Resolvents of convex functions and approximation of a minimizer on a geodesic space
測地距離空間上の凸関数に関するリゾルベントとそれを用いた最小点近似

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Abstract

The aim of this paper is to propose a new resolvent and to study the asymptotic behavior of a sequence generated by Mann iteration in complete geodesic space with negative curvature.

1 Introduction

Let X be an admissible complete CAT(1) space and f a proper lower semicontinuous convex function from X into $]-\infty, \infty]$. In this case, a resolvent of f is defined by

$$J_f x = \operatorname*{argmin}_{y \in X} \{ f(y) + \tan d(y, x) \sin d(y, x) \}$$

for all $x \in X$. In 2016, Kimura and Kohsaka proved its well-definedness. Let X be a complete CAT(-1) space and f a proper lower semicontinuous convex function of X to $]-\infty, \infty]$. Then the resolvent of f is defined by

$$I_f x = \operatorname*{argmin}_{y \in X} \{ f(y) + \tanh d(y, x) \sinh d(y, x) \}$$

for all $x \in X$. In 2019, Kajimura and Kimura showed that it is well-defined. The resolvent J_f corresponds to I_f in a complete CAT(-1) space.

Let X be a complete CAT(1) space and f a proper lower semicontinuous convex function from X into $]-\infty, \infty]$. A resolvent for f is defined by

$$Q_f x = \operatorname*{argmin}_{y \in X} \{ f(y) - \log \cos d(y, x) \}$$

for all $x \in X$. In 2019, Kajimura and Kimura provided its well-definedness. In 2023, Nakadai [7] showed the following theorem.

Theorem 1.1. *Let X be an admissible complete* CAT(1) *space and f a proper lower semicontinuous convex function from X into $]-\infty, \infty]$. Let $Q_{\eta f}$ the resolvent of ηf for all $\eta > 0$ and $\{x_n\}$ a sequence defined by $x_1 \in \operatorname{dom}(f)$ and*

$$x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n) Q_{\lambda_n f} x_n,$$

where $\{\alpha_n\}$ is a sequence in $[0, 1[$ and $\{\lambda_n\}$ is a sequence of positive real numbers such that $\sum_{n=1}^{\infty}(1 - \alpha_n)\lambda_n = \infty$. Then the following hold.

(i) *The set $\operatorname{argmin}_X f$ is nonempty if and only if $\{Q_{\lambda_n f} x_n\}$ is spherically bounded;*
(ii) *If $\operatorname{argmin}_X f$ is nonempty and $\sup_n \alpha_n < 1$, then both $\{x_n\}$ and $\{Q_{\lambda_n f} x_n\}$ are Δ-convergent to an element x_0 of $\operatorname{argmin}_X f$.*

In this paper, we propose a new resolvent corresponding Q_f in a complete CAT(-1) space and we show that it is well-defined as a single-valued mapping. Moreover, we study the asymptotic behavior of a sequence generated by Mann iteration.

2 Preliminaries

Let X be a metric space with metric d. We denote by $\mathcal{F}(T)$ the set of all fixed points of a mapping T of X into itself. For $x, y \in X$, a continuous mapping $c\colon [0, l] \to X$ is called a geodesic joining x and y if c satisfies $c(0) = x$, $c(l) = y$ and $d(c(s), c(t)) = |s - t|$ for all $s, t \in [0, 1]$. Its image, which is denoted by $[x, y]$, is called a geodesic segment with endpoints x and y. X is said to be a geodesic space if there exists a geodesic joining any two points in X. In this paper, for a geodesic metric space X, a geodesic joining any two points of X is always assumed to be unique.

Let X be geodesic metric space. For all $x, y \in X$ and $\alpha \in [0, 1]$, there exists a unique point $z \in X$ such that $d(x, z) = (1 - \alpha)d(x, y)$ and $d(z, y) = \alpha d(x, y)$. This point is called a convex combination of x and y, denoted by $\alpha x \oplus (1 - \alpha)y$. A subset $C \subset X$ is said to be convex if $[x, y] \subset X$ for all $x, y \in C$.

