 \leq j), D''_1 \cap \cdots \cap D''_j \neq \emptyset\}$ for $j = 0, 1, \cdots, i$.
- Let $D_j \in S^j_{\mathrm{res}}$ $(j = 0, 1, \cdots, i)$ be codimension j smooth irreducible subschemes of X such that $D_j \supset D_{j+1}$ $(j = 0, 1, \cdots, i-1)$.
- Define ω_j (by induction on j) to be the residue of ω_{j-1} on D_j for each $j = 1, \cdots, i$.
- Put $D'_{j+1} = D_j \cap (\cup_{D \in S_{\mathrm{res}}, D \neq D_j} D)$.

Throughout this section, we will make the following assumption.

Assumption 1. The divisor $\cup_{D \in S_{\mathrm{res}}} D$ is a simple normal crossing divisor.

By definition, it is obvious that there exists a canonical (uniquely determined) isomorphism

$$\mathrm{Isom}_{\Omega^0_{D_i}} : \Gamma(D_i, \Omega^0_{D_i}) \overset{\cong}{\to} H^0(D_i, O_{D_i}). \tag{28}$$

Throughout this section, we will make the following assumption.

Assumption 2. The image of ω_i by the isomorphism $\mathrm{Isom}_{\Omega^0_{D_i}}$ is contained in $H^0(D_i, \mathbb{Z})$ $(\subset H^0(D_i, O_{D_i}))$. Here, by abuse of notation, $\mathbb{Z}$ denotes the constant sheaf of the group $\mathbb{Z}$ (consisting of all integers).

Definition 18. We will denote by Z_0 the element $\mathrm{Isom}_{\Omega^0_{D_i}}(\omega_i) \pmod{l^n}$ $(\in H^0(D_i, \mathbb{Z}/l^n\mathbb{Z}))$.

We hope that we can construct the elements of the étale cohomology group $H^i(X - D'_1, \mathbb{Z}/l^n\mathbb{Z}(i))$ by the following inductive method. At first, we construct Z_0 from ω_i (as above), and construct $Z_{j,\mathrm{supp}}$ from Z_{j-1} and construct Z_j from $Z_{j,\mathrm{supp}}$ for $j = 1, 2, \cdots, i$ inductively (see the following picture).

$$\begin{array}{cccccc}
X & \omega_0 & Z_i \in H^i(X - D'_1, \mathbb{Z}/l^n\mathbb{Z}(i)) & \leftarrow & Z_{i,\mathrm{supp}} \in H^{i+1}_{D'_1}(X, \mathbb{Z}/l^n\mathbb{Z}(i)) \\
 & \downarrow & & \nearrow & \\
D_1 & \omega_1 & Z_{i-1} \in H^{i-1}(D_1 - D'_2, \mathbb{Z}/l^n\mathbb{Z}(i-1)) & \leftarrow & Z_{i-1,\mathrm{supp}} \in H^i_{D'_2}(D_1, \mathbb{Z}/l^n\mathbb{Z}(i-1)) \\
 & \downarrow & & \nearrow & \\
D_2 & \omega_2 & Z_{i-2} \in H^{i-2}(D_2 - D'_3, \mathbb{Z}/l^n\mathbb{Z}(i-2)) & \leftarrow & Z_{i-2,\mathrm{supp}} \in H^{i-1}_{D'_3}(D_2, \mathbb{Z}/l^n\mathbb{Z}(i-2)) \\
 & \downarrow & & \nearrow & \\
 & \cdots & & \cdots & \\
 & \downarrow & & \nearrow & \\
D_{i-1} & \omega_{i-1} & Z_1 \in H^1(D_{i-1} - D'_i, \mathbb{Z}/l^n\mathbb{Z}(1)) & \leftarrow & Z_{1,\mathrm{supp}} \in H^2_{D'_i}(D_{i-1}, \mathbb{Z}/l^n\mathbb{Z}(1)) \\
 & \downarrow & & \nearrow & \\
D_i & \omega_i & Z_0 \in H^0(D_i, \mathbb{Z}/l^n\mathbb{Z}) & &
\end{array} \tag{29}$$

We hope we can construct $Z_j \in H^j(D_{i-j} - D'_{i-j+1}, \mathbb{Z}/l^n\mathbb{Z}(j))$ and $Z_{j,supp} \in H^{j+1}_{D'_{i-j+1}}(D_{i-j}, \mathbb{Z}/l^n\mathbb{Z}(j))$ for any $j = 1, \cdots, i$.

Remark 2. We have to show $\mathrm{Gys}_{j+1}(Z_{j,\mathrm{supp}}) = 0$ to construct Z_j from $Z_{j,\mathrm{supp}}$. We hope that it is proved by the Eichler-Shimura relation when X is a Shimura variety (we have to replace ω_0 by $C\omega_0$ for some non-zero integer C if necessary). The Eichler-Shimura relations for many cases are proved in [18] and the reference given there.

9.2. **General method using an analogy of the Eichler-Shimura relations.**
In this section, we use the following notations unless otherwise stated.

Notation 6.

- Let p be a prime number different from l.
- Let K be a number field, and O_K be a discrete valuation ring with mixed characteristic $(0, p)$ such that $\mathrm{Frac}(O_K) = K$.
- Let q be the number of all elements of the residue field of O_K. Note that this notation is different from that of other section.
- Let $\mathcal{X}$ be a proper smooth scheme over O_K such that $\mathcal{X} \otimes \mathrm{Spec}(\mathbb{Q}) = X$.
- For each $D \in S_{\mathrm{res}}$, we choose a closed subscheme $\mathcal{D}$ of $\mathcal{X}$ such that $\mathcal{D} \otimes_{\mathcal{X}} X = D$. For $D'' = D_1'' \cap D_2'' \cap \cdots \cap D_j'' \in S_{\mathrm{res}}^j$, we put $\mathcal{D} = \mathcal{D}_1'' \otimes_{\mathcal{X}} \cdots \otimes_{\mathcal{X}} \mathcal{D}_j''$, where $\mathcal{D}_1'', \cdots, \mathcal{D}_j''$ are the closed subscheme of $\mathcal{X}$ chosen above.
- Let j be an integer such that $0 \le j \le i$.
- Let $\mathbb{Z}_{(i,j)} = \{j' \in \mathbb{Z} \mid 2(i-j) \le j' \le 2i - j + 1\}$.

In this section, we will make the following assumption.

