

Approximation of a common fixed point using a balanced mapping in Hadamard spaces and its application

均衡写像を用いた共通不動点近似定理とその応用

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Abstract

In this paper, we introduce a Mann iteration of a balanced mapping of a countable family of nonexpansive mappings in Hadamard spaces. Further, we prove a strong convergence theorem with the combining projection method of balanced type using a finite family of mappings in a real Hilbert ball.

1 Introduction

Approximation of a fixed point is studied by many researchers in various spaces. As a most famous method of approximation technique, we know the projection method. In 2003, Nakajo and Takahashi introduced *Nakajo–Takahashi projection method* in Hilbert spaces [9]. In 2005, Takahashi et al. introduced *the shrinking projection method* in Hilbert spaces [11].

In 2011, Kimura et al. introduced another projection method, which is called *the combining projection method*.

Theorem 1.1 (Kimura et al. [7]). *Let C be a nonempty closed convex subset C of a Hilbert space. Let $I_N = \{1, 2, \dots, N\}$ and T_j a nonexpansive mapping of C into itself for $j \in I_N$ such that $\bigcap_{j=1}^N \text{Fix} T_j \neq \emptyset$. Let $\{\alpha_n \mid n \in \mathbb{N}\} \subset [0, 1]$, $\{\beta_n^j \mid j \in I_N, n \in \mathbb{N}\} \subset [0, 1]$ such that $\sum_{j=1}^N \beta_n^j = 1$ for $n \in \mathbb{N}$, $\{\gamma_{n,k} \mid n, k \in \mathbb{N}, k \leq n\}$ such that $\sum_{k=1}^n \gamma_{n,k} = 1$ for $n \in \mathbb{N}$, and $\{\delta_n \mid n \in \mathbb{N}\} \subset [0, 1]$. Define a sequence*

$\{x_n\}$ by $u, x_1 \in C$ and

$$\begin{aligned} y_n^j &= \alpha_n x_n + (1 - \alpha_n) T_j x_n \text{ for } j \in I_N; \\ C_n^j &= \{z \in C \mid \|z - y_n^j\| \leq \|z - x_n\|\} \text{ for } j \in I_N; \\ x_{n+1} &= \delta_n u + (1 - \delta_n) \sum_{k=1}^n \gamma_{n,k} \sum_{j=1}^N \beta_k^j P_{C_k^j} x_n \end{aligned}$$

for each $n \in \mathbb{N}$, where P_K is the metric projection of H onto a nonempty closed convex subset K of H . Suppose the following conditions hold:

- (i) $\liminf_{n \rightarrow \infty} \alpha_n < 1$;
- (ii) $\beta_n^j > 0$ for all $j \in I_N$;
- (iii) $\lim_{n \rightarrow \infty} \gamma_{n,k} > 0$ for all $k \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \sum_{k=1}^n |\gamma_{n+1,k} - \gamma_{n,k}| < \infty$;
- (iv) $\lim_{n \rightarrow \infty} \delta_n = 0$, $\sum_{n=1}^{\infty} \delta_n = \infty$ and $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$.

Then, $\{x_n\}$ converges strongly to $P_{\bigcap_{j=1}^N \text{Fix } T_j} u$.

In geodesic spaces, the convex combination of more than three points are order-dependent in general. In 2018, Hasegawa and Kimura [5] introduced another definition of convex combination which is order-independent for three points. Using this notion, Kimura and Ogihara [6] introduced *the combining projection method of balanced type* and proved a convergence to a fixed point of a nonexpansive mapping.

In this paper, we consider approximating a common fixed point of a countable or finite family of mappings in Hadamard spaces. In section 3, we introduce a Mann iterative scheme of a balanced mapping of a countable family of nonexpansive mappings and prove a delta-convergent to a common fixed point. In section 4, we introduce the combining projection method of balanced type of a finite family of mappings in a real Hilbert ball.

