A Halpern type proximal point algorithm for a convex function on a geodesic space
測地距離空間上の凸関数に対する Halpern 型近接点法

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Abstract

In a geodesic space with curvature bounded above by a positive real number, a coefficient condition of the Halpern type iterative sequence is different from one on geodesic spaces with nonpositive curvature. In this paper, we consider a modified proximal point algorithm with an anchor point and resolvent operators of a convex function.

1 Introduction

To find a minimiser of a convex function, we often use iterative approximation methods which generate a sequence converging to a minimiser of the convex function, or a fixed point of a resolvent operator. The proximal point algorithm is a canonical minimiser approximation method, and it has many modified types. These schemes have been investigated on Banach spaces such as function spaces. Recently, they are also studied in geodesic spaces which are called CAT(κ) spaces. Particularly, a CAT(1) space includes the infinite dimensional unit sphere. Kimura and Kohsaka obtained the following pioneering result:

Theorem 1.1 (Kimura–Kohsaka [4]). *Let X be an admissible complete* CAT(1) *space and f a proper lower semicontinuous convex function from X into $]-\infty, \infty]$ which has a minimiser. Let $\{\lambda_n\}$ be a positive real sequence such that $\inf_{k \in \mathbb{N}} \lambda_k > 0$. Let $\{\alpha_n\}$ be a real sequence of $]0,1[$ such that $\lim_{n\to\infty} \alpha_n = 0$ and that $\sum_{n=1}^{\infty} \alpha_n^2 = \infty$. For an anchor point $u \in X$ and an initial point $x_1 \in X$, generate a sequence $\{x_n\}$ of X as follows:*

$$\begin{aligned} R_{\lambda_n f} x_n &= \operatorname*{Argmin}_{z \in X} \left(\lambda_n f(z) + \tan d(z, x_n) \sin d(z, x_n)\right); \\ x_{n+1} &= \alpha_n u \oplus (1 - \alpha_n) R_{\lambda_n f} x_n \end{aligned}$$

for each $n \in \mathbb{N}$. Then, $\{x_n\}$ converges to the closest minimiser to the anchor point u.

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In this theorem, we need to suppose that $\sum_{n=1}^{\infty} \alpha_n^2 = \infty$. However, for instance, in Hilbert space, we assumed that $\sum_{n=1}^{\infty} \alpha_n = \infty$. In this paper, we consider a new coefficient condition for the Halpern type proximal point algorithm.

2 Preliminaries

Let (X, d) be a metric space and $D \in [0, \infty[$. We call X a uniquely D-geodesic space if there exists a unique geodesic for each two points in X, namely, for each $x, y \in X$, there is a unique isometric mapping γ_{xy} from $[0, d(x, y)]$ into X such that $\gamma_{xy}(0) = x$ and $\gamma_{xy}(d(x, y)) = y$. In this case, we denote a point $\gamma_{xy}((1-t)d(x, y))$ by $tx \oplus (1-t)y$, and call it a convex combination for x and y with a ratio t.

Let X be a uniquely π-geodesic space. The canonical definition of a CAT(1) space uses geodesic triangles and their comparison triangles in the two-dimensional sphere. However, we can define a CAT(1) space as follows: We call X a CAT(1) space if

$$\cos d(tx \oplus (1-t)y, z) \sin l \geq \cos d(x, z) \sin(tl) + \cos d(y, z) \sin((1-t)l)$$

for every $x, y, z \in X$ with $d(y, z) + d(z, x) + l < 2\pi$ and $t \in [0, 1]$, where $l = d(x, y)$. Moreover, X is said to be admissible if $d(u, v) < \pi/2$ for any $u, v \in X$.

Let X be an admissible CAT(1) space and C a subset of X. We say C is convex if $tx \oplus (1-t)y \in C$ for every $x, y \in C$ and $t \in [0, 1]$.

Let X be an admissible CAT(1) space, $x, y \in X$ and $t \in [0, 1]$. Then, a real valued function defined by

$$z \mapsto t \cos d(z, x) + (1-t) \cos d(z, y)$$

for $z \in X$ has a unique maximiser [6]. We denote such a point by

$$tx \overset{1}{\oplus} (1-t)y$$

and call it 1-convex combination for x and y with a ratio t.