Assumption 3. For any $j' = 0, 1, 2, \cdots, i$ and any $D = D_1'' \cap D_2'' \cap \cdots \cap D_{j'}'' \in S_{\mathrm{res}}^{j'}$, $\mathcal{D}$ is a proper smooth scheme over $\mathrm{Spec}(O_K)$.

It is convenient to introduce the following notations.

Notation 7. Let $T/(D_{i-j}, D'_{i-j+1})$, T/D_{i-j}, $T/(\overline{D_{i-j}}, j')$ (for each $j' \in \mathbb{Z}_{(i,j)}$), and $T/((\mathcal{D}_{i-j})_{\bar{k}}, j')$ (for each $j' \in \mathbb{Z}_{(i,j)}$) be endomorphisms of $\oplus_{D_{i-j} \in S_{\mathrm{res}}^{i-j}} H^{2i-j+1}_{D'_{i-j+1}}(X, \mathbb{Z}/l^n\mathbb{Z}(i))$, $\oplus_{D_{i-j} \in S_{\mathrm{res}}^{i-j}} H^{2i-j+1}_{D_{i-j}}(X, \mathbb{Z}/l^n\mathbb{Z}(i))$, $\oplus_{D_{i-j} \in S_{\mathrm{res}}^{i-j}} H^{j'}_{\overline{D_{i-j}}}(\bar{X}, \mathbb{Z}/l^n\mathbb{Z}(i))$, and $\oplus_{D_{i-j} \in S_{\mathrm{res}}^{i-j}} H^{j'}_{(\mathcal{D}_{i-j})_{\bar{k}}}(\mathcal{X}_{\bar{k}}, \mathbb{Z}/l^n\mathbb{Z}(i))$ respectively. By abuse of notation, we let T stands for all these endomorphisms, and by abuse of language, we say T is an endomorphism of $\oplus_{D_{i-j} \in S_{res}^{i-j}} H^?_?$ in this situation.

Note that there are following canonical isomorphisms obtained by the purity theorem.

$$(30)\quad \begin{aligned} \mathrm{Isom}_{(D_{i-j}, D'_{i-j+1})} &: H^{2i-j+1}_{D'_{i-j+1}}(X, \mathbb{Z}/l^n\mathbb{Z}(i)) \overset{\cong}{\to} H^{j+1}_{D'_{i-j+1}}(D_{i-j}, \mathbb{Z}/l^n\mathbb{Z}(j)) \\ \mathrm{Isom}_{D_{i-j}} &: H^{2i-j+1}_{D_{i-j}}(X, \mathbb{Z}/l^n\mathbb{Z}(i)) \overset{\cong}{\to} H^{j+1}(D_{i-j}, \mathbb{Z}/l^n\mathbb{Z}(j)) \\ \mathrm{Isom}_{\overline{D_{i-j}}, j'} &: H^{j'}_{\overline{D_{i-j}}}(\bar{X}, \mathbb{Z}/l^n\mathbb{Z}(i)) \overset{\cong}{\to} H^{j'-2(i-j)}(\overline{D_{i-j}}, \mathbb{Z}/l^n\mathbb{Z}(j)) \text{ for each } j' \in \mathbb{Z}_{(i,j)} \\ \mathrm{Isom}_{(\mathcal{D}_{i-j})_{\bar{k}}, j'} &: H^{j'}_{(\mathcal{D}_{i-j})_{\bar{k}}}(\mathcal{X}_{\bar{k}}, \mathbb{Z}/l^n\mathbb{Z}(i)) \overset{\cong}{\to} H^{j'-2(i-j)}((\mathcal{D}_{i-j})_{\bar{k}}, \mathbb{Z}/l^n\mathbb{Z}(j)) \text{ for each } j' \in \mathbb{Z}_{(i,j)}\ . \end{aligned}$$

We often regard $T/(D_{i-j}, D'_{i-j+1})$, T/D_{i-j}, $T/(\overline{D_{i-j}}, j')$ ($j' \in \mathbb{Z}_{(i,j)}$) and $T/((\mathcal{D}_{i-j})_{\bar{k}}, j')$ ($j' \in \mathbb{Z}_{(i,j)}$) as the endomorphisms of $\oplus_{D_{i-j} \in S_{\mathrm{res}}^{i-j}} H^{j+1}_{D'_{i-j+1}}(D_{i-j}, \mathbb{Z}/l^n\mathbb{Z}(j))$, $\oplus_{D_{i-j} \in S_{\mathrm{res}}^{i-j}} H^{j+1}(D_{i-j}, \mathbb{Z}/l^n\mathbb{Z}(j))$, $\oplus_{D_{i-j} \in S_{\mathrm{res}}^{i-j}} H^{j'-2(i-j)}(\overline{D_{i-j}}, \mathbb{Z}/l^n\mathbb{Z}(j))$, and $\oplus_{D_{i-j} \in S_{\mathrm{res}}^{i-j}} H^{j'-2(i-j)}((\mathcal{D}_{i-j})_{\bar{k}}, \mathbb{Z}/l^n\mathbb{Z}(j))$ respectively using the above isomorphisms.

Definition 19. Endomorphisms $T/(D_{i-j}, D'_{i-j+1})$, T/D_{i-j}, $T/(\overline{D_{i-j}}, j')$ (for all $j' \in \mathbb{Z}_{(i,j)}$) and $T/((\mathcal{D}_{i-j})_{\bar{k}}, j')$ (for all $j' \in \mathbb{Z}_{(i,j)}$) are said to be compatible if the following three properties hold.

- The following diagram is commutative.

$$(31)\quad \begin{array}{ccc} \oplus_{D_{i-j} \in S_{\mathrm{res}}^{i-j}} H^{j+1}_{D'_{i-j+1}}(D_{i-j}, \mathbb{Z}/l^n\mathbb{Z}(j)) & \overset{\oplus_{D_{i-j} \in S_{\mathrm{res}}^{i-j}} \mathrm{Gys}_{j+1}}{\to} & \oplus_{D_{i-j} \in S_{\mathrm{res}}^{i-j}} H^{j+1}(D_{i-j}, \mathbb{Z}/l^n\mathbb{Z}(j)) \\ \downarrow T/(D_{i-j}, D'_{i-j+1}) & & \downarrow T/D_{i-j} \\ \oplus_{D_{i-j} \in S_{\mathrm{res}}^{i-j}} H^{j+1}_{D'_{i-j+1}}(D_{i-j}, \mathbb{Z}/l^n\mathbb{Z}(j)) & \overset{\oplus_{D_{i-j} \in S_{\mathrm{res}}^{i-j}} \mathrm{Gys}_{j+1}}{\to} & \oplus_{D_{i-j} \in S_{\mathrm{res}}^{i-j}} H^{j+1}(D_{i-j}, \mathbb{Z}/l^n\mathbb{Z}(j)) \end{array}$$

