Asymptotic behavior of a resolvent for a sequence of monotone operators in a complete geodesic space
完備測地距離空間における単調作用素列のリゾルベントの漸近的挙動

八戸工業高等専門学校・総合科学教育科 新藤圭介
Keisuke Shindo
Department of General Science and Education
Hachinohe National College of Technology

Abstract

The notion of resolvent is used to solve various problems in nonlinear analysis and convex analysis. In particular, resolvents defined for a maximal monotone operator are deeply related with many problems. In this paper, we consider asymptotic behavior of a resolvent for a sequence of maximal monotone operators on a complete geodesic space.

1 Introduction

In nonlinear analysis and convex analysis, the concept of resolvent is very important, and it is a useful tool to solve problems such as convex minimization problems, equilibrium problems and so on. For f a proper lower semicontinuous convex function from Hilbert space H to $]-\infty,\infty]$, a resolvent $R_{\lambda f}\colon X\to X$ with parameter $\lambda>0$ is defined by

$$R_{\lambda f}(x)=\operatorname*{argmin}_{y\in X}\left\{\lambda f(y)+d(y,x)^2\right\},$$

where the $\operatorname*{argmin}_{y\in X} g$ is the set of all minimizers of the function $g\colon X\to\,]-\infty,\infty]$. We know the following result about the asymptotic behavior of the resolvent for a convex function.

Theorem 1.1 (See [6]). *Let H be a Hilbert space, $f\colon H\to[-\infty,\infty[$ a proper lower semicontinuous convex function and $x\in X$. If* $\operatorname{argmin} f\neq\emptyset$*, then*

$$\lim_{\lambda\to\infty}R_{\lambda f}(x)=P_{\operatorname{argmin} f}(x),$$

where P_C is the metric projection onto a subset C which is a nonempty closed convex subset of X.

The Mosco convergence of the sequence of sets are introduced by [5]. Let $C_0, C_1, C_2 \cdots$ be nonempty closed convex subsets of a Hilbert space H. For each $x \in X$, we consider the sequence $\{P_{C_n}(x)\}$ of the nearest points. About the relation between convergence of $\{C_n\}$ and the convergence of $\{P_{C_n}(x)\}$, we know the following result.

Theorem 1.2 (See [7]). *Let H be a Hilbert space, $C_0, C_1, C_2 \cdots$ be nonempty closed convex subsets of H, and $x \in X$. If $\{C_n\}$ is convergent to C_0 as a sense of Mosco, then*

$$\lim_{n\to\infty} P_{C_n}(x) = P_{C_0}(x).$$

Geodesic space is a metric space which has some convex structures. One of the examples of geodesic spaces is Hadamard space and its class is generalization of Hilbert spaces. It is known that above theorems in a Hilbert space are extended in this space (See [1], [3]). To connect these theorems, we consider the sequence $\{R_{\lambda_n f_n}(x)\}$ of resolvents for a sequence $\{f_n\}$ of convex functions. We know the following theorem about convergence of this sequence.

Theorem 1.3 (See [4]). *Let X be a Hadamard space, $f_0, f_1, f_2 \cdots$ proper lower semicontinuous convex functions from X to $]0, \infty]$, and $x \in X$. If $\{f_n\}$ satisfies the following conditions*

(1) $\emptyset \neq \operatorname{argmin} f_0 \subset d\text{-Li} \operatorname{argmin} f_n$;
(2) $x_0 \in X$ *belongs to* $\operatorname{argmin} f_0$ *whenever a subsequence of* $\{R_{\lambda_n f_n}(x)\}$ *is* Δ-*convergent to* x_0,

then

$$\lim_{n\to\infty} R_{\lambda_n f_n}(x) = P_{\operatorname{argmin} f}(x).$$

The concept of the dual space of a Hadamard space is introduced by [2] and we can consider monotone operators on a Hadamard space. Further, for a monotone operator, we can define its resolvent. We know that problems of monotone operators and their resolvents are greatly related with problems of convex functions and their resolvents. In this paper, we consider a resolvent for a sequence of maximal monotone operators and describe the relation between the convergence of a sequence of maximal monotone operators and convergence of the resolvent.