2 Preliminaries

Let X be a metric space. A set $\text{Fix } T$ is all fixed points of a mapping T of X into itself. A mapping T is *nonexpansive* if the inequality $d(Tx, Ty) \leq d(x, y)$ holds for all $x, y \in X$. A mapping T is *quasinonexpansive* if $\text{Fix } T \neq \emptyset$ and the inequality $d(Tx, z) \leq d(x, z)$ for $x \in X$ and $z \in \text{Fix } T$. A nonexpansive mapping T with $\text{Fix } T \neq \emptyset$ is quasinonexpansive. Indeed, for $x \in X$ and $z \in \text{Fix } T$, by nonexpansiveness of T , it follows that

$$d(Tx, z) = d(Tx, Tz) \leq d(x, z)$$

and hence T is quasinonexpansive.

Let f be a function of X into \mathbb{R} . A set $\text{Argmin}_{y \in X} f(y)$ is defined by

$$\text{Argmin}_{y \in X} f(y) = \left\{ z \in X \mid f(z) = \inf_{y \in X} f(y) \right\}.$$

Let $\{x_n\}$ be a bounded sequence of X and $x_0 \in X$. Then, x_0 is an asymptotic center of $\{x_n\}$ if the equality

$$\limsup_{n \rightarrow \infty} d(x_n, x_0) = \inf_{y \in X} \limsup_{n \rightarrow \infty} d(x_n, y)$$

holds. The set of all asymptotic centers of $\{x_n\}$ is denoted by $\text{AC}(\{x_n\})$. Further, $\{x_n\}$ is *delta-convergent* to x_0 if for all subsequence $\{x_{n_i}\}$ of $\{x_n\}$, $\text{AC}(\{x_{n_i}\}) = \{x_0\}$, which is denoted by $x_n \xrightarrow{\Delta} x_0$.

A metric space X is a *uniquely geodesic space* if for all $x, y \in X$, there exists a unique mapping γ of $[0, d(x, y)]$ into X such that $\gamma(0) = x$, $\gamma(d(x, y)) = y$ and $d(\gamma(s), \gamma(t)) = |s - t|$ for all $s, t \in [0, d(x, y)]$. Let $x, y \in X$. Then, we can take a unique point $z = \gamma((1 - t)d(x, y))$ for each $t \in [0, 1]$, which is called a *convex combination between x and y* and is denoted by $z = tx \oplus (1 - t)y$.

Let X be a uniquely geodesic space and $x, y, z \in X$. Then, a *geodesic triangle of vertices x, y, z* is defined by $\text{Im } \gamma_{xy} \cup \text{Im } \gamma_{yz} \cup \text{Im } \gamma_{zx}$, which is denoted by $\Delta(x, y, z)$. For $x, y, z \in X$, a *comparison triangle* to $\Delta(x, y, z) \subset X$ of vertices $\bar{x}, \bar{y}, \bar{z} \in \mathbb{E}^2$ is defined by $\text{Im } \gamma_{\bar{x}\bar{y}} \cup \text{Im } \gamma_{\bar{y}\bar{z}} \cup \text{Im } \gamma_{\bar{z}\bar{x}}$ with $d(x, y) = d_{\mathbb{E}^2}(\bar{x}, \bar{y})$, $d(y, z) = d_{\mathbb{E}^2}(\bar{y}, \bar{z})$ and $d(z, x) = d_{\mathbb{E}^2}(\bar{z}, \bar{x})$, which is denoted by $\bar{\Delta}(\bar{x}, \bar{y}, \bar{z})$. A point $\bar{p} \in \text{Im } \gamma_{\bar{x}\bar{y}}$ is called a *comparison point of $p \in \text{Im } \gamma_{xy}$* if $d(x, p) = d_{\mathbb{E}^2}(\bar{x}, \bar{p})$. A uniquely geodesic space X is called a *CAT(0) space* if for all $x, y, z \in X$, $p, q \in \Delta(x, y, z)$ and their comparison points $\bar{p}, \bar{q} \in \bar{\Delta}(\bar{x}, \bar{y}, \bar{z})$, it follows that $d(p, q) \leq d_{\mathbb{E}^2}(\bar{p}, \bar{q})$. A complete CAT(0) space is called a *Hadamard space*.