Let M_κ^2 be a two dimensional model space for $\kappa \in \mathbb{R}$. For example, $M_0^2 = \mathbb{R}^2$, $M_1^2 = \mathbb{S}^2$ and $M_{-1}^2 = \mathbb{H}^2$. A geodesic triangle with vertices $x, y, z \in X$ is defined by $[x, y] \cup [y, z] \cup [z, x]$, which is denoted by $\triangle(x, y, z)$. A comparison triangle to $\triangle(x, y, z)$ with vertices $\overline{x}, \overline{y}, \overline{z} \in M_\kappa^2$ is defined by $[\overline{x}, \overline{y}] \cup [\overline{y}, \overline{z}] \cup [\overline{z}, \overline{x}]$ with $d(x, y) = d(\overline{x}, \overline{y})$, $d(y, z) = d(\overline{y}, \overline{z})$ and $d(z, x) = d(\overline{z}, \overline{x})$, which is denoted by $\overline{\triangle}(\overline{x}, \overline{y}, \overline{z})$. For $\kappa \in \mathbb{R}$, X is called a CAT(κ) space if $d(p, q) \le d(\overline{p}, \overline{q})$ holds whenever $\overline{p}$ and $\overline{q} \in \overline{\triangle}$ are comparison points for p and $q \in \triangle$, respectively. If $\kappa < \kappa'$ then the CAT(κ) spaces are CAT(κ') spaces. We know that the following lemmas hold.

Lemma 2.1. *Let X be a* CAT(-1) *space, $x_1, x_2, x_3 \in X$ and $\alpha \in [0, 1]$. Then*

$$\cosh d(\alpha x_1 \oplus (1 - \alpha)x_2, x_3) \le \alpha \cosh d(x_1, x_3) + (1 - \alpha)\cosh d(x_2, x_3).$$

Lemma 2.2. *Let X be a* CAT(-1) *space, $x_1, x_2, x_3 \in X$ and $\alpha \in [0, 1]$. Then*

$$\begin{aligned}
&\cosh d(\alpha x_1 \oplus (1 - \alpha)x_2, x_3) \sinh d(x_1, x_2) \\
&\le \cosh d(x_1, x_3) \sinh \alpha d(x_1, x_2) + \cosh d(x_2, x_3) \sinh (1 - \alpha) d(x_1, x_2).
\end{aligned}$$

Lemma 2.3. *Let X, x_1, x_2, x_3, and α be the same as in Lemma 2.2. Then*

$$\cosh d(\frac{1}{2}x_1 \oplus \frac{1}{2}x_2, x_3) \cosh \frac{1}{2}d(x_1, x_2) \le \frac{1}{2}\cosh d(x_1, x_3) + \frac{1}{2}\cosh d(x_2, x_3).$$

Let X be a metric space and $\{x_n\}$ a sequence in X. The asymptotic center $\mathcal{A}(\{x_n\})$ of $\{x_n\}$ is defined by

$$\mathcal{A}(\{x_n\}) = \left\{ z \in X \;\middle|\; \limsup_{n\to\infty} d(z, x_n) = \inf_{y\in X} \limsup_{n\to\infty} d(y, x_n) \right\}.$$

A sequence $\{x_n\}$ is said to be Δ-convergent to $p \in X$ if

$$\mathcal{A}(\{x_{n_i}\}) = \{p\}$$

holds for each subsequence $\{x_{n_i}\}$ of $\{x_n\}$. In this case, $\{x_n\}$ is bounded and its subsequence is also Δ-convergent to p.

Theorem 2.4. *Let X be a complete CAT(0) space and f a proper lower semicontinuous convex function of X into $]-\infty, \infty]$. Suppose that $f(x) \to \infty$ whenever $d(x, p) \to \infty$ for some $p \in X$. Then* $\operatorname{argmin}_X f$ *is nonempty. Further, if*

$$f\left(\frac{1}{2}x \oplus \frac{1}{2}y\right) < \frac{1}{2}f(x) + \frac{1}{2}f(y),$$

holds for all $x, y \in X$ with $x \ne y$, then $\operatorname{argmin}_X f$ *is consists of one point.*

3 Resolvents for convex function in complete CAT(-1) spaces

In this section, we show that a new resolvent

$$R_f x \coloneqq \operatorname*{argmin}_{y\in X}\{f(y) + \log\cosh d(y, x)\}$$

is well-defined.