2 Preliminaries

Let X be a metric space. For $x, y \in X$, a geodesic $c_{xy}\colon [0, d(x,y)] \to \mathbb{R}$ with endpoints x and y is an isometric which satisfies $c_{xy}(0) = x$, $c_{xy}(d(x,y)) = y$. We call X a uniquely geodesic space if a geodesic exists uniquely for all $x, y \in X$. Let X be a uniquely geodesic space. We call the image of geodesic c_{xy} the geodesic segment joining x and y, and denote it by $[x, y]$. For $x, y \in X$ and $t \in [0,1]$, the convex combination $z = tx \oplus (1-t)y$ between x and y is the point with $d(z,x) = (1-t)d(x,y)$ and $d(z,y) = td(x,y)$. The subset C of X is said to be convex if $tx \oplus (1-t)y \in C$ for all $x, y \in C$. For $x_1, x_2, x_3 \in X$, the geodesic triangle $\triangle(x_1, x_2, x_3)$ is defined by $\triangle(x_1, x_2, x_3) = [x_1, x_2] \cup [x_2, x_3] \cup [x_3, x_1]$. For $\triangle(x_1, x_2, x_3)$, the comparison triangle $\overline{\triangle}(\overline{x_1}, \overline{x_2}, \overline{x_3}) \in \mathbb{R}^2$ is a triangle which satisfies $d_{\mathbb{R}^2}(\overline{x}_1, \overline{x}_2) = d(x_1, x_2)$, $d_{\mathbb{R}^2}(\overline{x}_2, \overline{x}_3) = d(x_2, x_3)$, $d_{\mathbb{R}^2}(\overline{x}_3, \overline{x}_1) = d(x_3, x_1)$, and the comparison point $\overline{p} \in [\overline{x}_i, \overline{x}_j]$ for $p \in [x_i, x_j]$ is the point with $d(\overline{x}_i, \overline{p})_{\mathbb{R}^2} = d(x_i, p)$ $(i, j = 1, 2, 3)$. X is called a CAT(0) space if for all $\triangle(x_1, x_2, x_3)$, $p, q \in \triangle(x_1, x_2, x_3)$ and $\overline{p}, \overline{q} \in \overline{\triangle}(\overline{x_1}, \overline{x_2}, \overline{x_3})$, it holds that $d(p,q) \leq d(\overline{p}, \overline{q})$. We call a complete CAT(0) space a Hadamard space. In what follows, we let X a Hadamard space. We denote the pair $(a,b) \in X$ by $\overrightarrow{ab}$ and call it a vector. The quazilinearization mapping $\langle \cdot, \cdot \rangle \colon (X \times X) \times (X \times X) \to \mathbb{R}$ is defined by

$$\left\langle \overrightarrow{ab}, \overrightarrow{cd} \right\rangle = \frac{1}{2}\{d(a,d)^2 + d(b,c)^2 - d(a,c)^2 - d(b,d)^2\}$$

for $a, b, c, d \in X$. Then, we have that

(1) $\left\langle \overrightarrow{ab}, \overrightarrow{ab} \right\rangle = d(a,b)^2$ for $a, b \in X$;
(2) $\left\langle \overrightarrow{ba}, \overrightarrow{cd} \right\rangle = -\left\langle \overrightarrow{ab}, \overrightarrow{cd} \right\rangle$ for $a, b, c, d \in X$;
(3) $\left\langle \overrightarrow{ab}, \overrightarrow{cd} \right\rangle = \left\langle \overrightarrow{ae}, \overrightarrow{cd} \right\rangle + \left\langle \overrightarrow{eb}, \overrightarrow{cd} \right\rangle$ for $a, b, c, d, e \in X$;
(4) $\left\langle \overrightarrow{ab}, \overrightarrow{cd} \right\rangle \leq d(a,b)d(c,d)$ for $a, b, c, d \in X$.