The following lemmas are important properties of a CAT(0) space.

Lemma 2.1. *Let X be a CAT(0) space. Then,*

$$d(tx \oplus (1 - t)y, z)^2 \leq td(x, z)^2 + (1 - t)d(y, z)^2 - t(1 - t)d(x, y)^2$$

for each $x, y, z \in X$ and $t \in [0, 1]$.

Lemma 2.2 (Kirk and Panyanak [8]). *Let X be a Hadamard space. Then every bounded sequence has a subsequence which is delta-convergent to $x_0 \in X$.*

Lemma 2.3 (Dhompongsa, Kirk and Sims [2]). *Let X be a Hadamard space and $\{x_n\}$ a bounded sequence of X . Then the asymptotic center of $\{x_n\}$ consists of one point.*

In 2018, Hasegawa and Kimura introduced a notion of a *balanced mapping* and consider its properties in Hadamard spaces.

Theorem 2.1 (Hasegawa and Kimura [5]). *Let X be a Hadamard space, T_i a non-expansive mapping of X into itself for $i = 1, 2, \dots, N$ such that $\bigcap_{i=1}^N \text{Fix } T_i \neq \emptyset$, and $\{\alpha^i : i = 1, 2, \dots, N\} \subset]0, 1[$ such that $\sum_{i=1}^N \alpha^i = 1$. Let*

$$Ux = \text{Argmin}_{y \in X} \sum_{i=1}^N \alpha^i d(T_i x, y)^2$$

for each $x \in X$. Then, the following conditions hold:

- (i) U is nonexpansive;
- (ii) $\text{Fix } U = \bigcap_{i=1}^N \text{Fix } T_i$.

In the following theorem, Kimura and Ogihara [6] prove properties of a balanced mapping.

Theorem 2.2 (Kimura and Ogihara [6]). *Let X be a Hadamard space, C a nonempty bounded subset of X , T_k a nonexpansive mapping of X into itself for $k \in \mathbb{N}$ with $\bigcap_{k=1}^{\infty} \text{Fix } T_k \neq \emptyset$, $\{\alpha_n^k \mid n, k \in \mathbb{N}, k \leq n\} \subset]0, 1[$ such that $\sum_{k=1}^n \alpha_n^k = 1$ for all $n \in \mathbb{N}$. Let*

$$U_n x = \underset{y \in X}{\text{Argmin}} \sum_{k=1}^n \alpha_n^k d(T_k x, y)$$

for each $n \in \mathbb{N}$ and $x \in X$. Suppose the following conditions:

- $\lim_{n \rightarrow \infty} \alpha_n^k > 0$ for $k \in \mathbb{N}$;
- $\sum_{n=1}^{\infty} \sum_{k=1}^n |\alpha_{n+1}^k - \alpha_n^k| < \infty$.

Then the following conditions hold:

- (i) $\sum_{n=1}^{\infty} \sup_{x \in C} d(U_{n+1} x, U_n x) < \infty$;
- (ii) there exists a mapping $U: X \rightarrow X$ such that $Ux = \lim_{n \rightarrow \infty} U_n x$ for each $x \in X$;
- (iii) $\lim_{n \rightarrow \infty} \sup_{x \in C} d(U_n x, Ux) = 0$;
- (iv) U is nonexpansive and $\text{Fix } U = \bigcap_{k=1}^{\infty} \text{Fix } T_k$.

In the following, we introduced a Halpern iteration with a balanced mapping of countable family of nonexpansive mappings in Hadamard spaces.