- The Hochschild-Serre spectral sequence is "compatible" with the action of T. Namely, $\mathrm{Fil}^{i'} := \oplus_{D_{i-j} \in S_{res}^{i-j}} \mathrm{Fil}^{i'}(H^{j+1}(D_{i-j}, \mathbb{Z}/l^n\mathbb{Z}(j)))$ is preserved by T/D_{i-j} for each $i' = 0, 1, \cdots, j + 1$ and the action of T/D_{i-j} on $\mathrm{Fil}^{i'}/\mathrm{Fil}^{i'+1}$ is induced by the action of $T/(\overline{D_{i-j}}, 2i - j + 1 - i')$ on $\oplus_{D_{i-j} \in S_{res}^{i-j}} H^{i'}(G_K, H^{j+1-i'}(\overline{D_{i-j}}, \mathbb{Z}/l^n\mathbb{Z}(j)))$ for each $i' = 0, 1, \cdots, j + 1$. Here

$\mathrm{Fil}^{i'}(H^{j+1}(X,\mathbb{Z}/l^n\mathbb{Z}(j))$ denotes the usual filtration of $H^{j+1}(X,\mathbb{Z}/l^n\mathbb{Z}(j))$ determined by the Hochschild-Serre Spectral sequence for $i' = 0, 1, \cdots, j+2$. (In many papers, the notation $E^{j+1}_{i'}$ is used instead of $\mathrm{Fil}^{i'}(H^{j+1}(X,\mathbb{Z}/l^n\mathbb{Z}(j))$.)

- The following diagram is commutative.

$$\begin{array}{ccc} \oplus_{D_{i-j}\in S^{i-j}_{\mathrm{res}}} H^{j''}(\overline{D_{i-j}}, \mathbb{Z}/l^n\mathbb{Z}(j)) & \overset{\cong}{\to} & \oplus_{D_{i-j}\in S^{i-j}_{\mathrm{res}}} H^{j''}((\mathcal{D}_{i-j})_{\bar{k}}, \mathbb{Z}/l^n\mathbb{Z}(j)) \\ \downarrow T/(\overline{D_{i-j}}, j''+2(i-j)) & & \downarrow T/((\mathcal{D}_{i-j})_{\bar{k}}, j''+2(i-j)) \\ \oplus_{D_{i-j}\in S^{i-j}_{\mathrm{res}}} H^{j''}(\overline{D_{i-j}}, \mathbb{Z}/l^n\mathbb{Z}(j)) & \overset{\cong}{\to} & \oplus_{D_{i-j}\in S^{i-j}_{\mathrm{res}}} H^{j''}((\mathcal{D}_{i-j})_{\bar{k}}, \mathbb{Z}/l^n\mathbb{Z}(j)) \end{array} \tag{32}$$

for each $j'' = 0, 1, \cdots, j+1$. Here the two horizontal isomorphisms are induced by the proper and smooth base change theorem (see Proposition 4).

By abuse of language, we say T is a compatible endomorphism of $\oplus_{D_{i-j}\in S^{i-j}_{\mathrm{res}}} H^?_?$ if $T/(D_{i-j}, D'_{i-j+1})$, T/D_{i-j}, $T/(\overline{D_{i-j}}, j')$ (for all $j' \in \mathbb{Z}_{(i,j)}$) and $T/((\mathcal{D}_{i-j})_{\bar{k}}, j')$ (for all $j' \in \mathbb{Z}_{(i,j)}$) are compatible. It is expected that every correspondence T' (satisfying some assumptions) of X induces a compactible endomorphism of $\oplus_{D_{i-j}\in S^{i-j}_{\mathrm{res}}} H^?_?$.

Let d be an integer and $T_1, \cdots, T_d$ be compatible endomorphisms of $\oplus_{D_{i-j}\in S^{i-j}_{\mathrm{res}}} H^?_?$. In this subsection, we will make the following assumptions.

Assumption 4.

- The compatible endomorphisms $T_1, \cdots, T_d$ are commutative each other.
- There exists a polynomial $f(X_1, \cdots, X_d, X_{d+1}) \in \mathbb{Z}[X_1, \cdots, X_{d+1}]$ such that $f(T_1, T_2, \cdots, T_d, \mathrm{Fr})$ induces the zero map on $\oplus_{D_{i-j}\in S^{i-j}_{\mathrm{res}}} H^{j'}_{(\mathcal{D}_{i-j})_{\bar{k}}}(\mathcal{X}_{\bar{k}}, \mathbb{Z}/l^n\mathbb{Z}(i))$ for each $j' \in \mathbb{Z}_{(i,j)}$. Here the action of Fr is the usual action (see Section 13 for the definition of Fr).
- There exist $Z_{j,\mathrm{supp},D_{i-j}} \in H^{j+1}_{D'_{i-j+1}}(D_{i-j}, \mathbb{Z}/l^n\mathbb{Z}(j))$ $(\cong H^{2i-j+1}_{D'_{i-j+1}}(X, \mathbb{Z}/l^n\mathbb{Z}(i)))$ for all $D_{i-j} \in S^{i-j}_{\mathrm{res}}$ and $\lambda_1, \cdots, \lambda_d \in \mathbb{Z}$ such that

$$T_m Z_{j,\mathrm{supp},all} = \lambda_m Z_{j,\mathrm{supp},all} \tag{33}$$

for each $m = 1, \cdots, d$, where we put $Z_{j,\mathrm{supp},all} = (Z_{j,\mathrm{supp},D_{i-j}})_{D_{i-j}\in S^{i-j}_{\mathrm{res}}}$.

We want to show that there exists a non-zero integer C such that $\mathrm{Gys}_{j+1}(CZ_{j,\mathrm{supp},D_{i-j}}) = 0$ for any $D_{i-j} \in S^{i-j}_{\mathrm{res}}$. We will prove this below under some assumptions (see Proposition 6). Before stating the proposition, we introduce the following definition.