We define the mapping $\Theta \colon \mathbb{R} \times X \times X \to C(X, \mathbb{R})$

$$\Theta(t, a, b)(x) = t\left\langle \overrightarrow{ab}, \overrightarrow{ax} \right\rangle$$

for $t \in \mathbb{R}$ and $a, b, x \in X$. Further, we define the mapping d^* on $(R \times X \times X) \times (R \times X \times X) \to \mathbb{R}$ by

$$d^*((t,a,b),(s,c,d)) = L(\Theta(t,a,b) - \Theta(s,c,d))$$

for $(t,a,b), (s,c,d) \in \mathbb{R} \times X \times X$, where L is Lipchitz seminorm. Then, d^* is the pseudometric on $\mathbb{R} \times X \times X$. We consider the equivalence relation $\sim$ on $\mathbb{R} \times X \times X$ as $(t,a,b) \sim (s,c,d)$ if and only if $d^*((t,a,b),(s,c,d))$=0. Then, the set

$$X^* = ((\mathbb{R} \times X \times X)/\sim, d^*)$$

is a metric space and we call it a dual space of X. For simplicity, we use next symbols

(1) $\left[t\overrightarrow{ab}\right] = \left\{ s\overrightarrow{cd} \;\middle|\; d^*((t,a,b),(s,c,d)) = 0 \right\}$;
(2) $\langle x^*, \overrightarrow{xy} \rangle = t\left\langle \overrightarrow{ab}, \overrightarrow{xy} \right\rangle$; for $x, y \in X$, $x^* = \left[t\overrightarrow{ab}\right] \in X^*$;
(3) $\mathbf{0} = [t\overrightarrow{aa}]$ for $t \in \mathbb{R}$, $a \in X$;

(4) $\langle tx^* + sy^*, \overrightarrow{xy} \rangle = t \langle x^*, \overrightarrow{xy} \rangle + s \langle y^*, \overrightarrow{xy} \rangle$ for $t, s \in \mathbb{R}$, $x, y \in X$, $x^*, y^* \in X^*$.

Let $\{x_n\}$ be a sequence of X. The asymptotic center $\mathrm{AC}(\{x_n\})$ of $\{x_n\}$ is defined by

$$\mathrm{AC}(\{x_n\}) = \left\{ z \in X \;\middle|\; \limsup_{n\to\infty} d(z, x_n) = \inf_{y\in X} \limsup_{n\to\infty} d(y, x_n) \right\}.$$

If a bounded sequence $\{x_n\}$ satisfies that $\mathrm{AC}(\{x_{n_i}\}) = x_0$ for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, it is said to be Δ-convergent to $x_0 \in X$ and denoted by $x_n \overset{\Delta}{\rightharpoonup} x_0$. Whenever $\{x_n\}$ is bounded, $\mathrm{AC}(\{x_n\})$ is a singleton and $\{x_n\}$ has a Δ-convergent subsequence. For a sequence $\{C_n\}$ of nonempty subsets of X, we define $d\text{-}\mathrm{Li}\, C_n$ as $x \in d\text{-}\mathrm{Li}\, C_n$ if and only if there exists $\{x_n\}$ such that $x_n \to x$ and $x_n \in C_n$ for each $n \in \mathbb{N}$. We also define $\Delta\text{-}\mathrm{Ls}\, C_n$ as $x \in \Delta\text{-}\mathrm{Ls}\, C_n$ if and only if there exists a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{x_i\}$ of X such that $x_i \in C_{n_i}$ for each $i \in \mathbb{N}$ and $x \in \mathrm{AC}(\{x_i\})$ for all $i \in \mathbb{N}$. If $d\text{-}\mathrm{Li}\, C_n = C_0 = \Delta\text{-}\mathrm{Ls}\, C_n$, we say that $\{C_n\}$ is Δ-Mosco convergent to C_0 and denote it by $\text{M-lim}\, C_n = C_0$.

Let C be a closed convex subset of X. We define the indicator function $i_C \colon X \to [-\infty, \infty[$ for C by $i_C(x) = 0$ if $x \in C$ and $i_C(x) = \infty$ if $x \notin C$. For $x \in X$, there exists $z \in C$ such that $d(z, x) = \inf_{y\in X} d(y, x)$. Therefore, we can define $P_C \colon X \to C$ such that

$$P_C(x) = \operatorname*{argmin}_{y\in C} d(y, x)$$

for $x \in X$ and we call it the metric projection onto C.

