# Study on some special convex functions
# ある種の特殊な凸関数について

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Abstract

In this paper, we consider a special type of convex combination and convex function on uniquely geodesic spaces.

## 1 Introduction

A uniquely geodesic space is a metric space such that there exists a unique geodesic $\gamma_{x,y}$ joining each two points $x$ and $y$. In the uniquely geodesic space $(X,d)$, we can consider the convex combination $tx \oplus (1-t)y$ as an internal dividing point of the image of $\gamma_{x,y}$ with the ratio $(1-t):t$ by the metric $d$.

We call a uniquely geodesic space which has a curvature bounded above by $\kappa \in \mathbb{R}$ a $\mathrm{CAT}(\kappa)$ space. In 2020, Kimura and the author proposed another type of convex combination $\overset{-1}{\oplus}$. This was defined in [2] to show a fixed point approximation theorem for Halpern type iteration with multiple anchor points on a complete $\mathrm{CAT}(-1)$ space as follows:

**Theorem 1.1** (Kimura and Sasaki [2])**.** *Let $X$ be a complete $\mathrm{CAT}(-1)$ space and $R, S, T\colon X \to X$ strongly quasinonexpansive and $\Delta$-demiclosed mappings with $F = \mathrm{Fix}(R) \cap \mathrm{Fix}(S) \cap \mathrm{Fix}(T) \neq \emptyset$. Let $\{\alpha_n\} \subset\ ]0,1[$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Let $\{\beta_n\}, \{\gamma_n\} \subset\ ]0,1[$ such that $\lim_{n\to\infty} \beta_n = \beta \in\ ]0,1[$, $\lim_{n\to\infty} \gamma_n = \gamma \in\ ]0,1[$. Let $u, v, w, x_1 \in X$ and define a sequence $\{x_n\}$ on $X$ by*

$$\begin{cases} r_n = \alpha_n u \overset{-1}{\oplus} (1-\alpha_n) R x_n, \\ s_n = \alpha_n v \overset{-1}{\oplus} (1-\alpha_n) S x_n, \\ t_n = \alpha_n w \overset{-1}{\oplus} (1-\alpha_n) T x_n, \\ x_{n+1} = \beta_n r_n \overset{-1}{\oplus} (1-\beta_n)(\gamma_n s_n \overset{-1}{\oplus} (1-\gamma_n) t_n) \end{cases}$$

*for every $n \in \mathbb{N}$. Then $\{x_n\}$ converges to a point $p = \mathrm{argmin}_{x\in F}\big(\beta \cosh d(u,x) + (1-\beta)(\gamma \cosh d(v,x) + (1-\gamma)\cosh d(w,x))\big)$.*

In the theorem above, if we use a usual convex combination $\oplus$ instead of $\overset{-1}{\oplus}$, a sequence $\{x_n\}$ does not converges to the point $p$ in general.

Incidentally, in 2021, Kimura and the author [3] defined another type of convex combination $\overset{1}{\oplus}$. Later, Kimura and the author [4] obtained a nature which implies that $\overset{1}{\oplus}$ has better properties than $\oplus$ on the unit sphere in a Hilbert space.

In this paper, we consider a behavior of $\overset{-1}{\oplus}$ on the hyperbolic plane. Moreover, we consider another type of convex functions using $\overset{-1}{\oplus}$ on uniquely geodesic spaces.

# 2 Preliminaries

Let $X$ be a nonempty set and $f$ a function from $X$ into $]-\infty, \infty]$. Define an effective domain of $f$ by $\mathrm{dom}(f) = \{z \in X \mid f(z) \neq \infty\}$. A function $f$ is said to be *proper* if $\mathrm{dom}(f)$ is nonempty. In this paper, $\mathrm{argmin}_{z \in X} f(z)$ denotes the unique minimizer of $f$, if it exists. We write the restriction of $f$ to $A \subset X$ as $f|_A \colon A \to ]-\infty, \infty]$.

Let $(X, d)$ be a metric space. $X$ is called a *uniquely geodesic space* if there exists a unique geodesic $\gamma_{u,v} \colon [0, 1] \to X$ joining each two points $u, v \in X$, where $\gamma_{u,v}$ is called a *geodesic* joining $x$ and $y$ if $\gamma_{u,v}(0) = v$, $\gamma_{u,v}(1) = u$, and $d(\gamma_{u,v}(s), \gamma_{u,v}(t)) = |s - t| d(u, v)$ for any $s, t \in [0, 1]$.