Theorem 2.3 (Hasegawa [4], Kimura and Ogihara [6]). *Let X be a Hadamard space, T_k a nonexpansive mapping of X into itself for $k \in \mathbb{N}$ such that $F = \bigcap_{k=1}^{\infty} \text{Fix } T_k \neq \emptyset$, $\{\alpha_n^k \mid n, k \in \mathbb{N}, k \leq n\} \subset [0, 1]$ such that $\sum_{k=1}^n \alpha_n^k = 1$ for all $n \in \mathbb{N}$, and $\{\delta_n \mid n \in \mathbb{N}\} \subset [0, 1]$. Let*

$$U_n x = \underset{y \in X}{\text{Argmin}} \sum_{k=1}^n \alpha_n^k d(T_k x, y)^2$$

for all $x \in X$ and $n \in \mathbb{N}$. Define a sequence $\{x_n\}$ by $u, x_1 \in X$ and

$$x_{n+1} = \delta_n u \oplus (1 - \delta_n) U_n x_n$$

for each $n \in \mathbb{N}$. Suppose the following conditions hold:

- (i) $\lim_{n \rightarrow \infty} \alpha_n^k > 0$ for $k \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \sum_{k=1}^n |\alpha_{n+1}^k - \alpha_n^k| < \infty$;
- (ii) $\lim_{n \rightarrow \infty} \delta_n = 0$, $\sum_{n=1}^{\infty} \delta_n = \infty$ and $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$.

Then, $\{x_n\}$ is convergent to $P_F u$, where P_F is the metric projection of X onto F .

The following lemma is important to prove a delta-convergence theorem of a countable family of nonexpansive mappings.

Lemma 2.4 (Tan and Xu [12]). *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers such that $a_{n+1} \leq a_n + b_n$ for all $n \in \mathbb{N}$. If $\sum_{k=1}^{\infty} b_k < \infty$, then $\lim_{n \rightarrow \infty} a_n$ exists.*

3 Delta-convergence theorem using a balanced mapping

In this section, we prove an approximation theorem of a common fixed point of a countable family of nonexpansive mappings in Hadamard spaces.

Theorem 3.1. *Let X be a Hadamard space, T_k a nonexpansive mapping of X into itself for $k \in \mathbb{N}$ such that $F = \bigcap_{k=1}^{\infty} \text{Fix } T_k \neq \emptyset$, $\{\alpha_n^k \mid k, n \in \mathbb{N}, k \leq n\} \subset [0, 1]$ such that $\sum_{k=1}^n \alpha_n^k = 1$ for $n \in \mathbb{N}$, and $\{t_n \mid n \in \mathbb{N}\} \subset [0, 1[$. Let*

$$U_n = \underset{y \in X}{\text{Argmin}} \sum_{k=1}^n \alpha_n^k d(T_k x, y)^2$$

for each $n \in \mathbb{N}$ and $x \in X$. Define a sequence $\{x_n\}$ of X by $x_1 \in X$ and

$$x_{n+1} = t_n x_n \oplus (1 - t_n) U_n x_n$$

for all $n \in \mathbb{N}$. Suppose the following conditions:

- (i) $\lim_{n \rightarrow \infty} \alpha_n^k > 0$ for $k \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \sum_{k=1}^n |\alpha_{n+1}^k - \alpha_n^k| < \infty$;
- (ii) $\sum_{n=1}^{\infty} t_n(1 - t_n) = \infty$.

Then $\{x_n\}$ is delta-convergent to $x_0 \in F$.

Proof. Let $p \in F$. Then $p \in \bigcap_{k=1}^n \text{Fix } T_k$ for $n \in \mathbb{N}$. Since U_n is quasinonexpansive for $n \in \mathbb{N}$, we get

$$d(x_{n+1}, p) \leq t_n d(x_n, p) + (1 - t_n) d(U_n x_n, p) \leq d(x_n, p)$$

and hence $\{d(x_n, p)\}$ is nonincreasing. Then, there exists $\lim_{n \rightarrow \infty} d(x_n, p)$. Further, we get

$$\begin{aligned} d(x_{n+1}, p)^2 &\leq t_n d(x_n, p)^2 + (1 - t_n) d(U_n x_n, p)^2 - t_n(1 - t_n) d(x_n, U_n x_n)^2 \\ &\leq d(x_n, p)^2 - t_n(1 - t_n) d(x_n, U_n x_n)^2 \end{aligned}$$