Theorem 2.1 (Kimura–Sasaki [6])**.** *Let X be an admissible* CAT(1) *space. Then,*

$$\cos d(tx \overset{1}{\oplus} (1-t)y, z) \geq \frac{t \cos d(x, z) + (1-t) \cos d(y, z)}{\sqrt{t^2 + (1-t)^2 + t(1-t) \cos d(x, y)}}$$

for each $x, y, z \in X$ and $t \in [0, 1]$.

Let X be a metric space and T a mapping on X. We call T a quasinonexpansive mapping if its fixed point set $\operatorname{Fix} T = \{x \in X \mid Tx = x\}$ is nonempty and

$$d(p, Tx) \leq d(p, x)$$

for any $p \in \operatorname{Fix} T$ and $x \in X$. On an admissible CAT(1) space, the fixed point set of a quasinonexpansive mapping is closed and convex.

Let C be a nonempty closed convex subset of an admissible complete CAT(1) space X. Then, for $x \in X$, there exists a unique point $p_x \in C$ such that $d(x, p_x) = \inf_{y \in C} d(x, y)$. We call such a mapping P_C defined by $P_C x = p_x$ a metric projection onto C. The metric projection P_C is quasinonexpansive with the fixed point set $\operatorname{Fix} P_C = C$.

Let X be a metric space and $\{x_n\}$ a bounded sequence of X. An asymptotic centre $\mathrm{AC}(\{x_n\})$ of $\{x_n\}$ is defined by

$$\mathrm{AC}(\{x_n\}) = \left\{ z \in X \;\middle|\; \limsup_{n\to\infty} d(z, x_n) = \inf_{y\in X} \limsup_{n\to\infty} d(y, x_n) \right\}.$$

Let $\{x_n\}$ be a sequence of X and $x_0 \in X$. We say that $\{x_n\}$ Δ-converges to a Δ-limit x_0 if $\{x_0\} = \mathrm{AC}(\{x_{n_i}\})$ for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$. A sequence $\{x_n\}$ of a CAT(1) space X is said to be spherically bounded if

$$\inf_{x\in X} \limsup_{n\to\infty} d(x, x_n) < \frac{\pi}{2}.$$

We know the following lemmas about Δ-convergence:

Lemma 2.2 (Espínola–Fernández-León [1], Kirk–Panyanak [7]). *Let X be a complete* CAT(1) *space and $\{x_n\}$ a spherically bounded sequence of X. Then,* $\mathrm{AC}(\{x_n\})$ *is a singleton and $\{x_n\}$ has a Δ-convergent subsequence.*

Lemma 2.3 (He–Fang–Lopez–Li [2]). *Let X be an admissible complete* CAT(1) *space. Then,*

$$d(x_0, z) \le \liminf_{n\to\infty} d(x_n, z)$$

for all $z \in X$ whenever a spherically bounded sequence $\{x_n\}$ Δ-converges to $x_0 \in X$.

Let X be an admissible complete CAT(1) space and f a function from X into $]-\infty, \infty]$. We say that f is proper if its effective domain $\operatorname{dom} f = \{x \in X \mid f(x) < \infty\}$ is nonempty. We say that f is lower semicontinuous if its level set is closed everywhere. Moreover, f is said to be convex if

$$f(tx \oplus (1-t)y) \le tf(x) + (1-t)f(y)$$

for any $x, y \in X$ and $t \in]0, 1[$.

Lemma 2.4 (Kimura–Kohsaka [3]). *Let X be an admissible complete* CAT(1) *space and f a proper lower semicontinuous convex function from X into $]-\infty, \infty]$. Then,*

$$f(x_0) \le \liminf_{n\to\infty} f(x_n)$$

whenever a spherically bounded sequence $\{x_n\}$ of X is Δ-convergent to $x_0 \in X$.