For a multivalued operator A, we denote the domain of A by $D(A)$ and the range of A by $R(A)$. A multivalued operator $A \colon X \rightrightarrows X^*$ is called a monotone operator if it satisfies

$$0 \le \langle x^* - y^*, \overrightarrow{yx} \rangle$$

for all $(x, x^*), (y, y^*) \in A$. A monotone operator is said to be maximal if no monotone operator includes it properly. For a monotone operator A, we define the resolvent $J_{\lambda A} \colon X \rightrightarrows X$ with a parameter $\lambda > 0$ by

$$J_{\lambda A} = \left\{ z \in X \;\middle|\; \left[\frac{1}{\lambda} \overrightarrow{zx} \right] \in Az \right\}.$$

Then, the following holds

(1) $R(J_{\lambda A}) \subset D(A)$;
(2) $\mathrm{Fix}(J_{\lambda A}) = A^{-1}\mathbf{0}$;
(3) If A is monotone, $J_{\lambda A}$ consists of at most one point;
(4) $d(J_{\lambda A}(x), J_{\lambda A}(y)) \le d(x, y)$ for all $x, y \in D(A)$.

In what follows, we assume a maximal monotone A satisfies $D(S_{\lambda A}) = X$ for all $\lambda > 0$. Under this assumption, $J_{\lambda A}$ is a single-valued mapping from X to X for a maximal monotone operator A.

3 Main result

We consider the condition of a sequence of maximal monotone operators such that the sequence of resolvents is convergent to a point in the limit set.

Theorem 3.1. *Let X be a Hadamard space and X^* its dual, $A_0, A_1, A_2, \cdots$ maximal monotone operators from X to X^*, $\{\lambda_n\}$ a positive real sequence and $x \in X$. If $\{A_n\}$ satisfies*

(1) $\emptyset \neq C_0 \subset d\text{-Li}\, A_n^{-1}\mathbf{0}$;
(2) $x_0 \in X$ belongs to C_0 whenever a subsequence of $\{J_{\lambda_n A_n}(x)\}$ is Δ-convergent to x_0,

where C_0 is a nonempty closed convex subset, then

$$\lim_{n\to\infty} J_{\lambda_n A_n}(x) = P_{C_0}(x).$$

Proof. We put $p = P_{C_0}(x)$ and $x_n = J_{\lambda_n A_n}(x)$ for $n \in \mathbb{N}$. By the first assumption of $\{A_n\}$, there exists $\{u_n\}$ which satisfies $u_n \to p$ and $u_n \in A_n^{-1}\mathbf{0}$ for each $n \in \mathbb{N}$. From the definition of the resolvent, we have

$$\left[\frac{1}{\lambda_n}\overrightarrow{x_n x}\right] \in A_n x_n.$$

The monotonicity of A_n implies

$$\begin{aligned} 0 &\leq \left\langle \left[\frac{1}{\lambda_n}\overrightarrow{x_n x}\right] - \mathbf{0}, \overrightarrow{u_n x_n} \right\rangle \\ &= \frac{1}{\lambda_n}\langle \overrightarrow{x_n x}, \overrightarrow{u_n x_n}\rangle \\ &= \frac{1}{2\lambda_n}\left(d(u_n, x)^2 - d(x_n, x)^2 - d(u_n, x_n)^2\right) \end{aligned}$$

and hence

$$d(x_n, x)^2 \leq d(u_n, x)^2 - d(u_n, x_n)^2.$$

Therefore, we get $d(x_n, x) \leq d(u_n, x)$. Since $\{u_n\}$ is convergent, $\{x_n\}$ is bounded. We take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ arbitrarily. By the boundedness of $\{x_{n_i}\}$, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which is Δ-convergent to $x_0 \in X$. From the second assumption of $\{A_n\}$, we get $x_0 \in C_0$. Since $d(x_{n_{i_j}}, x) \leq d(u_{n_{i_j}}, x)$, letting $j \to \infty$, we get

$$d(x_0, x) \leq \liminf_{j\to\infty} d(x_{n_{i_j}}, x) \leq \limsup_{j\to\infty} d(x_{n_{i_j}}, x) \leq \limsup_{j\to\infty} d(u_{n_{i_j}}, x) = d(p, x).$$

Hence, $x_0 = p$. This implies $d(x_{n_{i_j}}, x) \to d(p, x)$ and $x_{n_{i_j}} \overset{\Delta}{\rightharpoonup} p$. Therefore, for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ converging to p, which is equivalent to $x_n \to p$. This is the desired result. $\square$

Assuming that positive real sequence $\{\lambda_n\}$ diverges to ∞, we get the following result by this theorem.