Definition 20. Let K be a field such that l is invertible in K, C be a smooth curve over K, and Z be a finite union of closed points of C. Let deg denote the composite of the following three homomorphisms $H^2_Z(C, \mathbb{Z}/l^n\mathbb{Z}(1)) \to H^2_{\bar{Z}}(\bar{C}, \mathbb{Z}/l^n\mathbb{Z}(1)) \overset{\cong}{\to} \oplus_{P\in\bar{Z}} \mathbb{Z}/l^n\mathbb{Z} \overset{\sum}{\to} \mathbb{Z}/l^n\mathbb{Z}$, where the first homomorphism is the pullback by the morphism $\bar{C} \to C$, the second isomorphism is induced by the purity theorem, and the third homomorphism is the summation homomorphism. Here we put $\bar{C} = C \otimes_{\mathrm{Spec}(K)} \mathrm{Spec}(\bar{K})$ and the morphism $\bar{C} \to C$ used above is induced by the structure morphism $\mathrm{Spec}(\bar{K}) \to \mathrm{Spec}(K)$. We call deg the degree map.

Proposition 6. *Let us assume Assumption 3 and Assumption 4. If $j \neq 1$ and the absolute value of every root of $f(\lambda_1, \lambda_2, \cdots, \lambda_d, x) = 0$ (with respect to x) is different from both $q^{(2i-j)/2}$ and $q^{(2i-j-1)/2}$, then there exists a non-zero integer C such that $\mathrm{Gys}_{j+1}(CZ_{j,\mathrm{supp},D_{i-j}}) = 0$ for any $D_{i-j} \in S^{i-j}_{\mathrm{res}}$. If $j = 1$, $\deg(Z_{1,\mathrm{supp},D_{i-1}}) = 0$ for any $D_{i-1} \in S_{\mathrm{res}}$, and the absolute value of every root of $f(\lambda_1, \lambda_2, \cdots, \lambda_d, x) = 0$ (with respect to x) is different from $q^{(2i-1)/2}$, then there exists a non-zero integer C such that $\mathrm{Gys}_2(CZ_{1,\mathrm{supp},D_{i-1}}) = 0$ for any $D_{i-1} \in S^{i-1}_{\mathrm{res}}$.*

Proof . We follow the notations used in Assumption 4. At first we treat the case $j \neq 1$. To prove the proposition, using the spectral sequence and the fact $H^m(G_K, \mathbb{Z}/l^n\mathbb{Z}(j))$ is 2-torsion for $m \geq 3$, it is sufficient to show $H^0(G_K, H^{j+1}(\overline{D_{i-j}}, \mathbb{Z}/l^n\mathbb{Z}(j)))$ is C-torsion for some non-zero integer C (not depending on n) and $f(\lambda_1, \lambda_2, \cdots, \lambda_d, q^{i-j}\mathrm{Fr})$ induce injections on both $H^j(\overline{D_{i-j}}, \mathbb{Z}_l(j))$ and $H^{j-1}(\overline{D_{i-j}}, \mathbb{Z}_l(j))$ (cf. the proof of Proposition 5).

The former part of the sufficient condition of the previous sentence is satisfied because of the Weil conjecture if $j \neq 1$, and the latter part is satisfied because the absolute value of every eigenvalue of the action of Fr on $H^j(\overline{D_{i-j}}, \mathbb{Z}_l(j))$ (resp. $H^{j-1}(\overline{D_{i-j}}, \mathbb{Z}_l(j))$) is $q^{j/2}$ (resp. $q^{(j-1)/2}$) by the Weil conjecture.

If $j = 1$, it is sufficient to show $\mathrm{Gys}_2(CZ_{1,\mathrm{supp},D_{i-1}})$ is contained in $\mathrm{Fil}^1(H^2(D_{i-1}, \mathbb{Z}/l^n\mathbb{Z}(1)))$ for any $D_{i-1} \in S_{\mathrm{res}}^{i-1}$ and $f(T_1, \cdots, T_d, q^{i-1}\mathrm{Fr})$ induces an injection on $H^1(G_K, H^1(\overline{D_{i-1}}, \mathbb{Z}_l(1)))$ (cf. the proof of Proposition 1). The former part of the sufficient condition of the previous sentence follows from the fact $\deg(Z_{1,\mathrm{supp},D_{i-1}}) = 0$ for any $D_{i-1} \in S_{\mathrm{res}}^{i-1}$, and the latter part follows from the fact that the absolute value of every eigenvalue of the action of Fr on $H^1(\overline{D_{i-1}}, \mathbb{Z}_l(1))$ is $q^{1/2}$. Remark that, by considering the restriction to a suitable point, we see that we doesn't need to consider the term $H^2(G_K, H^0(\overline{D_{i-1}}, \mathbb{Z}/l^n\mathbb{Z}(j)))$ of the spectral sequence.

Summary 1. (The summary of this subsection, the preceding subsection, and so on)

Let T be a correspondence on X satisfying some assumptions. If $T\omega_0 = \lambda\omega_0$ with $\lambda \in \mathbb{Z}$, it is expected that the method of the preceding subsection enable us to construct $Z_{j,\mathrm{supp},D_{i-j}}$ corresponding to $C\omega_j$ (in the sense of the preceding subsection) for any $j = 1, 2, \cdots, i$ and any $D_{i-j} \in S_{\mathrm{res}}^{i-j}$. Here C is some non-zero integer. Moreover it is expected we can construct $Z_{j,\mathrm{supp},D_{i-j}}$ so that $TZ_{j,\mathrm{supp},all} = \lambda Z_{j,\mathrm{supp},all}$ is satisfied, where we put $Z_{j,\mathrm{supp},all} = (Z_{j,\mathrm{supp},D_{i-j}})_{D_{i-j} \in S_{\mathrm{res}}^{i-j}}$.

In addition, it is expected that there exist non-zero integer C (not depending on n and D_{i-j}) and $Z_{j,D_{i-j}} \in H^{j+1}(D_{i-j} - D'_{i-j+1}, \mathbb{Z}/l^n\mathbb{Z}(j))$ (for any $D_{i-j} \in S_{\mathrm{res}}^{i-j}$) such that $\mathrm{Res}(Z_{j,D_{i-j}}) = CZ_{j,\mathrm{supp},D_{i-j}}$ (for any $D_{i-j} \in S_{\mathrm{res}}^{i-j}$) and "$TZ_{j,all}$" $= \lambda Z_{j,all}$, where we put $Z_{j,all} = (Z_{j,D_{i-j}})_{D_{i-j} \in S_{\mathrm{res}}^{i-j}}$. Furthermore it is expected that we can choose $Z_{j,D_{i-j}}$ "uniquely" (for any $D_{i-j} \in S_{\mathrm{res}}^{i-j}$) in the sense of the main theorem (Theorem 6.1) of this paper.