Lemma 3.1. *Let f be a proper lower semicontinuous convex function from X into $]-\infty, \infty]$ and $p \in X$. If $g\colon X \to]-\infty, \infty]$ is defined by*

$$g(\cdot) = f(\cdot) + \log\cosh d(\cdot, p)$$

then g is a proper lower semicontinuous convex function from X into $]-\infty, \infty]$.

Proof. Let $t > 0$. We have

$$(\log(\cosh t))'' = (\tanh t)' = \frac{1}{\cosh^2 t} > 0.$$

Hence

$$\log\cosh(\alpha d_1 + (1-\alpha)d_2) \le \alpha \log\cosh d_1 + (1-\alpha)\log\cosh d_2$$

for all $d_1, d_2 \geq 0$ and $\alpha \in]0,1[$. It follows that

$$\begin{aligned}\log\cosh d(\alpha d(x,p) \oplus (1-\alpha)d(y,p)) &\leq \log\cosh d(\alpha d(x,p) + (1-\alpha)d(y,p)) \\ &\leq \alpha \log\cosh d(x,p) + (1-\alpha)\log\cosh d(y,p)\end{aligned}$$

for all $x, y \in X$ and $\alpha \in]0,1[$. Thus g is convex. On the other hand, it is obvious that g is proper and lower semicontinuous. $\square$

Lemma 3.2. *Let f be a proper lower semicontinuous convex function from X into $]-\infty, \infty]$ and $p \in X$. Suppose that f is bounded below. If g is defined by*

$$g(\cdot) = f(\cdot) + \log\cosh d(\cdot, p)$$

then $\operatorname{argmin}_X g$ *consists of one point.*

Proof. Let $\{z_n\}$ be a sequence of X with $\lim_{n\to\infty} d(z_n, p) = \infty$ for each $p \in X$. Then, it is obvious that $\lim_{n\to\infty} \log\cosh d(z_n, p) = \infty$. From Lemma 2.4 and Lemma 3.1, $\operatorname{argmin}_X g$ is nonempty.

We next show that $\operatorname{argmin}_X g$ consists of one point. Suppose that $u, v \in \operatorname{argmin}_X g$ with $u \neq v$. Suppose $d(u,p) \neq d(v,p)$. Then,

$$\begin{aligned}g(u) &\leq f\left(\frac{1}{2}u \oplus \frac{1}{2}v\right) + \log\cosh d\left(\frac{1}{2}u \oplus \frac{1}{2}v, p\right) \\ &\leq \frac{1}{2}f(u) + \frac{1}{2}f(v) + \log\cosh\left(\frac{1}{2}d(u,p) + \frac{1}{2}d(v,p)\right) \\ &< \frac{1}{2}f(u) + \frac{1}{2}f(v) + \frac{1}{2}\log\cosh d(u,p) + \frac{1}{2}\log\cosh d(v,p) = g(u).\end{aligned}$$

It is a contradiction. Suppose $d(u,p) = d(v,p)$. Then,

$$\begin{aligned}g(u) &\leq f\left(\frac{1}{2}u \oplus \frac{1}{2}v\right) + \log\cosh d\left(\frac{1}{2}u \oplus \frac{1}{2}v, p\right) \\ &= f\left(\frac{1}{2}u \oplus \frac{1}{2}v\right) + \log\left(\cosh d\left(\frac{1}{2}u \oplus \frac{1}{2}v, p\right)\cosh\frac{d(u,v)}{2}\right) - \log\cosh\frac{d(u,v)}{2}.\end{aligned}$$

From the convexity of f and Lemma 2.3,

$$g(u) \leq \frac{1}{2}f(u) + \frac{1}{2}f(v) + \log\left(\frac{1}{2}\cosh d(u,p) + \frac{1}{2}\cosh d(v,p)\right) - \log\cosh\frac{d(u,v)}{2}.$$