Theorem 3.2. *Let X be a Hadamard space and X^* its dual, $A_0, A_1, A_2, \cdots$ maximal monotone operators from X to X^*, and $\{\lambda_n\}$ a positive real sequence such that $\lambda_n \to \infty$. If $\{A_n\}$ satisfies*

(1) $\text{M-lim}\, A_n^{-1}\mathbf{0} = A_0^{-1}\mathbf{0} \neq \emptyset$;
(2) for all $(u_0, u_0^*) \in A_0$, there exists $\{(u_i, u_i^*)\}$ such that $(u_i, u_i^*) \in A_{n_i}$ for each $i \in \mathbb{N}$, $\{u_i\}$ is bounded, and $\limsup_{i\to\infty} \langle u_i^*, \overrightarrow{x_{n_i}u_i}\rangle \leq \langle u_0^*, \overrightarrow{x_0u_0}\rangle$,

then

$$\lim_{n\to\infty} J_{\lambda_n A_n}(x) = P_{A_0^{-1}\mathbf{0}}(x).$$

Proof. Obviously, it holds that $\emptyset \neq A_0^{-1}\mathbf{0} \subset d\text{-Li}\, A_n^{-1}\mathbf{0}$ from the assumption (1) of $\{A_n\}$. We show that the Δ-limit of any subsequence of resolvent belongs to $A_0^{-1}\mathbf{0}$ under the assumption (2) of $\{A_n\}$. Put p and x_n as in the proof of the previous theorem. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ which is Δ-convergent to $x_0 \in X$. For all $(u_0, u_0^*) \in A_0$, there exists $\{(u_i, u_i^*)\}$ which satisfies $(u_i, u_i^*) \in A_{n_i}$ for each $i \in \mathbb{N}$, $\{u_i\}$ is bounded, and $\limsup_{i\to\infty} \langle u_i^*, \overrightarrow{x_{n_i}u_i}\rangle \leq \langle u_0^*, \overrightarrow{x_0u_0}\rangle$. Then, from the monotonicity of A_{n_i}, we have

$$\begin{aligned}
0 &\leq \left\langle \left[\frac{1}{\lambda_{n_i}}\overrightarrow{x_{n_i}x}\right] - u_i^*, \overrightarrow{u_ix_{n_i}} \right\rangle \\
&= \frac{1}{\lambda_{n_i}} \langle \overrightarrow{x_{n_i}x}, \overrightarrow{u_ix_{n_i}}\rangle - \langle u_i^*, \overrightarrow{u_ix_{n_i}}\rangle \\
&\leq \frac{1}{\lambda_{n_i}} d(x_{n_i}, x) d(u_i, x_{n_i}) + \langle u_i^*, \overrightarrow{x_{n_i}u_i}\rangle
\end{aligned}$$

for all $i \in \mathbb{N}$. Letting $i \to \infty$, we get

$$0 \leq \limsup_{i\to\infty} \langle u_i^*, \overrightarrow{x_{n_i}u_i}\rangle \leq \langle u_0^*, \overrightarrow{x_0u_0}\rangle = \langle u_0^* - \mathbf{0}, \overrightarrow{x_0u_0}\rangle.$$

Then, maximality of A_0 implies that $x_0 \in A_0^{-1}\mathbf{0}$. Consequently, we get $x_n \to p$ from the previous theorem. It completes the proof. □

Next, we consider the case that the positive real sequence $\{\lambda_n\}$ is convergent to 0.

Theorem 3.3. *Let X be a Hadamard space and X^* its dual, $A_0, A_1, A_2, \cdots$ maximal monotone operators from X to X^*, and $\{\lambda_n\}$ a positive real sequence such that $\lambda_n \to 0$. If $\{A_n\}$ satisfies*

(1) $\text{M-lim}\, D(A_n) = \text{cl}\, D(A_0) \neq \emptyset$;
(2) there exists $\{(u_n, u_n^*)\}$ such that $(u_n, u_n^*) \in A_n$ for each $n \in \mathbb{N}$, $\{u_n\}$ is bounded, and $\limsup_{n\to\infty} \langle u_n^*, \overrightarrow{x_nu_n}\rangle < \infty$,