Let X be an admissible complete CAT(1) space and f a proper lower semicontinuous convex function from X into $]-\infty, \infty]$. We denote its minimiser set by

$$\operatorname{Min} f = \operatorname*{Argmin}_{z\in X} f(z) = \left\{ z \in X \;\middle|\; f(z) = \inf_{y\in X} f(y) \right\}.$$

Then, $\operatorname{Min} f$ is closed and convex. For $x \in X$, a function defined by

$$z \mapsto f(z) + \tan d(z, x) \sin d(z, x)$$

for each $z \in X$ has a unique minimiser [3]. We define a mapping R_f on X by

$$\{R_f x\} = \operatorname*{Argmin}_{z\in X} \left(f(z) + \tan d(z, x) \sin d(z, x) \right) \subset \operatorname{dom} f$$

for each $x \in X$, and call it a resolvent operator for f. Note that the fixed point set $\operatorname{Fix} R_f$ coincides with the minimiser set $\operatorname{Min} f$ and that R_f is quasinonexpansive if f has a minimiser. We show the following lemma:

Lemma 2.5. *Let X be an admissible complete* CAT(1) *space and f a proper lower semicontinuous convex function from X into $]-\infty, \infty]$. Then,*

$$f(R_f x) \le f(w) + \frac{l}{\sin l}\left(\frac{1}{\cos^2 d(R_f x, x)} + 1\right)(\cos l \cos d(R_f x, x) - \cos d(w, x))$$

for any $w, x \in X$ with $R_f x \ne w$, where $l = d(R_f x, w)$.

Proof. Fix $w, x \in X$ with $R_f x \ne w$ arbitrarily. Let $\tau \in]0,1[$ and $w_\tau = \tau w \oplus (1-\tau) R_f x$. Then, by the definition of R_f, we have

$$\begin{aligned}
&f(R_f x) + \tan d(R_f x, x) \sin d(R_f x, x) \\
&\le f(w_\tau) + \tan d(w_\tau, x) \sin d(w_\tau, x) \\
&\le \tau f(w) + (1-\tau) f(R_f x) + \tan d(w_\tau, x) \sin d(w_\tau, x)
\end{aligned}$$

and hence

$$\tau f(R_f x) \le \tau f(w) + \tan d(w_\tau, x) \sin d(w_\tau, x) - \tan d(R_f x, x) \sin d(R_f x, x).$$

Moreover, we get

$$\begin{aligned}
&\tan d(w_\tau, x) \sin d(w_\tau, x) - \tan d(R_f x, x) \sin d(R_f x, x) \\
&= \frac{\sin^2 d(w_\tau, x)}{\cos d(w_\tau, x)} - \frac{\sin^2 d(R_f x, x)}{\cos d(R_f x, x)} = \frac{1 - \cos^2 d(w_\tau, x)}{\cos d(w_\tau, x)} - \frac{1 - \cos^2 d(R_f x, x)}{\cos d(R_f x, x)} \\
&= \frac{1}{\cos d(w_\tau, x)} - \cos d(w_\tau, x) - \frac{1}{\cos d(R_f x, x)} + \cos d(R_f x, x) \\
&= \frac{\cos d(R_f x, x) - \cos d(w_\tau, x)}{\cos d(w_\tau, x) \cos d(R_f x, x)} + \cos d(R_f x, x) - \cos d(w_\tau, x) \\
&= \left(\frac{1}{\cos d(w_\tau, x) \cos d(R_f x, x)} + 1\right)(\cos d(R_f x, x) - \cos d(w_\tau, x)).
\end{aligned}$$

Thus, we obtain

$$\tau f(R_f x) \le \tau f(w) + \left(\frac{1}{\cos d(w_\tau, x) \cos d(R_f x, x)} + 1\right)(\cos d(R_f x, x) - \cos d(w_\tau, x)).$$

Dividing both sides by $\tau > 0$, we get

$$f(R_f x) \le f(w) + \left(\frac{1}{\cos d(w_\tau, x) \cos d(R_f x, x)} + 1\right)\left(\frac{\cos d(R_f x, x) - \cos d(w_\tau, x)}{\tau}\right).$$