9.3. Examples.

Throughout this subsection, we use the notation $X_{1,\mathrm{univ}}(N)$ defined in Section 5.2 (Definition 8).

Example 1. (The universal elliptic curve).

Let $X = X_{1,\mathrm{univ}}(N)$, U be the open subscheme of $X_{1,\mathrm{univ}}(N)$ defined in Section 6.2 (Notation 2), and $\omega_0 = E_{3,N} dz \wedge dw \in \Omega_U^2$. Here we define $E_{3,N}(z) = \sum_{(c,d)\in\mathbb{Z}^2,(c,d)=1,(c,d)\equiv(1,0) \mod N} (cz+d)^{-3}$ for z in the upper half plane, and z, w are the coordinates in Section 5.1 (Definition 1). Using the method of the preceding subsection, we can construct the element $Z_{1,\mathrm{supp},D_1} \in H^2_{D'_2}(D_1, \mathbb{Z}/l^n\mathbb{Z}(1))$ corresponding to ω_1 (in the sense of Section 9.1) for any $D_1 \in S_{\mathrm{res}}$. For simplicity, we put $Z_{1,\mathrm{supp},all} = (Z_{1,\mathrm{supp},D_1})_{D_1 \in S_{\mathrm{res}}}$.

Let p be a prime number such that $p \equiv 1 \mod N$ and $p \in S_{X,good}$, where $S_{X,good}$ is defined in Section 3. Then, we put $f(X_1, X_2, X_3) = X_3^2 - X_1X_3 + pX_2$. By the Eichler-Shimura relation, as a correspondence on $\mathcal{X} \otimes \mathrm{Spec}(\mathbb{F}_p)$, $f(T_{z,p}, T_{w,p}, \mathrm{Fr}) = 0$ is satisfied. Moreover, Hecke eigenvalues of the actions of $T_{z,p}$ and $T_{w,p}$ on $Z_{1,supp,all}$ are calculated as follows.

Lemma 8. *Let p be an odd prime number such that $p \equiv 1 \mod N$, $p \neq l$, and $p \in S_{X,good}$, where $S_{X,good}$ is defined in Section 3. Then, there exist compatible endomorphisms $T'_{z,p}$ and $T'_{w,p}$ of $\oplus_{D \in S_{\mathrm{res}}^1} H^?_?$ such that $T'_{z,p} Z_{1,\mathrm{supp},all} = (p^2+1) Z_{1,\mathrm{supp},all}$ and $T'_{w,p} Z_{1,\mathrm{supp},all} = pZ_{1,\mathrm{supp},all}$.*

Proof . We easily see $T_{z,p}$ and $T_{w,p}$ are both finite-to-finite correspondences on $\cup_{D \in S_{\mathrm{res}}} D$. Using the assumption that p is odd, we see both $T_{z,p}$ and $T_{w,p}$ preserve D_1 and D'_2 for any $D_1 \in S_{\mathrm{res}}$. Here we put $D'_2 = D_1 \cap (\cup_{D \in S_{res}, D \neq D_1} D)$. Hence there exist compatible endomorphisms $T'_{z,p}$ and $T'_{w,p}$ "canonically" induced by $T_{z,p}$ and $T_{w,p}$ respectively. (The details are left to the reader.) These endomorphisms satisfy $T'_{z,p} Z_{1,\mathrm{supp},all} = (p^2+1) Z_{1,\mathrm{supp},all}$ and $T'_{w,p} Z_{1,\mathrm{supp},all} = pZ_{1,\mathrm{supp},all}$ as desired.

Lemma 9. *There exists a non-zero integer C such that $\mathrm{Gys}_2(CZ_{1,\mathrm{supp},D_1}) = 0$ for any $D_1 \in S_{\mathrm{res}}$.*

Proof . Let p be an odd prime number such that $p \equiv 1 \mod N$, $p \neq l$, and $p \in S_{X,good}$. (It is easy to see that there exists such p.) Note that we see $q = p$ because $p \equiv 1 \mod N$. To prove the lemma, it is sufficient to verify the assumption of Proposition 6. Obviously, $\deg(Z_{1,\mathrm{supp},D_1}) = 0$ for any $D_1 \in S_{\mathrm{res}}$. Moreover, by the preceding lemma, $f(\lambda_1, \lambda_2, x) = f(1+p^2, p, x) = (x-1)(x-p^2)$ is satisfied. Hence, the absolute value of every root of f (with respect to x) is different from $p^{3/2}$ as desired.

The statement of Lemma 9, which is the main result of this example, is already proved. For example, it follows from the results of Section 6.1 (Definition 12 and Lemma 5). But the method of the proof is important. The general strategy in this section can apply to general Shimura varieties.

Example 2. (The powers of the universal elliptic curve)

Let k be a positive integer. To construct Beilinson's l-adic Eisenstein classes of the k-th power of the universal elliptic curve, we have to consider the case $X = X_{1,\text{univ}}(N) \otimes_{X(N)} X_{1,\text{univ}}(N) \otimes_{X(N)} \cdots \otimes_{X(N)} X_{1,\text{univ}}(N)$ (k-times) and $\omega_0 = CE_{k+2,N} dz \wedge dw_1 \wedge dw_2 \wedge \cdots \wedge dw_k$, where z is the coordinate of $X(N)$ and $dw_1, \cdots, dw_k$ are the coordinates of the fiber. (We omit the exact definition.) Here we define $E_{m,N}(z) = \sum_{(c,d)\in\mathbb{Z}^2,(c,d)=1,(c,d)\equiv(1,0) \mod N} (cz+d)^{-m}$ for each integer m greater than 2 and all z in the upper half-plane. Put $X^k_{1,univ}(N) = X_{1,univ}(N) \otimes \cdots \otimes X_{1,univ}(N)$ (k-times) and $D^k = D \otimes \cdots \otimes D$ (k-times), where D is defined in Lemma 4. We regard D^k as the closed subscheme of $X^k_{1,univ}(N)$. By the universal property of the tensor product, we obtain the natural morphism $i : X \to X^k_{1,univ}(N)$. By this morphism, we regard X as the smooth closed subscheme of $X^k_{1,univ}(N)$. We also regard D^k as a closed subscheme of X. Put $\text{Eis}_{X^k_{1,\text{univ}}(N)} = \text{Eis}_N \cup \cdots \cup \text{Eis}_N$ (k-times) $\in H^{3k}_{D^k}(X^k_{1,\text{univ}}(N), \mathbb{Z}/l^n\mathbb{Z}(2k))$, where $\cup$ means the cup product. By the purity theorem and the Leray spectral sequence (of the composite of the functors $i^!$ and $\text{Ker}(\Gamma(X,\cdot) \to \Gamma(X - D^k, \cdot))$, we see that there exists the natural isomorphism $H^{3k}_{D^k}(X^k_{1,\text{univ}}(N), \mathbb{Z}/l^n\mathbb{Z}(2k)) \overset{\cong}{\to} H^{k+2}_{D^k}(X, \mathbb{Z}/l^n\mathbb{Z}(k+1))$. We define Eis^k_N to be the element of $H^{k+2}_{D^k}(X, \mathbb{Z}/l^n\mathbb{Z}(k+1))$ corresponding to $\text{Eis}_{X^k_{1,\text{univ}}(N)}$ by the preceding isomorphism. For a positive integer m, let $T_{z,m}$,$T_{w_1,m}$, $\cdots$, $T_{w_k,m}$ denote the Hecke operators. (The definitions of these Hecke operators are similar to those of the Hecke operators $T_{z,m}$, $T_{w,m}$ of the universal elliptic curve and we omit the definition.) The following lemma holds.