Let $(X, d)$ be a uniquely geodesic space and $u, v \in X$. Then a point $z = \gamma_{u,v}(t)$ is called a *convex combination* of $u$ and $v$ with a ratio $t \in [0, 1]$, and write it by $z = tu \oplus (1 - t)v$. We know that the convex combination $tx \oplus (1 - t)y$ satisfies

$$tu \oplus (1 - t)v = \underset{z \in X}{\mathrm{argmin}} \left( t d(u, z)^2 + (1 - t) d(v, z)^2 \right).$$

A uniquely geodesic space $(X, d)$ is called a $\mathrm{CAT}(-1)$ space if an inequality $\cosh d(tx \oplus (1 - t)y, z) \sinh d(x, y) \leq \cosh d(x, z) \sinh(t d(x, y)) + \cosh d(y, z) \sinh((1 - t) d(x, y))$ holds for every $x, y, z \in X$ and $t \in [0, 1]$. This is not an original definition of a $\mathrm{CAT}(-1)$ space, but this is an equivalent definition of such a space.

**Lemma 2.1** (Mayer [5]). *Let $X$ be a complete* $\mathrm{CAT}(0)$ *space and $f$ a lower semicontinuous convex function from $X$ into $\mathbb{R}$. Then there exists $L \in ]-\infty, 0]$ such that for any $v \in X$,*

$$\liminf_{d(v,z) \to \infty} \frac{f(z)}{d(v, z)} \geq L.$$

Note that every $\mathrm{CAT}(-1)$ space is a $\mathrm{CAT}(0)$ space; see for example, [1].

# 3 $(-1)$-convex combination

## 3.1 $(-1)$-convex combination in geodesic spaces

In 2020, Kimura and the author defined the $(-1)$-convex combination as follows.

**Definition 3.1** ([2])**.** Let $X$ be a uniquely geodesic space. Then for any two points $u, v \in X$ and $t \in [0,1]$, put

$$tu \overset{-1}{\oplus} (1-t)v = \operatorname*{argmin}_{z \in X} (t \cosh d(u,z) + (1-t)\cosh d(v,z)).$$

We call this point a $(-1)$-*convex combination* of $u$ and $v$.

Then we obtain that $tu \overset{-1}{\oplus} (1-t)v \in \gamma_{u,v}([0,1])$. Therefore, $\overset{-1}{\oplus}$ can be expressed by a usual convex combination $\oplus$ by changing the coefficients of $u$ and $v$ as follows.

**Theorem 3.2** ([2])**.** *Let $X$ be a uniquely geodesic space and $t, \tau \in [0,1]$. Take $u, v \in X$ and suppose that $u \neq v$. Then the following are equivalent:*

(a) $tx \overset{-1}{\oplus} (1-t)y = \tau x \oplus (1-\tau)y$;

(b) $\tau = \dfrac{1}{d(u,v)} \tanh^{-1} \dfrac{t \sinh d(u,v)}{1-t+t\cosh d(u,v)}$;

(c) $1-\tau = \dfrac{1}{d(u,v)} \tanh^{-1} \dfrac{(1-t)\sinh d(u,v)}{t+(1-t)\cosh d(u,v)}$;

(d) $t = \dfrac{\sinh(\tau d(u,v))}{\sinh(\tau d(u,v)) + \sinh((1-\tau)d(u,v))}$.

**Corollary 3.3.** *Let $X$ be a uniquely geodesic space. Then for any $u, v \in X$,*

$$\frac{1}{2}u \oplus \frac{1}{2}v = \frac{1}{2}u \overset{-1}{\oplus} \frac{1}{2}v.$$

In CAT$(-1)$ spaces, the following two inequalities hold.

**Theorem 3.4.** *Let $X$ be a* CAT$(-1)$ *space. Then for any $x, y, z \in X$ such that $x \neq y$ and $t \in [0,1]$,*

$$\cosh d(tx \oplus (1-t)y, z) \le \frac{\sinh(td(x,y))\cosh d(x,z) + \sinh((1-t)d(x,y))\cosh d(y,z)}{\sinh d(x,y)}.$$

**Theorem 3.5** ([2])**.** *Let $X$ be a* CAT$(-1)$ *space. Then for any $x, y, z \in X$ and $t \in [0,1]$,*

$$\cosh d(tx \overset{-1}{\oplus} (1-t)y, z) \le \frac{t\cosh d(x,z) + (1-t)\cosh d(y,z)}{\sqrt{t^2 + 2t(1-t)\cosh d(x,y) + (1-t)^2}}.$$

Comparing two inequalities above, we can find that $\overset{-1}{\oplus}$ has the useful property that the ratio of the coefficients of the two terms on the right-hand side is exactly $t : (1-t)$.