Since

$$\frac{1}{2}\cosh d(u,p) + \frac{1}{2}\cosh d(v,p) = \cosh\frac{d(u,p)+d(v,p)}{2}\cosh\frac{d(u,p)-d(v,p)}{2},$$

we have

$$g(u) \leq \frac{1}{2}f(u) + \frac{1}{2}f(v) + \log\cosh\frac{d(u,p)+d(v,p)}{2} - \log\cosh\frac{d(u,v)}{2}$$

and hence

$$
\begin{aligned}
0 < \log\cosh\frac{d(u,v)}{2} &\le \frac{1}{2}f(u) + \frac{1}{2}f(v) + \log\cosh\frac{d(u,p)+d(v,p)}{2} - g(u) \\
&\le \frac{1}{2}f(u) + \frac{1}{2}f(v) + \frac{1}{2}\log\cosh d(u,p) + \frac{1}{2}\log\cosh d(v,p) - g(u) \\
&= g(u) - g(u) = 0.
\end{aligned}
$$

It is a contradiction. Consequently, $\operatorname{argmin}_X g$ consists of one point. □

Definition 3.3. Let f be a proper lower semicontinuous convex function from X into $]-\infty,\infty]$. Suppose that f is bounded below. Then we define a new resolvent $R_f\colon X \to X$ by

$$
R_f x = \operatorname*{argmin}_{y\in X}\{f(y) + \log\cosh d(y,x)\}
$$

for all $x \in X$. By Lemma 3.2, we know that R_f is well-defined.

4 Fundamental properties of resolvents in CAT(−1) spaces

Lemma 4.1. *Let X be a complete* CAT(−1) *space, f a proper lower semicontinuous convex function of X into $]-\infty,\infty]$. Suppose that f is bounded below. Let $R_{\eta f}$ be the resolvent of ηf for all $\eta > 0$. If $\lambda,\mu > 0$ and $x,y \in X$, then the inequality*

$$
(\lambda+\mu)\cosh d(R_{\lambda f}x, R_{\mu f}y) \le \frac{\mu\cosh d(R_{\mu f}y, x)}{\cosh d(R_{\lambda f}x, x)} + \frac{\lambda\cosh d(R_{\lambda f}x, y)}{\cosh d(R_{\mu f}y, y)}
$$

holds.

Proof. Let $\lambda,\mu > 0$ and $x,y \in X$ be given. Set $D = d(R_{\lambda f}x, R_{\mu f}y)$ and

$$
z_t = tR_{\mu f}y \oplus (1-t)R_{\lambda f}x
$$

for all $t \in]0,1[$. By the definition of $R_{\lambda f}$ and the convexity of f, we have

$$
\begin{aligned}
&\lambda f(R_{\lambda f}x) + \log\cosh d(R_{\lambda f}x, x) \\
&\le \lambda f(z_t) + \log\cosh d(z_t, x) \\
&\le t\lambda f(R_{\mu f}y) + (1-t)\lambda f(R_{\lambda f}x) + \log\cosh d(z_t, x).
\end{aligned}
$$

On the other hand, Lemma 2.2 implies that

$$
\begin{aligned}
&\cosh d(tR_{\mu f}y \oplus (1-t)R_{\lambda f}x, x)\sinh D \\
&\le \cosh d(R_{\mu f}y, x)\sinh tD + \cosh d(R_{\lambda f}x, x)\sinh(1-t)D.
\end{aligned}
$$

If $D > 0$, we have

$$
\begin{aligned}
&t\lambda(f(R_{\lambda f}x) - f(R_{\mu f}y)) \\
&\le \log(\cosh d(tR_{\mu f}y, x)\sinh tD + \cosh d(R_{\lambda f}x, x)\sinh(1-t)D) - \log\sinh D
\end{aligned}
$$

and hence

$$\lambda(f(R_{\lambda f}x) - f(R_{\mu f}y)) \\ \leq \frac{\log(\cosh d(R_{\mu f}y, x) \sinh tD + \cosh d(R_{\lambda f}x, x) \sinh(1-t)D) - \log \sinh D}{t}.$$