Then,

$$\frac{\cos d(R_f x, x) - \cos d(w_\tau, x)}{\tau}$$

$$
\begin{aligned}
&\leq \frac{\cos d(R_f x, x)\sin l - \cos d(w,x)\sin(\tau l) - \cos d(R_f x, x)\sin((1-\tau)l)}{\tau \sin l} \\
&= \frac{(\sin l - \sin((1-\tau)l))\cos d(R_f x, x) - \cos d(w,x)\sin(\tau l)}{\tau \sin l}
\end{aligned}
$$

and thus

$$
(*) \qquad f(R_f x) \leq f(w) + \left(\frac{1}{\cos d(w_\tau, x)\cos d(R_f x, x)} + 1\right)\frac{L(\tau)}{\tau \sin l},
$$

where

$$
L(\tau) = (\sin l - \sin((1-\tau)l))\cos d(R_f x, x) - \cos d(w,x)\sin(\tau l).
$$

Note that $L(\tau)$ tends to 0 as $\tau \searrow 0$. From l'Hospital's rule, we have

$$
\begin{aligned}
\lim_{\tau\searrow 0}\frac{L(\tau)}{\tau \sin l} &= \lim_{\tau\searrow 0}\frac{L'(\tau)}{\sin l} = \lim_{\tau\searrow 0}\frac{l\cos((1-\tau)l)\cos d(R_f x, x) - l\cos d(w,x)\cos(\tau l)}{\sin l} \\
&= \frac{l}{\sin l}\cdot(\cos l \cos d(R_f x, x) - \cos d(w,x)).
\end{aligned}
$$

Thus, letting $\tau \searrow 0$ for the inequality $(*)$, we obtain

$$
f(R_f x) \leq f(w) + \frac{l}{\sin l}\left(\frac{1}{\cos^2 d(R_f x, x)} + 1\right)(\cos l \cos d(R_f x, x) - \cos d(w,x)).
$$

It completes the proof. □

Corollary 2.6. *Let X be an admissible complete* CAT(1) *space and f a proper lower semicontinuous convex function from X into $]-\infty,\infty]$. Then,*

$$
f(R_f x) \leq f(w) + 2\left(\frac{1}{\cos^2 d(R_f x, x)} + 1\right)|\cos d(R_f x, w) - \cos d(w,x)|
$$

for any $w, x \in X$.

Proof. For $l \in\,]0, \pi/2[$, we know that

$$
\frac{l}{\sin l} \leq \frac{\pi}{2} < 2.
$$

The desired inequality holds if $w = R_f x$. Suppose that $w \neq R_f x$. Then,

$$
\begin{aligned}
&\frac{d(w, R_f x)}{\sin d(w, R_f x)}\left(\frac{1}{\cos^2 d(R_f x, x)} + 1\right)(\cos d(w, R_f x)\cos d(R_f x, x) - \cos d(w,x)) \\
&\leq \frac{d(w, R_f x)}{\sin d(w, R_f x)}\left(\frac{1}{\cos^2 d(R_f x, x)} + 1\right)(\cos d(w, R_f x) - \cos d(w,x)) \\
&\leq \frac{d(w, R_f x)}{\sin d(w, R_f x)}\left(\frac{1}{\cos^2 d(R_f x, x)} + 1\right)|\cos d(w, R_f x) - \cos d(w,x)| \\
&\leq 2\left(\frac{1}{\cos^2 d(R_f x, x)} + 1\right)|\cos d(w, R_f x) - \cos d(w,x)|.
\end{aligned}
$$

From the previous lemma, we obtain

$$f(R_f x) \le f(w) + 2\left(\frac{1}{\cos^2 d(R_f x, x)} + 1\right) \left|\cos d(w, R_f x) - \cos d(w, x)\right|,$$

which completes the proof. $\square$

3 A Halpern type proximal point algorithm

The following lemma plays an important role for a proof of Halpern type convergence theorems:

Lemma 3.1 (Kimura–Saejung [5], Saejung–Yotkaew [8]). *Let $\{s_n\}$ be a nonnegative real sequence of and $\{t_n\}$ a real sequence. Let $\{\beta_n\}$ be a real sequence of $]0,1]$ such that $\sum_{n=1}^{\infty} \beta_n = \infty$. Suppose that*

$$s_{n+1} \le (1-\beta_n)s_n + \beta_n t_n$$

for all $n \in \mathbb{N}$ and that $\limsup_{i\to\infty} t_{n_i} \le 0$ for every subsequence $\{s_{n_i}\}$ of $\{s_n\}$ satisfying that $\limsup_{i\to\infty}(s_{n_i} - s_{n_i+1}) \le 0$. Then, $\lim_{n\to\infty} s_n = 0$.