Lemma 10. *Let q be a prime number congruent to* 1 *modulo N. Then, the equalities $T_{z,q}\text{Eis}^k_N = (q^{k+1}+1)\text{Eis}^k_N$, $T_{w_i,q}\text{Eis}^k_N = q\text{Eis}^k_N$ (for each $i = 1, 2, \cdots, k$) are satisfied.*

Proof . This lemma is proved by direct computations. The details are left to the reader.

Lemma 11. *There exists a non-zero integer C such that $Gys_{k+2}(C\text{Eis}^k_N) = 0$.*

Proof. Let p be a prime number such that $p \equiv 1 \mod N$ and $p \in S_{X,good}$. The Eichler-Shimura relation of the powers of universal elliptic curve is the following relation as a correspondence on $\mathcal{X} \otimes \text{Spec}(\mathbb{F}_p)$.

$$T_{z,p} = \text{Fr} + T_{w_1,p} \cdots T_{w_k,p}\text{Fr}^*. \tag{34}$$

Put $d = k+2$ and $f(X_1, X_2, \cdots, X_{k+2}) = X^2_{k+2} - X_1X_{k+2} + pX_2X_3\cdots X_{k+1}$. The Eichler-Shimura relation implies $f(T_{z,p}, T_{w_1,p}, \cdots, T_{w_k,p}, \text{Fr}) = 0$ as a correspondence on $\mathcal{X} \otimes \text{Spec}(\mathbb{F}_p)$, and easy computation (using the preceding lemma) show $f(T_{z,p}, T_{w_1,p}, \cdots, T_{w_k,p}, x)\text{Eis}^k_N = f(p^{k+1}+1, p, \cdots, p, x)\text{Eis}^k_N = (x-1)(x-p^{k+1})\text{Eis}^k_N$. Hence we see $Gys_{k+2}(\text{Eis}^k_N) = 0$ by Proposition 6 as desired. □

9.4. **Problem.** To construct some elements of étale cohomology using the methods of subsection 9.1 and subsection 9.2, the following problem appears.

Problem 1. (Algebraic geometry)

Let X be a Shimura variety.

(1) Find a Hecke operator T such that T is finite-to-finite. Here we regard T as a correspondence.
(2) Find a Hecke operator T such that there exists a compatible endomorphism of $\oplus_{D_{i-j}\in S^{i-j}_{res}} H^?_?$ "corresponding" to T?

10. Determination of the constant C

Let X be a scheme, Z be its divisor, i be an integer and Eis_X be an element of $H^{i+1}_Z(X, \mathbb{Z}/l^n\mathbb{Z}(i))$.

Problem 2.

(1) Does $\text{Gys}_{i+1}(C\text{Eis}_X) = 0$ hold for some non-zero integer C?
(2) If (1) is true, determine C such that $\text{Gys}_{i+1}(C\text{Eis}_X) = 0$. Especially, determine C when X is the universal elliptic curve (or the powers of the universal elliptic curve).
(3) Determine the set of all Hecke operators T such that $TH^{i+1}(X, \mathbb{Z}_l(i)) = 0$ when X is a Shimura variety.
(4) Determine the set of all Hecke operators T such that $TH^j(\bar{X}, \mathbb{Z}_l) = 0$ for each non-negative integer j when X is a Shimura variety.

Comment 1. Probably we can prove (1) of Problem 2 in many cases of Shimura varieties using the Eichler-Shimura relations.

Example 3. Let $X = X(N)$. Then, C is related to special values of the Dirichlet L-functions. The details are left to the reader.

Problem 3.

(1) Determine the set of all Hecke operators T such that $T(H^1(\overline{X(N)}, \mathbb{Z}/l^n\mathbb{Z}(1))) = 0$. This is related to the Galois representation $G_{\mathbb{Q}(\zeta_N)} \to \mathrm{End}(H^1(\overline{X(N)}, \mathbb{Z}/l^n\mathbb{Z}(1)))$, where End means the endomorphism group.
(2) Let $\mathrm{Div}_{\mathrm{cusp}}$ be the group consisting of the divisors of $X(N)$ generated by all cusps. Determine the group $\mathrm{Div}_{\mathrm{cusp}}/ < Siegel >$, where $< Siegel >$ is the group generated by all Siegel units of $X(N)$.

11. Remaining topics

In this section, we describe some remaining topics of this paper.