By l'Hospital's rule, we have

$$\begin{aligned}
&\lambda(f(R_{\lambda f}x) - f(R_{\mu f}y)) \\
&\leq \lim_{t \to 0} \frac{\log(\cosh d(R_{\mu f}y, x) \sinh tD + \cosh d(R_{\lambda f}x, x) \sinh(1-t)D) - \log \sinh D}{t} \\
&= \lim_{t \to 0} \frac{D(\cosh d(R_{\mu f}y, x) \cosh tD - \cosh d(R_{\lambda f}x, x) \cosh(1-t)D)}{\cosh d(R_{\mu f}y, x) \sinh tD + \cosh d(R_{\lambda f}x, x) \sinh(1-t)D} \\
&= \frac{D}{\sinh D} \left(\frac{\cosh d(R_{\mu f}y, x)}{\cosh d(R_{\mu f}y, y)} - \cosh D \right).
\end{aligned}$$

It implies that

$$\lambda(f(R_{\lambda f}x) - f(R_{\mu f}y)) \leq \frac{D}{\sinh D} \left(\frac{\cosh d(R_{\mu f}y, x)}{\cosh d(R_{\mu f}y, y)} - \cosh D \right) \tag{1}$$

and that

$$\mu(f(R_{\mu f}y) - f(R_{\lambda f}x)) \leq \frac{D}{\sinh D} \left(\frac{\cosh d(R_{\lambda f}x, y)}{\cosh d(R_{\lambda f}x, x)} - \cosh D \right). \tag{2}$$

Multiplying (1) by μ and (2) by λ, and adding them, we obtain

$$(\lambda + \mu) \cosh d(R_{\lambda f}x, R_{\mu f}y) \leq \frac{\mu \cosh d(R_{\mu f}y, x)}{\cosh d(R_{\lambda f}x, x)} + \frac{\lambda \cosh d(R_{\lambda f}x, y)}{\cosh d(R_{\mu f}y, y)}.$$

This is the desired result. □

Corollary 4.2. *Suppose that X and f are the same as the previous lemma. Then $\mathcal{F}(R_f) = \operatorname{argmin}_X f$.*

Proof. Let $u \in \operatorname{argmin}_X f$ and $y \in X$. By the definition of R_f, we have

$$f(u) + \log(\cosh d(u, u)) = f(u) \leq f(y) \leq f(y) + \log(\cosh d(y, u)).$$

Thus $u \in \mathcal{F}(R_f)$. Let $u \in \mathcal{F}(R_f)$ and $y \in X$. By Lemma 4.1, we have

$$f(R_f u) - f(y) \leq \frac{d(R_f u, y)}{\sinh d(R_f u, y)} \left(\frac{\cosh d(u, y)}{\cosh d(R_f u, u)} - \cosh d(R_f u, y) \right)$$

and hence

$$f(u) - f(y) \leq \frac{d(u, y)}{\sinh d(u, y)} \left(\frac{\cosh d(u, y)}{\cosh d(u, u)} - \cosh d(u, y) \right) = 0.$$

It follows that $f(u) \leq f(y)$. It implies $u \in \operatorname{argmin}_X f$. □

Corollary 4.3. *Suppose that X and f are the same as the previous lemma. Then*

$$\cosh d(y, R_{\lambda f}x) \cosh d(R_{\lambda f}x, x) \leq \cosh d(y, x)$$

for each $y \in \operatorname{argmin}_X f$.

Corollary 4.4. *Suppose that X and f are the same as the previous lemma. If $\operatorname{argmin}_X f$ is nonempty, then $R_{\lambda f}$ is quasinonexpansive.*

5 Δ-convergent proximal-type algorithm

Theorem 5.1. *Let X be a complete* $\mathrm{CAT}(-1)$ *space, f a proper lower semicontinuous convex function of X into $]-\infty, \infty]$. Suppose that f is bounded below. Let $R_{\eta f}$ the resolvent of ηf for all $\eta > 0$ and $\{x_n\}$ a sequence defined by $x_1 \in X$ and*

$$x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n) R_{\lambda_n f} x_n,$$

where $\{\alpha_n\}$ is a sequence in $[0, 1[$ and $\{\lambda_n\}$ is a sequence of positive real numbers such that $\sum_{n=1}^{\infty}(1 - \alpha_n)\lambda_n = \infty$. If $\operatorname{argmin}_X f$ is nonempty and $\sup_n \alpha_n < 1$, then both $\{x_n\}$ and $\{R_{\lambda_n f} x_n\}$ are Δ-convergent to an element x_0 of $\operatorname{argmin}_X f$.