## 3.2 $(-1)$-convex combination in Hyperbolic spaces

Let $H = \{(z,x,y) \in \mathbb{R}^3 \mid z^2 - x^2 - y^2 = 1,\ z > 0\}$ be an upper half hyperboloid. Define a function $d \colon H \times H \to [0, \infty[$ by $d(\boldsymbol{u}, \boldsymbol{v}) = \cosh^{-1}(u_z v_z - u_x v_x - u_y v_y)$ for

$\boldsymbol{u} = (u_z, u_x, u_y) \in H$ and $\boldsymbol{v} = (v_z, v_x, v_y) \in H$. Then $(H, d)$ is a metric space. Moreover, for any two points $\boldsymbol{u}, \boldsymbol{v} \in H$, a mapping $\gamma_{\boldsymbol{u},\boldsymbol{v}} \colon [0,1] \to H$ defined by

$$\gamma_{\boldsymbol{u},\boldsymbol{v}}(t) = \begin{cases} \dfrac{\sinh(td(\boldsymbol{u},\boldsymbol{v}))}{\sinh d(\boldsymbol{u},\boldsymbol{v})}\boldsymbol{u} + \dfrac{\sinh((1-t)d(\boldsymbol{u},\boldsymbol{v}))}{\sinh d(\boldsymbol{u},\boldsymbol{v})}\boldsymbol{v} & (\text{if } \boldsymbol{u} \neq \boldsymbol{v}) \\ \boldsymbol{u} & (\text{if } \boldsymbol{u} = \boldsymbol{v}) \end{cases}$$

for each $t \in [0,1]$ is a unique geodesic joining $\boldsymbol{u}$ and $\boldsymbol{v}$, where a symbol $+$ stands for a usual addition on the Euclidean space $\mathbb{R}^3$. Hence $(H, d)$ is a uniquely geodesic space. Then a convex combination $t\boldsymbol{u} \oplus (1-t)\boldsymbol{v}$ coincides with $\gamma_{\boldsymbol{u},\boldsymbol{v}}(t)$. We know that $(H, d)$ is one of the model of the hyperbolic space named a hyperboloid model. In what follows, $(H, d)$ always represents such a model.

Define $B \colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ and $Q \colon \mathbb{R}^3 \to \mathbb{R}$ by $B(\boldsymbol{p}, \boldsymbol{q}) = p_z q_z - p_x q_x - p_y q_y$ and $Q(\boldsymbol{p}) = B(\boldsymbol{p}, \boldsymbol{p}) = p_z^2 - p_x^2 - p_y^2$ for any $\boldsymbol{p} = (p_z, p_x, p_y)$, $\boldsymbol{q} = (q_z, q_x, q_y) \in \mathbb{R}^3$. Then we have $H = \{\boldsymbol{p} \in \mathbb{R}^3 \mid Q(\boldsymbol{p}) = 1\}$, and $d(\boldsymbol{u}, \boldsymbol{v}) = \cosh^{-1} B(\boldsymbol{u}, \boldsymbol{v})$ for any $\boldsymbol{u}, \boldsymbol{v} \in H$. The following are immediately obtained for any $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^3$ and $s, t \in \mathbb{R}$:

- $B(\boldsymbol{u}, \boldsymbol{v}) = B(\boldsymbol{v}, \boldsymbol{u})$;
- $B(s\boldsymbol{u} + t\boldsymbol{v}, \boldsymbol{w}) = sB(\boldsymbol{u}, \boldsymbol{w}) + tB(\boldsymbol{v}, \boldsymbol{w})$;
- $Q(s\boldsymbol{u} + t\boldsymbol{v}) = s^2 Q(\boldsymbol{u}) + 2stB(\boldsymbol{u}, \boldsymbol{v}) + t^2 Q(\boldsymbol{v})$.

**Lemma 3.6.** *Take $\boldsymbol{p} = (p_z, p_x, p_y)$, $\boldsymbol{q} = (q_z, q_x, q_y) \in \mathbb{R}^3$ and suppose that $Q(\boldsymbol{p}) \geq 0$, $Q(\boldsymbol{q}) \geq 0$, $p_z > 0$, and $q_z > 0$. Then $B(\boldsymbol{p}, \boldsymbol{q}) \geq \sqrt{Q(\boldsymbol{p})}\sqrt{Q(\boldsymbol{q})}$ holds.*