Now, we can prove the following theorem:

Theorem 3.2. *Let X be an admissible complete $\mathrm{CAT}(1)$ space and f a proper lower semicontinuous convex function from X into $]-\infty,\infty]$ such that $\operatorname{Min} f$ is nonempty. Let $\{\lambda_n\}$ be a positive real sequence such that $\inf_{k\in\mathbb{N}} \lambda_k > 0$. Let $\{\varepsilon_n\}$ be a real sequence of $]0,1[$ such that $\lim_{n\to\infty} \varepsilon_n = 0$ and that $\sum_{n=1}^{\infty} \varepsilon_n = \infty$. For an anchor point $u \in X$ and an initial point $x_1 \in X$, generate a sequence $\{x_n\}$ of X as follows:*

$$\begin{aligned} R_{\lambda_n f} x_n &= \operatorname*{Argmin}_{z\in X} \left(\lambda_n f(z) + \tan d(z, x_n) \sin d(z, x_n)\right); \\ \alpha_n &\in \left[\sqrt{\varepsilon_n + \frac{\cos^2 d(u, R_{\lambda_n f} x)}{4}} - \frac{\cos d(u, R_{\lambda_n f} x)}{2}, \sqrt{\varepsilon_n}\right] \subset\,]0,1[; \\ x_{n+1} &= \alpha_n u \overset{1}{\oplus} (1-\alpha_n) R_{\lambda_n f} x_n \end{aligned}$$

for each $n \in \mathbb{N}$. Then, the generated sequence $\{x_n\}$ converges to a minimiser $P_{\operatorname{Min} f} u$, where $P_{\operatorname{Min} f}$ is a metric projection onto $\operatorname{Min} f$.

Proof. For $\varepsilon \in\,]0,1[$ and $c > 0$, we have

$$0 < \sqrt{\varepsilon + c^2} - c < \sqrt{\varepsilon} < 1.$$

Indeed,

$$\sqrt{\varepsilon + c^2} - c > \sqrt{c^2} - c = 0$$

and

$$\left(\sqrt{\varepsilon + c^2} - c\right)^2 = \varepsilon + 2c^2 - 2c\sqrt{\varepsilon + c^2} < \varepsilon + 2c^2 - 2c\sqrt{c^2} = \varepsilon < 1.$$

Therefore, the sequence $\{x_n\}$ is well defined and $\lim_{n\to\infty} \alpha_n = 0$. Moreover,

$$\alpha_n^2 + \alpha_n \cos d(u, R_{\lambda_n f} x_n) \ge \varepsilon_n$$

for all $n \in \mathbb{N}$. Set $p = P_{\operatorname{Min} f} u$ and $R_n = R_{\lambda_n f}$ for each $n \in \mathbb{N}$. Since R_n is quasinonexpansive for each $n \in \mathbb{N}$,

$$\begin{aligned}\cos d(p, x_{n+1}) &\geq \alpha_n \cos d(p, u) + (1 - \alpha_n) \cos d(p, R_n x_n) \\ &\geq \alpha_n \cos d(p, u) + (1 - \alpha_n) \cos d(p, x_n) \geq \min\{\cos d(p, u), \cos d(p, x_n)\}.\end{aligned}$$

Therefore, for all $n \in \mathbb{N}$,

$$d(p, R_n x_n) \leq d(p, x_n) \leq \max\{d(p, u), d(p, x_1)\} < \frac{\pi}{2},$$

which implies that $\{R_n x_n\}$ is spherically bounded. Set $l_n = d(u, R_n x_n)$ for each $n \in \mathbb{N}$. Let

$$s_n = 1 - \cos d(p, x_n)$$

and

$$t_n = 1 - \frac{\left(1 - \alpha_n + \sqrt{\alpha_n^2 + (1 - \alpha_n)^2 + 2\alpha_n (1 - \alpha_n) \cos l_n}\right) \cos d(u, p)}{\alpha_n + 2\alpha_n (1 - \alpha_n) \cos l_n}$$