Proof. Suppose that $\operatorname{argmin}_X f$ is nonempty and $\sup_n \alpha_n < 1$. Let $u \in \operatorname{argmin}_X f$ be given. By Lemma 2.1 and Lemma 4.4, we have

$$\cosh d(u, x_{n+1}) \leq \alpha_n \cosh d(u, x_n) + (1 - \alpha_n) \cosh d(u, R_{\lambda_n f} x_n) \leq \cosh d(u, x_n).$$

and hence

$$d(u, x_{n+1}) \leq d(u, x_n).$$

Thus, $\{d(u, x_n)\}$ converges to some $\beta \in [0, \infty[$. By Lemma 2.1 and Lemma 4.3, we have

$$\begin{aligned}
\cosh d(u, x_{n+1}) &\leq \alpha_n \cosh d(u, x_n) + (1 - \alpha_n) \cosh d(u, R_{\lambda_n f} x_n) \\
&\leq \alpha_n \cosh d(u, x_n) + (1 - \alpha_n) \cdot \frac{\cosh d(u, x_n)}{\cosh d(x_n, R_{\lambda_n f} x_n)} \\
&\leq \cosh d(u, x_n) + (1 - \alpha_n) \cosh d(u, x_n) \left(\frac{1}{\cosh d(x_n, R_{\lambda_n f} x_n)} - 1 \right)
\end{aligned}$$

and hence

$$\begin{aligned}
0 &\geq (1 - \alpha_n) \cosh d(u, x_n) \left(\frac{1}{\cosh d(x_n, R_{\lambda_n f} x_n)} - 1 \right) \\
&\geq \frac{\cosh d(u, x_{n+1})}{\cosh d(u, x_n)} - 1 \to \frac{\cosh \beta}{\cosh \beta} - 1 = 0.
\end{aligned}$$

as $n \to \infty$. Since $\sup_n \alpha_n < 1$, we have

$$\lim_{n \to \infty} d(x_n, R_{\lambda_n f} x_n) = 0.$$

On the other hand, it follows from Lemma 4.1 that

$$\lambda_n(f(R_{\lambda_n f}x_n) - f(u)) \leq \cosh d(u, x_n) - \cosh d(u, R_{\lambda_n f}x_n)$$

for all $n \in \mathbb{N}$. It then follows from Lemma 2.1 that

$$(1 - \alpha_n)\lambda_n(f(R_{\lambda_n f}x_n) - f(u)) \leq \cosh d(u, x_n) - \cosh d(u, x_{n+1})$$

and hence

$$\sum_{n=1}^{\infty}(1 - \alpha_n)\lambda_n(f(R_{\lambda_n f}x_n) - f(u)) \leq \cosh d(u, x_1) - \cosh\beta < \infty.$$

Since $\sum_{n=1}^{\infty}(1-\alpha_n)\lambda = \infty$, it follows that

$$\liminf_{n\to\infty} f(R_{\lambda_n f}x_n) - f(u) = 0.$$

By the definition of $\{x_n\}$ and $\{R_{\lambda_n f}x_n\}$ and the convexity of f, we also have

$$-\infty < \inf f(X) \leq f(R_{\lambda_n f}x_n) \leq f(R_{\lambda_n f}x_n) + \log\cosh d(R_{\lambda_n f}x_n, x_n) \leq f(x_n)$$

and

$$-\infty < \inf f(X) \leq f(x_{n+1}) \leq \alpha_n f(x_n) + (1-\alpha_n)f(R_{\lambda_n f}x_n) \leq f(x_n)$$