*Proof.* Put $w_1 = \sqrt{p_x^2 + p_y^2}$ and $w_2 = \sqrt{q_x^2 + q_y^2}$. Then we get $p_x p_y + q_x q_y \leq w_1 w_2 \leq p_z q_z$. It follows that

$$\begin{aligned} B(\boldsymbol{p}, \boldsymbol{q}) = p_z q_z - p_x q_x - p_y q_y &\geq p_z q_z - w_1 w_2 \\ &= \sqrt{\left(p_z^2 - w_1^2\right)\left(q_z^2 - w_2^2\right) + \left(w_1 q_z - w_2 p_z\right)^2} \\ &\geq \sqrt{\left(p_z^2 - w_1^2\right)\left(q_z^2 - w_2^2\right)} = \sqrt{Q(\boldsymbol{p})}\sqrt{Q(\boldsymbol{q})}. \end{aligned}$$

It is the desired result. □

**Lemma 3.7.** *For any $\boldsymbol{u}, \boldsymbol{v} \in H$ and $t \in [0,1]$, inequalities $B(\boldsymbol{u}, \boldsymbol{v}) \geq 1$ and $Q(t\boldsymbol{u} + (1-t)\boldsymbol{v}) \geq 1$ hold.*

*Proof.* We obtain $B(\boldsymbol{u}, \boldsymbol{v}) \geq \sqrt{Q(\boldsymbol{u})}\sqrt{Q(\boldsymbol{v})} = 1$ by Lemma 3.6. Moreover, since $Q(\boldsymbol{u}) = Q(\boldsymbol{v}) = 1$ and $B(\boldsymbol{u}, \boldsymbol{v}) \geq 1$, we have $Q(t\boldsymbol{u} + (1-t)\boldsymbol{v}) = t^2 + 2t(1-t)B(\boldsymbol{u}, \boldsymbol{v}) + (1-t)^2 \geq 1$, which is the desired result. □

**Theorem 3.8.** *For any $\boldsymbol{u}, \boldsymbol{v} \in H$ and $t \in [0,1]$,*

$$t\boldsymbol{u} \overset{-1}{\oplus} (1-t)\boldsymbol{v} = \frac{t\boldsymbol{u} + (1-t)\boldsymbol{v}}{\sqrt{Q(t\boldsymbol{u} + (1-t)\boldsymbol{v})}}.$$

*Proof.* By the definition of $(-1)$-convex combination, we get $t\boldsymbol{u} \overset{-1}{\oplus} (1-t)\boldsymbol{v} = \operatorname{argmin}_{\boldsymbol{z} \in H} \left(t \cosh d(\boldsymbol{u}, \boldsymbol{z}) + (1-t) \cosh d(\boldsymbol{v}, \boldsymbol{z})\right) = \operatorname{argmin}_{\boldsymbol{z} \in H} B(t\boldsymbol{u} + (1-t)\boldsymbol{v}, \boldsymbol{z})$.

Set $\boldsymbol{p} = t\boldsymbol{u} + (1-t)\boldsymbol{v}$ and $\boldsymbol{w} = \boldsymbol{p}/\sqrt{Q(\boldsymbol{p})}$. Then we get $Q(\boldsymbol{w}) = 1$, and hence $\boldsymbol{w} \in H$. Furthermore, for any $\boldsymbol{z} \in H$, we have

$$B(t\boldsymbol{u} + (1-t)\boldsymbol{v}, \boldsymbol{z}) = B(\boldsymbol{p}, \boldsymbol{z}) \geq \sqrt{Q(\boldsymbol{p})}\sqrt{Q(\boldsymbol{z})} = \sqrt{Q(\boldsymbol{p})} = B(t\boldsymbol{u} + (1-t)\boldsymbol{v}, \boldsymbol{w})$$

by Lemma 3.7. Consequently, we obtain $t\boldsymbol{u} \overset{-1}{\oplus} (1-t)\boldsymbol{v} = \boldsymbol{w}$. □

# 4 New type of convex functions

Using the $(-1)$-convex combination, we can consider another type of convex function on uniquely geodesic spaces as follows.

**Definition 4.1.** Let $X$ be a uniquely geodesic space and $f\colon X \to ]-\infty, \infty]$. We call $f$ a $(-1)$-*convex* function if an inequality

$$f(tx \overset{-1}{\oplus} (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for any $x, y \in X$ and $t \in ]0, 1[$.

Then we get the following proposition hold:

- If $f, g\colon X \to ]-\infty, \infty]$ is $(-1)$-convex, then so is $f + g$.
- If $f\colon X \to ]-\infty, \infty]$ is $(-1)$-convex, then so is $kf$ for any $k \geq 0$.
- If $f\colon \mathbb{R} \to ]-\infty, \infty]$ is $(-1)$-convex, then so is $g\colon \mathbb{R} \ni t \longmapsto f(-t)$.
- If $f\colon \mathbb{R} \to ]-\infty, \infty]$ is $(-1)$-convex, then so is $g\colon \mathbb{R} \ni t \longmapsto f(t+c)$ for any $c \in \mathbb{R}$.