and hence

$$t_n(1 - t_n) d(x_n, U_n x_n)^2 \leq d(x_n, p)^2 - d(x_{n+1}, p)^2.$$

Since $\sum_{k=1}^{\infty} t_k(1 - t_k) = \infty$, we get $\liminf_{n \rightarrow \infty} d(x_n, U_n x_n) = 0$. We next show $\lim_{n \rightarrow \infty} d(x_n, U_n x_n) = 0$. Let C be a nonempty bounded subset of X including $\{x_n\}$. By the nonexpansiveness of U_n for $n \in \mathbb{N}$, we get

$$d(x_{n+1}, U_{n+1} x_{n+1}) \leq d(x_{n+1}, U_n x_n) + d(U_n x_n, U_n x_{n+1}) + d(U_n x_{n+1}, U_{n+1} x_{n+1})$$

$$\begin{aligned}
&\leq d(x_{n+1}, U_n x_n) + d(x_{n+1}, x_n) + d(U_n x_{n+1}, U_{n+1} x_{n+1}) \\
&= d(x_n, U_n x_n) + d(U_n x_{n+1}, U_{n+1} x_{n+1}) \\
&\leq d(x_n, U_n x_n) + \sup_{x \in C} d(U_n x, U_{n+1} x)
\end{aligned}$$

for all $n \in \mathbb{N}$. By Theorem 2.2, $\sum_{n=1}^{\infty} \sup_{x \in C} d(U_n x, U_{n+1} x) < \infty$. By Lemma 2.4, there exists $\lim_{n \rightarrow \infty} d(x_n, U_n x_n)$ and hence $\lim_{n \rightarrow \infty} d(x_n, U_n x_n) = 0$. By Theorem 2.2, there exists a mapping $U: X \rightarrow X$ such that $Ux = \lim_{n \rightarrow \infty} U_n x$ for each $x \in X$, U is nonexpansive and $\text{Fix } U = F$. Take a subsequence $\{x_{n_i}\}$ of X with $\text{AC}(\{x_{n_i}\}) = \{y_0\}$ arbitrary. Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that $x_{n_{i_j}} \xrightarrow{\Delta} z_0 \in X$. Then, we get

$$\begin{aligned}
d(x_{n_{i_j}}, U x_{n_{i_j}}) &\leq d(x_{n_{i_j}}, U_{n_{i_j}} x_{n_{i_j}}) + d(U_{n_{i_j}} x_{n_{i_j}}, U x_{n_{i_j}}) \\
&\leq d(x_{n_{i_j}}, U_{n_{i_j}} x_{n_{i_j}}) + \sup_{x \in C} d(U_{n_{i_j}} x, U x).
\end{aligned}$$

From (iii) of Theorem 2.2, we get $\lim_{n \rightarrow \infty} d(x_{n_{i_j}}, U x_{n_{i_j}}) = 0$. Then, we get

$$\begin{aligned}
\limsup_{j \rightarrow \infty} d(x_{n_{i_j}}, U z_0) &\leq \limsup_{j \rightarrow \infty} (d(x_{n_{i_j}}, U x_{n_{i_j}}) + d(U x_{n_{i_j}}, U z_0)) \\
&= \limsup_{j \rightarrow \infty} d(U x_{n_{i_j}}, U z_0) \\
&\leq \limsup_{j \rightarrow \infty} d(x_{n_{i_j}}, z_0)
\end{aligned}$$

and hence $z_0 \in \text{Fix } U = F$. Put $\text{AC}(\{x_n\}) = \{y'_0\}$. Then, we get

$$\begin{aligned}
\limsup_{n \rightarrow \infty} d(x_n, z_0) &= \limsup_{j \rightarrow \infty} d(x_{n_{i_j}}, z_0) \\
&\leq \limsup_{j \rightarrow \infty} d(x_{n_{i_j}}, y_0) \\
&\leq \limsup_{i \rightarrow \infty} d(x_{n_i}, y_0) \\
&\leq \limsup_{i \rightarrow \infty} d(x_{n_i}, y'_0) \\
&\leq \limsup_{n \rightarrow \infty} d(x_n, y'_0) \leq \limsup_{n \rightarrow \infty} d(x_n, y_0)
\end{aligned}$$

and hence $y'_0 = y_0 = z_0 \in F$. Consequently, we complete the proof. \square

4 The combining projection method of balanced type in a real Hilbert ball

In this section, we introduce the combining projection method of balanced type and prove its convergence to a common fixed point by applying Theorem 2.3 in a real Hilbert ball.