for each $n \in \mathbb{N}$. Further, let $M_n = \sqrt{\alpha_n^2 + (1 - \alpha_n)^2 + 2\alpha_n (1 - \alpha_n) \cos l_n}$ and

$$\beta_n = 1 - \frac{1 - \alpha_n}{M_n} \in \,]0, 1[$$

for each $n \in \mathbb{N}$. Note that

$$M_n = \sqrt{\alpha_n^2 + (1 - \alpha_n)^2 + 2\alpha_n (1 - \alpha_n) \cos l_n} \leq 2.$$

Since $\{\alpha_n\}$ converges to 0, there exists $n_0 \in \mathbb{N}$ such that $1 - \alpha_n \geq 1/2$ whenever $n \geq n_0$. Hence, for $n \in \mathbb{N}$ with $n \geq n_0$,

$$\begin{aligned}\beta_n = 1 - \frac{1 - \alpha_n}{M_n} &= \frac{M_n - (1 - \alpha_n)}{M_n} = \frac{M_n^2 - (1 - \alpha_n)^2}{M_n (M_n + (1 - \alpha_n))} \\ &\geq \frac{\alpha_n^2 + 2\alpha_n (1 - \alpha_n) \cos l_n}{6} \geq \frac{\alpha_n^2 + \alpha_n \cos l_n}{6} \geq \frac{\varepsilon_n}{6}\end{aligned}$$

which implies that $\sum_{n=1}^{\infty} \beta_n = \infty$. For arbitrary fixed $n \in \mathbb{N}$,

$$\begin{aligned}s_{n+1} &= 1 - \cos d(p, x_{n+1}) = 1 - \cos d(p, \alpha_n u \overset{1}{\oplus} (1 - \alpha_n) R_n x_n) \\ &\leq 1 - \frac{\alpha_n \cos d(p, u) + (1 - \alpha_n) \cos d(p, R_n x_n)}{M_n} \\ &\leq 1 - \frac{\alpha_n \cos d(p, u) + (1 - \alpha_n) \cos d(p, x_n)}{M_n} \\ &= \frac{M_n - \alpha_n \cos d(p, u) - (1 - \alpha_n) \cos d(p, x_n)}{M_n} \\ &= \frac{1 - \alpha_n}{M_n} (1 - \cos d(p, x_n)) + \frac{M_n - (1 - \alpha_n) - \alpha_n \cos d(p, u)}{M_n} \\ &= (1 - \beta_n) s_n + \frac{M_n - (1 - \alpha_n) - \alpha_n \cos d(p, u)}{M_n}.\end{aligned}$$

Moreover,

$$
\begin{aligned}
&\frac{M_n - (1-\alpha_n) - \alpha_n \cos d(p,u)}{M_n} \\
&= \beta_n \frac{M_n - (1-\alpha_n) - \alpha_n \cos d(p,u)}{\beta_n M_n} = \beta_n \frac{M_n - (1-\alpha_n) - \alpha_n \cos d(p,u)}{M_n - (1-\alpha_n)} \\
&= \beta_n \frac{M_n^2 - (1-\alpha_n)^2 - \alpha_n (M_n + 1 - \alpha_n) \cos d(p,u)}{M_n^2 - (1-\alpha_n)^2} \\
&= \beta_n \frac{\alpha_n^2 + 2\alpha_n(1-\alpha_n)\cos l_n - \alpha_n (M_n + 1 - \alpha_n) \cos d(p,u)}{\alpha_n^2 + 2\alpha_n(1-\alpha_n)\cos l_n} \\
&= \beta_n \left(1 - \frac{\alpha_n (M_n + 1 - \alpha_n) \cos d(p,u)}{\alpha_n^2 + 2\alpha_n(1-\alpha_n)\cos l_n}\right) = \beta_n \left(1 - \frac{(1 - \alpha_n + M_n) \cos d(p,u)}{\alpha_n + 2(1-\alpha_n)\cos l_n}\right) \\
&= \beta_n t_n
\end{aligned}
$$