The following are examples of $(-1)$-convex functions and non-$(-1)$-convex functions.

- Let $X$ be a CAT$(-1)$ space and $z \in X$. Then a function $f\colon X \to ]-\infty, \infty]$ defined by $f(\cdot) = \cosh d(\cdot, z)$ is $(-1)$-convex.
- A function $f\colon \mathbb{R} \ni t \longmapsto \cosh t$ is $(-1)$-convex.
- A function $f\colon \mathbb{R} \ni t \longmapsto \exp t$ is $(-1)$-convex.
- A function $f\colon ]-\infty, 0] \ni t \longmapsto \tanh(t/2)$ is $(-1)$-convex.
- For $b \in \mathbb{R}$, a function $f\colon \mathbb{R} \ni t \longmapsto b$ is $(-1)$-convex.
- For $a, b \in \mathbb{R}$ such that $a \neq 0$, a function $f\colon \mathbb{R} \ni t \longmapsto ax + b$ is not $(-1)$-convex.
- A function $f\colon \mathbb{R} \ni t \longmapsto x^2$ is not $(-1)$-convex.

**Theorem 4.2.** *Let $X$ be a uniquely geodesic space. Then a function $f\colon X \to ]-\infty, \infty]$ is $(-1)$-convex if and only if for any $x, y \in X$ with $D = d(x, y) > 0$ and $t \in ]0, 1[$,*

$$f(tx \oplus (1-t)y) \leq \frac{\sinh(tD)}{\sinh(tD) + \sinh((1-t)D)} f(x) + \frac{\sinh((1-t)D)}{\sinh(tD) + \sinh((1-t)D)} f(y).$$

*Proof.* If $x = y$, then $f(tx \overset{-1}{\oplus} (1-t)y) = tf(x) + (1-t)f(y)$ holds. Therefore, we obtain that $f$ is $(-1)$-convex if and only if $f(tx \overset{-1}{\oplus} (1-t)y) \leq tf(x) + (1-t)f(y)$ for any $x, y \in X$ with $x \neq y$ and $t \in ]0, 1[$. Therefore we get the conclusion by Theorem 3.2. □

We know that $(-1)$-convex functions have the following conditions.

**Theorem 4.3.** *Let $X$ be a uniquely geodesic space. Then every continuous $(-1)$-convex function $f\colon X \to ]-\infty, \infty]$ is convex.*

*Proof.* Assume that $f$ is continuous and $(-1)$-convex. Then we obtain from Corollary 3.3 that

$$f\Big(\frac{1}{2}x \oplus \frac{1}{2}y\Big) = f\Big(\frac{1}{2}x \overset{-1}{\oplus} \frac{1}{2}y\Big) \le \frac{1}{2}f(x) + \frac{1}{2}f(y)$$

for any $x, y \in X$. Therefore, since $f$ is continuous, we get the desired result. □

**Theorem 4.4.** *Let $X$ be a uniquely geodesic space and $f\colon X \to ]-\infty, \infty]$ a proper $(-1)$-convex function. Let $u, v \in \operatorname{dom}(f)$. Then $f|_{]u,v[}$ is continuous, where $]u, v[ = \{tu \oplus (1-t)v \mid t \in ]0, 1[\}$.*

**Theorem 4.5.** *Let $X$ be a uniquely geodesic space and $f\colon X \to ]-\infty, \infty]$ a proper $(-1)$-convex function. Then $f|_{\operatorname{dom}(f)}$ is convex.*

**Corollary 4.6.** *Every $(-1)$-convex function $f\colon \mathbb{R} \to \mathbb{R}$ is convex.*

**Theorem 4.7.** *Let $X$ be a uniquely geodesic space and $f\colon X \to ]-\infty, \infty]$ a proper $(-1)$-convex function. Then for any $u, v \in \operatorname{dom}(f)$ such that $f(u) \neq f(v)$ and $t \in ]0, 1[$, an inequality $f(tu \oplus (1-t)v) < tf(u) + (1-t)f(y)$ holds.*

The theorem above implies that, if a proper $(-1)$-convex function $f\colon X \to ]-\infty, \infty]$ satisfies

$$f\Big(\frac{1}{2}u \oplus \frac{1}{2}v\Big) = \frac{1}{2}f(u) + \frac{1}{2}f(v).$$

for any $u, v \in \operatorname{dom}(f)$, then $f|_{\operatorname{dom}(f)}$ is a constant function.