for all $n \in \mathbb{N}$. Thus $\{f(x_n)\}$ converges to $\gamma \in \mathbb{R}$ and $\{f(R_{\lambda_n f}x_n)\}$ is bounded. Let $\{n_i\}$ be any increaseing sequence in $\mathbb{N}$. Since $\sup_n \alpha_n < 1$, we have a subsequence $\{n_{i_j}\}$ of $\{n_i\}$ such that $\{\alpha_{n_{i_j}}\}$ converges to some $\delta \in [0,1[$. Then letting $j \to \infty$ in

$$\frac{1}{1-\alpha_{n_{i_j}}}\left(f(x_{n_{i_j}+1}) - \alpha_{n_{i_j}}f(x_{n_{i_j}})\right) \leq f(R_{\lambda_{n_{i_j}}f}x_{n_{i_j}}) \leq f(x_{n_{i_j}}),$$

Thus $\{f(R_{\lambda_{n_{i_j}}f}x_{n_{i_j}})\}$ also converges to γ. Consequently, it follows from

$$\lim_{n\to\infty}(f(R_{\lambda_n f}x_n) - f(u)) = 0$$

that

$$\lim_{n\to\infty} f(x_n) = \gamma = f(u) = \inf f(X).$$

Let $\{x_{n_i}\}$ be an arbitrary subsequence of $\{x_n\}$. Let

$$\{x_0\} = \mathcal{A}(\{x_n\}) \text{ and } \{z\} = \mathcal{A}(\{x_{n_i}\}).$$

There exists $\{x_{n_{i_j}}\} \subset \{x_{n_i}\}$ and $w \in X$ such that $x_{n_i} \overset{\Delta}{\to} w$. Since f is Δ-lower semicontinuous,

$$f(w) \leq \liminf_{j\to\infty} f(x_{n_{i_j}}) = f(u).$$

Thus $w \in \operatorname{argmin}_X f$. we also have

$$\begin{aligned}
\limsup_{n\to\infty} d(w, x_n) &= \limsup_{i\to\infty} d(w, x_{n_i}) \\
&= \limsup_{j\to\infty} d(w, x_{n_{i_j}}) \\
&\leq \limsup_{j\to\infty} d(z, x_{n_{i_j}}) \\
&\leq \limsup_{i\to\infty} d(z, x_{n_i}) \\
&\leq \limsup_{i\to\infty} d(x_0, x_{n_i}) \\
&\leq \limsup_{n\to\infty} d(x_0, x_n) \leq \limsup_{n\to\infty} d(w, x_n).
\end{aligned}$$

hence $z = x_0 = w \in \operatorname{argmin}_X f$. Thus $\{x_n\}$ Δ-converges to $x_0 \in \operatorname{argmin}_X f$. On the other hand, Let $\{q\} = \mathcal{A}(\{R_{\lambda_{n_i} f} x_{n_i}\})$ such that any $\{R_{\lambda_{n_i} f} x_{n_i}\} \subset \{R_{\lambda_n f} x_n\}$. It follows that

$$\begin{aligned}
\limsup_{i\to\infty} d(R_{\lambda_{n_i} f} x_{n_i}, q) &\leq \limsup_{i\to\infty} d(R_{\lambda_{n_i} f} x_{n_i}, x_0) \\
&\leq \limsup_{i\to\infty} d(R_{\lambda_{n_i} f} x_{n_i}, x_{n_i}) + \limsup_{i\to\infty} d(x_{n_i}, x_0) \\
&\leq \limsup_{i\to\infty} d(x_{n_i}, x_0) \\
&\leq \limsup_{n\to\infty} d(x_n, x_0) \\
&\leq \limsup_{n\to\infty} d(x_n, q) \\
&\leq \limsup_{n\to\infty} d(x_n, R_{\lambda_n f} x_n) + \limsup_{n\to\infty} d(R_{\lambda_n f} x_n, q) \\
&\leq \limsup_{n\to\infty} d(R_{\lambda_n f} x_n, q) \\
&= \limsup_{i\to\infty} d(R_{\lambda_{n_i} f} x_{n_i}, q)
\end{aligned}$$

Consequently, we conclude that both $\{x_n\}$ and $\{R_{\lambda_n f} x_n\}$ are Δ-convergent to an element x_0 of $\operatorname{argmin}_X f$. □

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