and therefore

$$s_{n+1} \le (1-\beta_n) s_n + \beta_n t_n.$$

Take a subsequence $\{s_{n_i}\}$ of $\{s_n\}$ such that

$$\limsup_{i\to\infty} (s_{n_i} - s_{n_i+1}) \le 0,$$

and we show that $\limsup_{i\to\infty} t_{n_i} \le 0$. Note that $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} M_n = 1$. If $\limsup_{i\to\infty} \cos l_{n_i} = 0$, then

$$\limsup_{i\to\infty} t_{n_i} = 1 - \liminf_{i\to\infty} \frac{(1 - \alpha_{n_i} + M_{n_i}) \cos d(u,p)}{\alpha_{n_i} + 2\alpha_{n_i}(1-\alpha_{n_i})\cos l_{n_i}} = -\infty < 0$$

Suppose that $\limsup_{i\to\infty} \cos l_{n_i} > 0$. Then,

$$
\begin{aligned}
0 &\ge \limsup_{i\to\infty} (s_{n_i} - s_{n_i+1}) = \limsup_{i\to\infty} (\cos d(p, x_{n_i+1}) - \cos d(p, x_{n_i})) \\
&= \limsup_{i\to\infty} \left(\cos d(p, \alpha_{n_i} u \overset{1}{\oplus} (1-\alpha_{n_i}) R_{n_i} x_{n_i}) - \cos d(p, x_{n_i})\right) \\
&\ge \limsup_{i\to\infty} (\alpha_{n_i} \cos d(p,u) + (1-\alpha_{n_i}) \cos d(p, R_{n_i} x_{n_i}) - \cos d(p, x_{n_i})) \\
&= \limsup_{i\to\infty} (\cos d(p, R_{n_i} x_{n_i}) - \cos d(p, x_{n_i})) \\
&\ge \liminf_{i\to\infty} (\cos d(p, R_{n_i} x_{n_i}) - \cos d(p, x_{n_i})) \ge 0
\end{aligned}
$$

and thus $\lim_{i\to\infty} (\cos d(p, R_{n_i} x_{n_i}) - \cos d(p, x_{n_i})) = 0$. Set $D_n = d(R_n x_n, x_n)$ for each $n \in \mathbb{N}$. From Lemma 2.5, when $d(R_{n_i} x_{n_i}, p) \ne 0$, we know that

$$
\begin{aligned}
&f(R_{n_i} x_{n_i}) \\
&\le f(p) + \frac{d(R_{n_i} x_{n_i}, p)}{\sin d(R_{n_i} x_{n_i}, p)} \left(\frac{1}{\cos^2 D_{n_i}} + 1\right) (\cos d(R_{n_i} x_{n_i}, p) \cos D_{n_i} - \cos d(p, x_{n_i})).
\end{aligned}
$$

Since $f(R_{n_i}x_{n_i}) - f(p) \geq 0$, we have

$$0 \leq \frac{d(R_{n_i}x_{n_i}, p)}{\sin d(R_{n_i}x_{n_i}, p)} \left(\frac{1}{\cos^2 D_{n_i}} + 1 \right) \left(\cos d(R_{n_i}x_{n_i}, p) \cos D_{n_i} - \cos d(p, x_{n_i}) \right).$$

Therefore,

$$0 \leq \cos d(R_{n_i}x_{n_i}, p) \cos D_{n_i} - \cos d(p, x_{n_i}).$$

This inequality also holds when $d(R_{n_i}x_{n_i}, p) = 0$. Further, this inequality implies that

$$0 \leq \cos d(R_{n_i}x_{n_i}, p)(\cos D_{n_i} - 1) + \cos d(R_{n_i}x_{n_i}, p) - \cos d(p, x_{n_i})$$

and hence

$$\begin{aligned}
\cos D_{n_i} &\geq 1 - \frac{\cos d(R_{n_i}x_{n_i}, p) - \cos d(p, x_{n_i})}{\cos d(R_{n_i}x_{n_i}, p)} \geq 1 - \frac{|\cos d(R_{n_i}x_{n_i}, p) - \cos d(p, x_{n_i})|}{\cos d(R_{n_i}x_{n_i}, p)} \\
&\geq 1 - \frac{|\cos d(R_{n_i}x_{n_i}, p) - \cos d(p, x_{n_i})|}{\cos\left(\max\{d(p, u), d(p, x_1)\}\right)}.
\end{aligned}$$