**Theorem 4.8.** *Let $X$ be a uniquely geodesic space and $f\colon X \to ]-\infty, \infty]$ a proper $(-1)$-convex function. For fixed $u, v \in \operatorname{dom}(f)$ such that $u \neq v$, suppose that $f(x) \neq f(y)$ for every $x, y \in [u, v]$. Then,*

$$f\Big(\frac{1}{2}u \oplus \frac{1}{2}v\Big) < \frac{1}{2}f(u) + \frac{1}{2}f(v).$$

In Theorem 4.8, if there exist $x, y \in [u, v]$ satisfying $f(x) = f(y)$, then $f$ does not always hold the conclusion of Theorem 4.8. For instance, a function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} 0 & (\text{if } t \le 0); \\ \cosh t - 1 & (\text{if } t \ge 0) \end{cases}$$

for $t \in \mathbb{R}$ is $(-1)$-convex, and also $f\big(\frac{1}{2}x + \frac{1}{2}y\big) = \frac{1}{2}f(x) + \frac{1}{2}f(y)$ for $x = -1$ and $y = 0$.

# 5 Natures of $(-1)$-convex functions on the real line

In this section, we consider $(-1)$-convex functions on $\mathbb{R}$. From Corollary 4.6, every $(-1)$-convex function on $\mathbb{R}$ is convex.

Let $I$ be a (bounded or unbounded) closed interval on $\mathbb{R}$, and let $f\colon I \to \mathbb{R}$. For arbitrary chosen $x, y \in I$ such that $x < y$, define a function $g_{x,y}\colon [x,y] \to \mathbb{R}$ by

$$\begin{aligned} g_{x,y}(z) &= \frac{\sinh(y-z)}{\sinh(y-z)+\sinh(z-x)} f(x) + \frac{\sinh(z-x)}{\sinh(y-z)+\sinh(z-x)} f(y) \\ &= \frac{f(x)+f(y)}{2} - \frac{\tanh\left(z - \frac{y+x}{2}\right)}{\tanh \frac{y-x}{2}} \cdot \frac{f(x)-f(y)}{2} \end{aligned}$$

for every $z \in [x,y]$. Then we clearly obtain that $g_{x,y}(x) = f(x)$ and $g_{x,y}(y) = f(y)$.

By Theorem 4.2, $f$ is $(-1)$-convex if and only if for any $x, y, z \in I$ such that $x < z < y$, an inequality $f(z) \leq g_{x,y}(z)$ holds. That is, $f$ is $(-1)$-convex if and only if

$$f(z) \leq \inf_{\substack{x<z<y \\ x,y\in I}} g_{x,y}(z) = \inf_{\substack{x\leq z\leq y \\ x<y \\ x,y\in I}} g_{x,y}(z)$$

for every $z \in I$. In other words, we can explain that the $(-1)$-convexity of $f$ means that $\operatorname{epi} f \supset \operatorname{epi} g_{x,y}$ for every $x, y \in I$ such that $x < y$.

**Theorem 5.1.** *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a $(-1)$-convex function. Then for any $v \in \mathbb{R}$,*

$$\liminf_{|u-v|\to\infty} \frac{f(u)}{|u-v|} \geq 0.$$

*Proof.* From Corollary 4.6, $f$ is continuous and convex, Thus, from Lemma 2.1, there exists $L \in \,]-\infty, 0]$ such that $\liminf_{|u-v|\to\infty} f(u)/|u-v| \geq L$ for any $v \in \mathbb{R}$. Suppose $L < 0$ and assume that there exists $v \in \mathbb{R}$ such that the inequality above holds as an equation. Then

$$\lim_{u\to\infty} \frac{f(u)}{u-v} = L \quad \text{or} \quad \lim_{u\to-\infty} \frac{f(u)}{-(u-v)} = L$$

holds. Without loss of generality, we may assume the first equation holds, which implies

$$\lim_{u\to\infty} \frac{f(u)}{u} = \lim_{u\to\infty} \left( \frac{f(u)}{u-v} \cdot \frac{u-v}{u} \right) = L.$$

Take a real number $\varepsilon$ such that $0 < \varepsilon < -L/7$. Then there exists $u_0 > 0$ such that for any $u \geq u_0$, an inequality $(L-\varepsilon)u < f(u) < (L+\varepsilon)u$ holds. Thus, for any $\lambda > 0$, we get

$$\begin{aligned} (L-\varepsilon)(u_0+3\lambda) < f(u_0+3\lambda) &\leq \frac{(\sinh\lambda)f(u_0) + (\sinh 3\lambda)f(u_0+4\lambda)}{\sinh\lambda + \sinh 3\lambda} \\ &\leq (L+\varepsilon) \cdot \frac{(\sinh\lambda)u_0 + (\sinh 3\lambda)(u_0+4\lambda)}{\sinh\lambda+\sinh 3\lambda} \\ &\leq (L+\varepsilon)u_0 + (4L+4\varepsilon) \cdot \frac{(\sinh 3\lambda)\lambda}{\sinh\lambda + \sinh 3\lambda}. \end{aligned}$$