(1) Over Spec($\mathbb{Q}$) vs. Spec($\mathbb{Z}[1/Nl]$). What is a difference?
To construct "True" Euler systems, we have to construct "True" elements of étale cohomology of a certain scheme not over Spec($\mathbb{Q}$) but over Spec($\mathbb{Z}[1/Nl]$) whose structure morphism is surjective. Harder obtained some results in [6]. (He obtained some motivic elements.)
(2) Euler systems.
Construct "l-adic Siegel Eisenstein classes of degree 2" (see Section 7) and determine the constant C exactly. If we can do that, then it may happen that we get an Euler system of the Galois module $\mathrm{Sym}^2(H^1(\overline{X(N)}, \mathbb{Z}_l)(3)$ by restricting the preceding Siegel Eisenstein classes to the diagonal.
(3) An proof of the Shimura-Taniyama conjecture.
If we can construct the Siegel Eisenstein classes of degree 2 corresponding to the Siegel Eisenstein series in [16], it may happen that we can get the "True" upper bounds of the "True" Selmer groups of the symmetric squares of modular forms.
(4) The l-adic Eisenstein classes of general Shimura varieties.
It is expected, under some assumptions, there exists an element of étale cohomology corresponding to each Eisenstein series of general Shimura varieties.
(5) The pullback formula (general degree).
The restriction to the diagonal of some conjectural l-adic Siegel Eisenstein class of Sp($4g$) is (conjecturally) related to some special values of L-functions by the pullback formula. How to interpret this algebraically (such as the Beilinson conjecture or Euler systems) ?
(6) The Beilinson conjecture.
Some Siegel Eisenstein series of Sp(4) is related to some special values of the symmetric square L-functions of modular forms by the pullback formula (see, for example, [19], [15] or [16]). It is expected that the restriction to the diagonal of some l-adic Siegel Eisenstein class of degree 2 is the element of $H^3(Y(N) \times Y(N), \mathbb{Z}/l^n\mathbb{Z}(3))$ predicted by the "l-adic Beilinson conjecture" of $X(N) \times X(N)$.

12. Corresponences

In this section we define the action of correspondences on étale cohomology. Especially we will define the action of Hecke operators on étale cohomology.

12.1. **The definition of the action of correpondences on étale cohomology.** The results of this subsection originated in a comment of Yoichi Mieda to our talk. In our talk, we assume the finiteness to define the action of correspondences on étale cohomology, but he teach me the assumption is not necessary. The author would like to thank him for the comment.

In this section, we use the following notations unless otherwise stated.

Notation 8.

- Let l be a prime number and K be a field such that l is invertible in K.

- Let X be a smooth equidimensional scheme over K, X' be an equidimensional scheme over K, $f : X' \to X$ be a morphism over K, and $g : X \to \mathrm{Spec}(K)$ be the structure morphism.
- Let i be a non-negative integer and j be an integer.
- Put $\Lambda = \mathbb{Z}/l^n\mathbb{Z}$.

We will make the following assumption in this section.

Assumption 5. The morphism $f : X' \to X$ is a compactifiable morphism (see [1] (tome 3, Chapter 17, Section 3, Definition 3.2.1. (pp305)) for the definition of compactifiable morphism).

This assumption is necessary to define $Rf_!$ and $Rf^!$ (see the definitions below).

In this section, we only treat the sheaves of Λ-module, and use the following definitions. For simplicity we call the reference [1] SGA 4.

Definition 21. For a scheme Y, let $D(Y)$ and $D^+(Y)$ denote the derived categories defined in SGA 4, tome 3. Let $Rf_! : D(X') \to D(X)$ be the functor defined in SGA 4, tome 3, Chapter 17, Section 6 (pp224) and $Rf^! : D^+(X) \to D^+(X')$ be the functor defined in SGA 4, tome 3, Chapter 18, Section 3 (pp316). We write Λ_X for the complex $\cdots \to 0 \to \Lambda \to 0 \to \cdots \in \mathrm{Ob}(D^+(X))$ (put Λ in degree 0). By abuse of notation, we write Λ instead of Λ_X when no confusion can arise.

Remark that, to be precise, for a scheme Y, we have to write $D(Y, \Lambda)$ and $D^+(Y, \Lambda)$ following the notations of SGA 4. But for simplicity of notations, we write $D(Y)$ and $D^+(Y)$ instead of $D(Y, \Lambda)$ and $D^+(Y, \Lambda)$ respectively.

Lemma 12. *Let $F \in \mathrm{Ob}(D(X'))$ and $G \in \mathrm{Ob}(D^+(X))$. The following "canonical" isomorphism exists.* $\mathrm{Hom}_{D(X')}(F, Rf^!G) \overset{\cong}{\to} \mathrm{Hom}_{D(X)}(Rf_!F, G)$. *(For the precise statement and the proof, see Remark 3.1.5. in SGA 4, tome 3 (pp571).)*

Lemma 13. *For any smooth morphism $g : X \to \mathrm{Spec}(K)$ and any compactifiable morphism $f : X' \to X$, the following equalities hold. (Note that every morphism $X \to \mathrm{Spec}(K)$ is compactifiable.)*

$Rg^!\Lambda = g^*\Lambda(d)[2d] = \Lambda(d)[2d]$. $Rf^!\Lambda \cong Rf^!(Rg^!\Lambda(-d)[-2d]) \cong R(g \circ f)^!\Lambda(-d)[2d]$. *Here we put $d = \dim(X)$.*

Proof . The former statement of the lemma follows from the fact g is the smooth morphism (see Theorem 3.2.5 (Poincaré duality) in SGA 4, tome 3 (pp585)), and the latter statement follows from the former statement and the fact $Rf^! \circ Rg^! = R(g \circ f)^!$.

Definition 22. For any morphism $h : X' \to \mathrm{Spec}(K)$ and $F \in \mathrm{Ob}(D^+(\mathrm{Spec}(K)))$, let $t_h : Rh^*F(d)[2d] \to Rh^!F$ be the homomorphism defined in SGA 4, tome 3, (3.2.1.2) (pp583) or Lemma 3.2.3. (pp583). Here we put $d = \dim(X')$. (Note that every morphism $X' \to \mathrm{Spec}(K)$ is compactifiable.)

Definition 23. Let $c_{\Lambda,f} : \Lambda \to Rf^!\Lambda$ be the morphism of the derived category $D(X')$ obtained from the map $t_{g\circ f} : \Lambda(d)[2d] \to R(g \circ f)^!\Lambda$ by twisting $(-d)[-2d]$. Here we put $d = \dim(X')$. Let ${c'}_{\Lambda,f} : Rf_!\Lambda \to \Lambda$ be the morphism of the derived category $D(X)$ corresponding to $c_{\Lambda,f}$ by the adjoint property of the functors $f^!$ and $f_!$ (see Lemma 12).

Definition 24. Let $t' : Rf_*\Lambda(j) \to \Lambda(j)$ be a morphism of the derived category $D(X)$. By taking $\Gamma(X, \cdot)$ (resp. $\mathrm{Ker}(\Gamma(X, \cdot) \to \Gamma(X - Z, \cdot))$), the homomorphism t' induces the homomorphism $H^i(X', \Lambda(j)) \to H^i(X, \Lambda(j))$ (resp. $H^i_{Z'}(X', \Lambda(j)) \to H^i_Z(X, \Lambda(j))$), where Z is a closed subscheme of X and $Z' = f^{-1}(Z)$. Let $f_*(t')$ denote both the above two homomorphisms. For a proper morphism f, $Rf_*\Lambda(j) = Rf_!\Lambda(j)$ is satisfied, and by abuse of notation, we write simply f_* instead of $f_*({c'}_{\Lambda,f}(j))$.