Let $(H, \|\cdot\|)$ be a Real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $B = \{x \in H \mid \|x\| < 1\}$. we define $\rho(\cdot, \cdot): B \times B \rightarrow \mathbb{R}$ by

$$\rho(x, y) = \tanh^{-1} \sqrt{1 - \frac{(1 - \|x\|^2)(1 - \|y\|^2)}{1 - |\langle x, y \rangle|^2}}$$

for $x, y \in B$. Then, (B, ρ) is a metric space, which is called a *real Hilbert ball*. Further, we know that a real Hilbert ball is an example of Hadamard spaces, and a half space $\{z \in B \mid \rho(x, z) \leq \rho(y, z)\}$ is convex for $x, y \in B$; see [1, 3, 10].

In the following theorem, we introduce a sequence generated by *the combining projection method of balanced type* using a finite mappings, and prove its convergence in a real Hilbert ball.

Theorem 4.1. *Let B be a real Hilbert ball with the metric with ρ , T_i a family of a quasinonexpansive and continuous mappings of B into itself for $i = 1, 2, \dots, N$ such that $F = \bigcap_{i=1}^N \text{Fix } T_i \neq \emptyset$. Let $\{\alpha_n \mid n \in \mathbb{N}\} \subset [0, 1]$, $\{\beta_n^i \mid i = 1, 2, \dots, N, n \in \mathbb{N}\} \subset]0, 1[$ such that $\sum_{i=1}^N \beta_n^i = 1$ for $n \in \mathbb{N}$, $\{\gamma_{n,k} \mid n, k \in \mathbb{N}, k \leq n\} \subset]0, 1[$ such that $\sum_{k=1}^n \gamma_{n,k} = 1$ for $n \in \mathbb{N}$, and $\{\gamma_n \mid n \in \mathbb{N}\} \subset [0, 1]$. Let $u \in B$ and define sequences $\{x_n\}$ and $\{y_n^i\}$, sequences $\{V_k\}$ and $\{U_n\}$ of mappings of B into itself, and a sequence $\{C_n^i\}$ of a subset of B by $x_1 \in B$ and*

$$\begin{aligned} y_n^i &= \alpha_n x_n \oplus (1 - \alpha_n) T_i x_n \text{ for } i = 1, 2, \dots, N; \\ C_n^i &= \{z \in B \mid \rho(y_n^i, z) \leq \rho(x_n, z)\} \text{ for } i = 1, 2, \dots, N \text{ and } n \in \mathbb{N}; \\ V_k x &= \underset{y \in B}{\text{Argmin}} \sum_{i=1}^N \beta_i^k \rho(P_{C_k^i} x, y)^2 \text{ for } k \leq n \text{ and } x \in X; \\ U_n x &= \underset{y \in B}{\text{Argmin}} \sum_{k=1}^n \gamma_{n,k} \rho(V_k x, y)^2 \text{ for } x \in X; \\ x_{n+1} &= \delta_n u \oplus (1 - \delta_n) U_n x_n \end{aligned}$$

for each $n \in \mathbb{N}$, where a mapping P_K is the metric projection of B into a nonempty closed convex subset K of B . Suppose the following conditions:

- (i) $\liminf_{n \rightarrow \infty} \alpha_n < 1$;
- (ii) $\beta_n^i > 0$ for $i = 1, 2, \dots, N$ and $n \in \mathbb{N}$;
- (iii) $\lim_{n \rightarrow \infty} \gamma_{n,k} > 0$ for $k \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \sum_{k=1}^n |\gamma_{n+1,k} - \gamma_{n,k}| < \infty$;
- (iv) $\lim_{n \rightarrow \infty} \gamma_n = 0$ and $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$.