It deduces that

$$
\begin{aligned}
0 &< \frac{1}{\lambda}\left((-L+\varepsilon)(u_0+3\lambda)+(L+\varepsilon)u_0+(4L+4\varepsilon)\cdot\frac{(\sinh 3\lambda)\lambda}{\sinh\lambda+\sinh 3\lambda}\right)\\
&< \frac{2\varepsilon}{\lambda}u_0-3L+3\varepsilon+(4L+4\varepsilon)\cdot\frac{\sinh 3\lambda}{\sinh\lambda+\sinh 3\lambda}\\
&\to L+7\varepsilon<0
\end{aligned}
$$

as $\lambda\to\infty$, which is a contradiction. Hence we get the conclusion. □

# 6 Numerical experiments for $(-1)$-convex functions on the real line

In this section, we consider numerical experiments for $(-1)$-convex functions on $\mathbb{R}$. First, we generate a "maximum" $(-1)$-convex function on $\mathbb{R}$ joining two points. Let $x_1, x_2, y_1, y_2$ be real numbers such that $x_1 < x_2$ and $y_1 < y_2$. Let $f_1\colon [x_1, x_2] \to [y_1, y_2]$ be an affine function such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Namely,

$$
f_1(t) = y_1 \cdot \frac{t-x_2}{x_1-x_2} + y_2 \cdot \frac{t-x_1}{x_2-x_1}
$$

for $t \in [x_1, x_2]$. Then $f_1$ is not $(-1)$-convex.

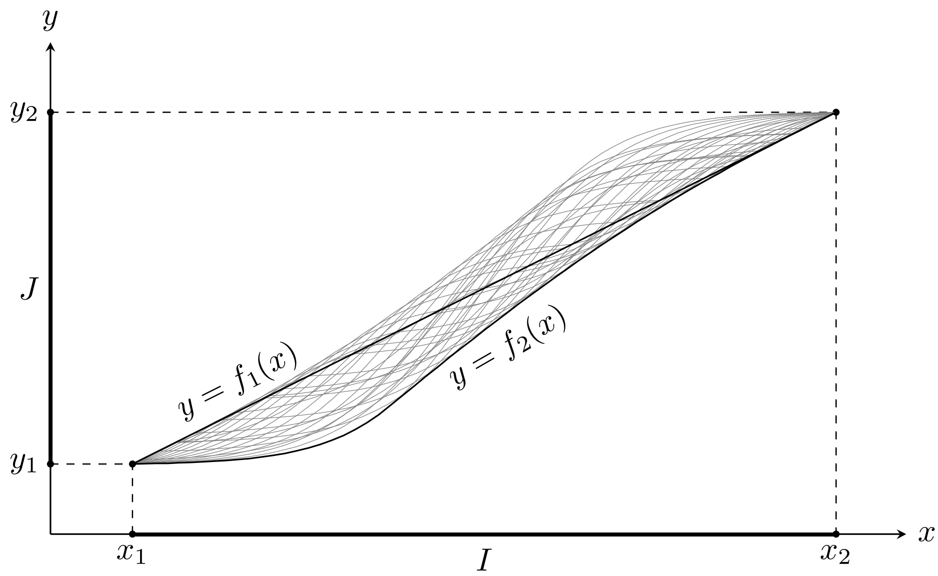
Starting with $f_1$, we attempt to create a sequence of mappings $\{f_n\}$ whose limit $\lim_{n\to\infty} f_n$ being $(-1)$-convex. Put $I = [x_1, x_2]$ and $J = [y_1, y_2]$. Define a function $f_2\colon I \to J$ by