It implies that $\lim_{i \to \infty} D_{n_i} = 0$. Moreover, From Corollary 2.6, we obtain

$$\begin{aligned}
f(R_{n_i}x_{n_i}) &\leq f(p) + 2 \left(\frac{1}{\cos^2 D_{n_i}} + 1 \right) \frac{|\cos d(R_{n_i}x_{n_i}, p) - \cos d(p, x_{n_i})|}{\lambda_{n_i}} \\
&\leq f(p) + 2 \left(\frac{1}{\cos^2 D_{n_i}} + 1 \right) \frac{|\cos d(R_{n_i}x_{n_i}, p) - \cos d(p, x_{n_i})|}{\inf_{k \in \mathbb{N}} \lambda_k}
\end{aligned}$$

for all $i \in \mathbb{N}$. Since $\{R_{n_i}x_{n_i}\}$ is spherically bounded, it has a Δ-convergent subsequence. Let $\{R_{n_{i_j}}x_{n_{i_j}}\}$ be a Δ-convergent subsequence of $\{R_{n_i}x_{n_i}\}$ such that

$$\liminf_{i \to \infty} l_{n_i} = \liminf_{i \to \infty} d(u, R_{n_i}x_{n_i}) = \lim_{j \to \infty} d(u, R_{n_{i_j}}x_{n_{i_j}}) = \lim_{j \to \infty} l_{n_{i_j}}.$$

Let $w \in X$ be its Δ-limit. Since

$$\lim_{j \to \infty} \Big(\cos d(p, R_{n_{i_j}}x_{n_{i_j}}) - \cos d(p, x_{n_{i_j}}) \Big) = 0,$$

letting $j \to \infty$, we have

$$\begin{aligned}
f(w) &\leq \liminf_{j \to \infty} f(R_{n_{i_j}}x_{n_{i_j}}) \\
&\leq f(p) + 2 \liminf_{j \to \infty} \left(\frac{1}{\cos^2 D_{n_{i_j}}} + 1 \right) \frac{\left|\cos d(R_{n_{i_j}}x_{n_{i_j}}, p) - \cos d(p, x_{n_{i_j}})\right|}{\inf_{k \in \mathbb{N}} \lambda_k} \\
&\leq f(p) = \inf_{y \in X} f(y)
\end{aligned}$$

and thus $w \in \operatorname{Min} f$. Since $p = P_{\operatorname{Min} f} u$ is the closest minimiser to u, we obtain

$$\liminf_{i \to \infty} l_{n_i} = \lim_{j \to \infty} l_{n_{i_j}} = \lim_{j \to \infty} d(u, R_{n_{i_j}}x_{n_{i_j}}) \geq d(u, w) \geq d(u, p).$$

Now, we have

$$\begin{aligned}\limsup_{i\to\infty} t_{n_i} &= 1-\liminf_{i\to\infty}\frac{(1-\alpha_{n_i}+M_{n_i})\cos d(u,p)}{\alpha_{n_i}+2\alpha_{n_i}(1-\alpha_{n_i})\cos l_{n_i}}\\ &= 1-\frac{\cos d(u,p)}{\limsup_{i\to\infty}\cos l_{n_i}} = 1-\frac{\cos d(u,p)}{\cos\left(\liminf_{i\to\infty} l_{n_i}\right)}\\ &\le 1-\frac{\cos d(u,p)}{\cos d(u,p)} = 0.\end{aligned}$$

Consequently, from Lemma 3.1, we obtain $\lim_{n\to\infty} s_n = 0$, namely, $\{x_n\}$ converges to $P_{\mathrm{Min}\, f}u$. □

In the iteration of Theorem 3.2, we can take $\{\alpha_n\}$ as $\{\sqrt{\varepsilon_n}\}$. In Theorem 1.1, the sum of square of the coefficients sequence of convex combination should diverge. Theorem 3.2 means that we can remove this assumption by using 1-convex combination. However, we have not known whether we can obtain some similar result in the case of the canonical convex combination yet.

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