Proposition 7. *Let C be a schemes over K. Let $c : C \to X \times X$ be a morphism of scheme over K, $p_1 : C \to X$ be its first projection and $p_2 : C \to X$ be the second projection. Assume that p_1 is compactifiable, proper and dim(C) = dim(X). Then two homomorphisms $p_{1*} \circ p_2^* : H^i(X, \Lambda(j)) \to H^i(X, \Lambda(j))$ and $H^i_Z(X, \Lambda(j)) \to H^i_{p_1(p_2^{-1}(Z)))}(X, \Lambda(j))$ are defined.*

Proof . The pullbacks $p_2^* : H^i(X, \Lambda(j)) \to H^i(C, \Lambda(j))$ and $H^i_Z(X, \Lambda(j)) \to H^i_{p_2^{-1}(Z)}(C, \Lambda(j))$ are always defined. Moreover, under the assumption of the proposition, we can define $p_{1*} : H^i(C, \Lambda(j)) \to H^i(X, \Lambda(j))$ and $H^i_{p_2^{-1}(Z)}(C, \Lambda(j)) \to H^i_{p_1(p_2^{-1}(Z))}(C, \Lambda(j))$ (see Definition 24). This completes the proof.

12.2. **Hecke operators as correspondences.**

We will define the closed subschemes $T_{z,n,open}$ (for $n \in \mathbb{Z}_{\geq 1}$) and $T_{w,m,open}$ (for $m \in \mathbb{Z}$) of $U_{1,\mathrm{univ}}(N) \times U_{1,\mathrm{univ}}(N)$ using the modular interpretation, where $U_{1,\mathrm{univ}}(N)$ is the open subscheme of $X_{1,\mathrm{univ}}(N)$ defined in Section 5.2 (Definition 7). Let F be a field extension of $\mathbb{Q}(\zeta_N)$, E be an elliptic curve defined over F, e_N be its F-point of exact order N, and P be a F-point of E. We regard (E, e_N) (resp. (E, e_N, P)) as the F-point of $Y_1(N)_{\mathbb{Q}(\zeta_N)}$ (resp. $U_{1,\mathrm{univ}}(N)$).

Definition 25. Let n be a positive integer and m be an integer.

- $T_{z,n,open} := \{((E, P, e_N), (E/L_n, \bar{P}, \bar{e_N})) \in U_{1,\mathrm{univ}}(N) \times U_{1,\mathrm{univ}}(N) \mid (E, e_N) \in Y_1(N)_{\mathbb{Q}(\zeta_N)}, P \in E, L_n : \textit{order } n \textit{ subgroup of } E\}$, where $\bar{P}$ (resp. $\bar{e_N}$) denotes the point of E/L_n corresponding to P modulo L_n (resp. e_N modulo L_n).
- $T_{w,m,open} := \{((E, P, e_N), (E, mP, e_N)) \in U_{1,\mathrm{univ}}(N) \times U_{1,\mathrm{univ}}(N) \mid (E, e_N) \in Y_1(N), P \in E\}$.
- $T_{z,n}$ is the Zariski closure of $T_{z,n,open}$ in $X_{1,\mathrm{univ}}(N)$.
- $T_{w,m}$ is the Zariski closure of $T_{w,m,open}$ in $X_{1,\mathrm{univ}}(N)$.

Definition 26. Using Proposition 7 in the case c is the closed immesion $T_{z,n} \to U_{1,\mathrm{univ}}(N) \times U_{1,\mathrm{univ}}(N)$ (resp. $T_{w,m} \to U_{1,\mathrm{univ}}(N) \times U_{1,\mathrm{univ}}(N)$), we get the actions of $T_{z,n}$ (resp. $T_{w,m}$) on $H^i(X_{1,\mathrm{univ}}(N), \Lambda(j))$ and $H^i_{D_1 \cup D_2}(X_{1,\mathrm{univ}}(N), \Lambda(j))$. Here D_1 and D_2 is defined in Section 5.2 (Definition 9).

Remark 3. The argument in the preceding subsection works for a proper smooth scheme $\mathcal{X}$ over $\mathrm{Spec}(O_K)$ such that $\mathcal{X} \otimes \mathrm{Spec}(\mathbb{Q}) = X_{1,\mathrm{univ}}(N)$. Here O_K is defined in Section 6.2 (Notation 2). The details are left to the reader.

13. Frobenius endomorphism, dual correspondences, the Eichler-Shimura relation

Definition 27.

- For a scheme X over a finite field k, let Fr denote the Frobenius endomorphism $\mathrm{Fr}_{X/k}$ defined in [13] (Chapter VI, Section 13 (pp290)). For example, for a ring A over k, $\mathrm{Fr}_{\mathrm{Spec}(A)/k}$ is the morphism of scheme induced by the $|k|$-th power map of A, where $|k|$ is the number of all elements of k. By abuse of notation, we use the same symbol Fr for the morphism $Fr_{X \otimes \mathrm{Spec}(\bar{k})/\bar{k}}$ defined in [13], where $\bar{k}$ is an algebraic closure of k. We often regard Fr as the correspondence on $X \times X$ (Namely, $\mathrm{Fr} = \{(x, \mathrm{Fr}(x)) \in X \times X \mid x \in X\}$).
- Let C be a closed subscheme of $X \times X$. Define $C^{dual} = p_2 \times p_1(C)$, where p_1 (resp. p_2) is the first (resp. second) projection. Namely, $C^{dual} = \{(y, x) \in X \times X \mid (x, y) \in C\}$. We often write C^* instead of C^{dual}. In many papers, the notation Ver is used instead of Fr^*. But in this paper, we use the notation Fr^*.

Theorem 13.1. *We follow the notation of Section 6.2 (Notation 2) (we may ignore the condition $p \neq l$).*

As a correspondence on $\mathcal{X} \otimes \mathrm{Spec}(\mathbb{F}_p)$, the relation $T_{z,p} = \mathrm{Fr} + T_{w,p}\mathrm{Fr}^*$ (the Eichler-Shimura relation of the universal elliptic curve) is satisfied.

Proof . See [18] for the proof.

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