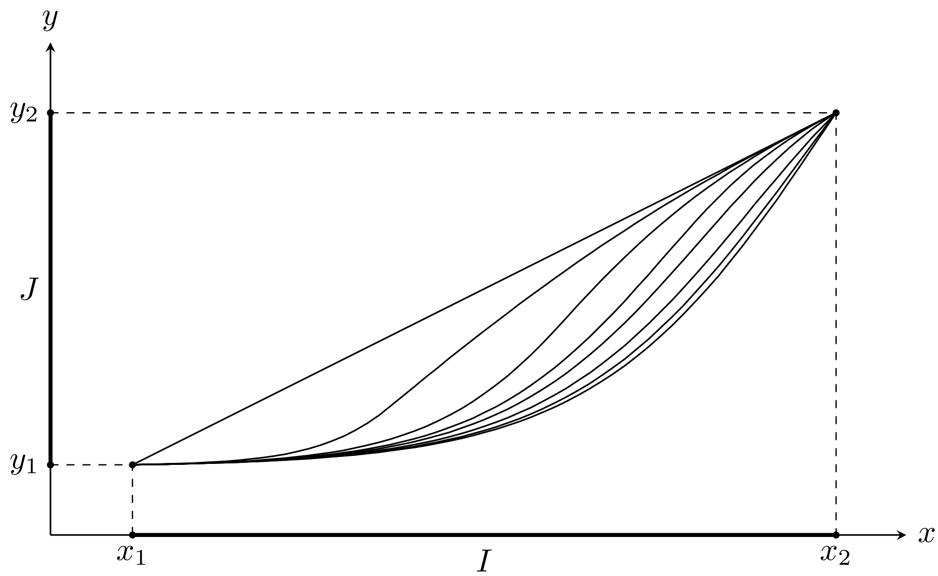
$$
f_2(t) = \min_{\substack{s\le t\le u\\ s<u\\ s,u\in I}} \left(\frac{\sinh(u-t)}{\sinh(u-t)+\sinh(t-s)}\, f_1(s) + \frac{\sinh(t-s)}{\sinh(u-t)+\sinh(t-s)}\, f_1(u)\right)
$$

for $t \in I$. Then $f_2$ is not $(-1)$-convex. In the same way, define a function $f_{n+1}\colon I \to J$ for each $n \in \mathbb{N}$ by

$$
f_{n+1}(t) = \min_{\substack{s\le t\le u\\ s<u\\ s,u\in I}} \left(\frac{\sinh(u-t)}{\sinh(u-t)+\sinh(t-s)}\, f_n(s) + \frac{\sinh(t-s)}{\sinh(u-t)+\sinh(t-s)}\, f_n(u)\right)
$$

inductively. Then we obtain that $y_1 \le f_{n+1}(t) \le f_n(t)$ holds for any $n \in \mathbb{N}$ and $t \in I$. This implies that there exists a limit $\lim_{n\to\infty} f_n(t)$ for each $t \in I$. Define a function $f_\infty\colon I \to J$ by $f_\infty(t) = \lim_{n\to\infty} f_n(t)$ for $t \in I$.




The above two figures describe the creation of functions $\{f_n\}$. The first figure makes a function $f_2$ from $f_1$. The second figure describes graphs of $f_1$, $f_2$, $f_3$, $f_4$, $f_5$, $f_8$, $f_{18}$, and $f_\infty$.

Then we expect the following holds.

**Conjecture 6.1.** *For any $t \in I$,*

$$f_\infty(t) = y_2 + (y_2 - y_1) \cdot \frac{\tanh \dfrac{t - x_2}{2}}{\tanh \dfrac{x_2 - x_1}{2}}.$$

If Conjecture 6.1 is true, then the function $f_\infty$ is $(-1)$-convex. Indeed, we obtain the following fact.

**Theorem 6.2.** *Let $x_1, x_2, y_1, y_2 \in \mathbb{R}$ such that $x_1 < x_2$ and $y_1 < y_2$. Define $g\colon ]-\infty, x_2] \to \mathbb{R}$ by*

$$g(t) = y_2 + (y_2 - y_1) \cdot \frac{\tanh \dfrac{t - x_2}{2}}{\tanh \dfrac{x_2 - x_1}{2}}$$

*for $t \in ]-\infty, x_2]$. Then $g$ is $(-1)$-convex.*

*Proof.* Define $h\colon ]-\infty, 0] \to \mathbb{R}$ by

$$h(t) = \frac{g(t + x_2) - y_2}{y_2 - y_1} \tanh \frac{x_2 - x_1}{2} = \tanh \frac{t}{2}$$

for $t \in ]-\infty, 0]$. Then $g$ is $(-1)$-convex if and only if $h$ is $(-1)$-convex. Since $h$ is $(-1)$-convex, we get the conclusion. □

We also propose the following conjectures:

**Conjecture 6.3.** *Let $f$ be a $(-1)$-convex function from $\mathbb{R}$ into itself. Then $f$ is bounded below.*

**Conjecture 6.4.** *Let $X$ be a* CAT$(-1)$ *space and $f$ a proper function from $X$ into $]-\infty, \infty]$. Then for any $u \in X$,*

$$\liminf_{d(u,v)\to\infty} \frac{f(v)}{d(u,v)} \geq 0.$$

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