Then, $\{x_n\}$ is convergent to $P_F u$.

Proof. Since $\{z \in B \mid \rho(z, u) \leq \rho(z, v)\}$ is convex for $u, v \in B$ and a metric ρ is continuous, C_n^i is closed and convex for $i = 1, 2, \dots, N$ and $n \in \mathbb{N}$. Let $p \in F$. Since T_i is quasinonexpansive for $i = 1, 2, \dots, N$, we have F is closed convex and we get

$$\rho(y_k^i, p) = \rho(\alpha_k x_k \oplus (1 - \alpha_k) T_i x_k, p)$$

$$\begin{aligned} &\leq \alpha_k \rho(x_k, p) + (1 - \alpha_k) \rho(T_i x_k, p) \\ &\leq \rho(x_k, p) \end{aligned}$$

and hence $p \in C_k^i$ for $i = 1, 2, \dots, N$ and $k \in \mathbb{N}$. This implies that

$$\emptyset \neq \bigcap_{i=1}^N \text{Fix } T_i \subset \bigcap_{k=1}^{\infty} \bigcap_{i=1}^N C_k^i.$$

Since a metric projection $P_{C_k^i}$ is nonexpansive for $i = 1, 2, \dots, N$ and $k \in \mathbb{N}$, and Theorem 2.1, we get V_k is nonexpansive for $k \in \mathbb{N}$ and

$$\text{Fix } V_k = \bigcap_{i=1}^N \text{Fix } P_{C_k^i} = \bigcap_{i=1}^N C_k^i$$

for $k \in \mathbb{N}$ and hence

$$\bigcap_{k=1}^{\infty} \text{Fix } V_k = \bigcap_{k=1}^{\infty} \bigcap_{i=1}^N C_k^i \supset \bigcap_{i=1}^N \text{Fix } T_i \neq \emptyset.$$

Put $C_0 = \bigcap_{k=1}^{\infty} \bigcap_{i=1}^N C_k^i$. Since Theorem 2.3, $\{x_n\}$ is convergent to $x_0 = P_{C_0} u$. Since $x_0 \in C_0$, we get

$$\rho(y_n^i, x_0) \leq \rho(x_n, x_0)$$

and hence $y_n^i \rightarrow x_0$ for $i = 1, 2, \dots, N$. By (i), there exists $\{\alpha_{n_j}\} \subset \{\alpha_n\}$ such that $\lim_{j \rightarrow \infty} \alpha_{n_j} \in [0, 1[$. Then, it follows that

$$\rho(x_{n_j}, T_i x_{n_j}) = \frac{1}{1 - \alpha_{n_j}} \rho(x_{n_j}, y_{n_j}^i) \leq \frac{1}{1 - \alpha_{n_j}} (\rho(x_{n_j}, x_0) + \rho(x_0, y_{n_j}^i))$$

for $i = 1, 2, \dots, N$. Letting $j \rightarrow \infty$, we get $\lim_{j \rightarrow \infty} \rho(x_{n_j}, T_i x_{n_j}) = 0$. Since T_i is continuous for $i = 1, 2, \dots, N$, we obtain

$$\rho(x_0, T_i x_0) \leq \rho(x_0, x_{n_j}) + \rho(x_{n_j}, T_i x_{n_j}) + \rho(T_i x_{n_j}, T_i x_0)$$

for $i = 1, 2, \dots, N$ and $j \in \mathbb{N}$. Letting $j \rightarrow \infty$, we get $x_0 = T_i x_0$ for $i = 1, 2, \dots, N$. This implies that $x_0 \in \bigcap_{i=1}^N \text{Fix } T_i$. Consequently, we complete the proof. \square

Remark. In Theorem 4.1, we can replace B to a Hadamard space by adding the condition that a half space

$$\{z \in X \mid d(z, x) \leq d(z, y)\}$$

is convex for $x, y \in X$.

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