then

$$\lim_{n\to\infty} J_{\lambda_n A_n}(x) = P_{\text{cl}\, D(A_0)}(x).$$

***Proof*.** Put $p = P_{\mathrm{cl}\, D(A_0)}(x)$ and $x_n = J_{\lambda_n A_n}(x)$ for $n \in \mathbb{N}$. Then, there exists $\{(u_n, u_n^*)\}$ which satisfies $(u_n, u_n^*) \in A_n$, $u_n \to P_{\mathrm{cl}\, D(A_0)}(x)$ and $\limsup_{i\to\infty} \langle u_n^*, \overrightarrow{x_n u_n} \rangle < \infty$ from the assumption (2). By the monotonicity of A_n, we have

$$\begin{aligned} 0 &\le \left\langle \left[\frac{1}{\lambda_n}\overrightarrow{x_n x}\right] - u_n^*, \overrightarrow{u_n x_n} \right\rangle \\ &= \frac{1}{\lambda_n} \langle \overrightarrow{x_n x}, \overrightarrow{u_n x_n} \rangle - \langle u_n^*, \overrightarrow{u_n x_n} \rangle \\ &= \frac{1}{\lambda_n} \left(d(x, u_n)^2 - d(x_n, u_n)^2 - d(x, x_n)^2 \right) - \langle u_n^*, \overrightarrow{u_n x_n} \rangle \end{aligned}$$

and thus

$$\begin{aligned} d(x, x_n)^2 &\le d(x, u_n)^2 - d(x_n, u_n)^2 + \lambda_n \langle u_n^*, \overrightarrow{u_n x_n} \rangle \\ &\le d(x, u_n)^2 + \lambda_n \langle u_n^*, \overrightarrow{u_n x_n} \rangle . \end{aligned}$$

Letting $n \to \infty$, we get

$$0 \le \limsup_{n\to\infty} d(x, x_n)^2 \le \limsup_{n\to\infty} d(x, u_n)^2 = d(x, p)^2.$$

Hence, $\{x_n\}$ is bounded, and therefore for all subsequence $\{x_{n_i}\}$ of $\{x_n\}$, we can take a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which is Δ-convergent to some $x_0 \in X$. Then, $x_0 = AC(\{x_{n_{i_j}}\})$ for every $i \in \mathbb{N}$ and $x_{n_{i_j}} \in D(A_{n_{i_j}})$. From the assumption (1) of $\{A_n\}$, x_0 belongs to $\mathrm{cl}\, D(A_0)$. Since $d(x, x_{n_{i_j}})^2 \le d(x, u_{n_{i_j}})^2 + \lambda_{n_{i_j}} \left\langle u_{n_{i_j}}^*, \overrightarrow{u_{n_{i_j}} x_{n_{i_j}}} \right\rangle$, we obtain

$$d(x, p) \le d(x, x_0) \le \liminf_{j\to\infty} d(x, x_{n_{i_j}}) \le \limsup_{j\to\infty} d(x_{n_{i_j}}, x) \le d(x, p).$$

This implies $x_{n_{i_j}} \xrightarrow{\Delta} p$, and $d(x_{n_{i_j}}, x) \to d(p, x)$ and thus we get $x_{n_{i_j}} \to p$. Therefore, for every subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there exists a subsequence $\{x_{n_{i_j}}\}$ converging to p. Thus, we get $x_n \to p$, which is the desired result. □

References

[1] M. Bačák, *Convex Analysis and optimaization in Hadamard spaces*, De Gruyter, Berlin, 2014.

[2] B. A. Kakavandi and M. Amini, *Duality and subdifferential for convex functions on complete* CAT(0) *metric spaces* Nonlinear Anal. **73** (2010) 3450–3455.

[3] Y. Kimura, *Convergence of a sequence of sets in a Hadamard space and the shrinking projection method for a real Hilbert ball*, Abstract and Applied Analysis 2010 (2010), Article ID 582475, 11 pages.

[4] Y. Kimura and K. Shindo, *Asymptotic behavior of resolvents of a convergent sequence of convex functions on complete geodesic spaces*, Axioms **11** (2022), 8.

[5] U. Mosco, *Convergence of convex sets and solutions of variational inequalities*, Adv. Math. 3(1969), 510–585.

[6] W. Takahashi, *Nonlinear functional analysis fixed point theory and its applications*, Yokohama Publishers, Yokohama, 2000.

[7] M. Tsukada, *Convergence of best approximations in a smooth Banach space*, J. Approx. Theory **40** (1984), no. 4, 301–309.