Multi-criteria comparison of intuitionistic fuzzy sets from the viewpoint of set optimization*

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Dedicated to the memory of Professor Kazimierz Goebel

1 Introduction

A fuzzy set is a mathematical concept introduced in 1965 as an extension of classical set theory. While classical sets allow an element to either fully belong or not belong to a set, fuzzy sets introduce the concept of partial membership. In other words, fuzzy sets allow for degrees of membership between 0 and 1, representing the extent to which an element belongs to a set.

Comparing entities are crucial in optimization, giving criteria to determine optimal solutions or to specify the constraints of a problem. When comparing fuzzy numbers, which are particular types of fuzzy sets, an inequality relation can be defined to determine the relative ordering between them. The most commonly used inequality relation for fuzzy numbers is the fuzzy max order which is an α-cut-based inequality relation. In [5], the fuzzy max order is extended for fuzzy sets by using orderings of level sets or α-cuts of fuzzy sets.

Ike and Tanaka [3] in 2018 give a new evaluation method via level sets of two fuzzy sets which is similar to the approach of Kon [5] in 2014. The method is based on set-relations which are order-like binary relations between crisp sets proposed by Kuroiwa, Tanaka, and Ha [4] using a vector ordering by a convex cone, and related studies are made in [2, 6, 7]. In a preordered vector space, set relations are widely used to compare the values of objective set-valued mappings.

In this paper, we intend to study a multi-criteria comparison of intuitionistic fuzzy sets which is a generalization of a fuzzy set introduced by Atanassov [1]

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characterized by two functions expressing degree of belongingness and the degree of nonbelongingness, respectively. In particular, correspondences between several types of intuitionistic fuzzy-set relations and their respective difference evaluation function are obtained under certain assumptions of compactness and continuity of set-valued maps.

2 Preliminaries

Let $\mathcal{P}(Z)$ denote the set of all nonempty subsets of a real topological vector space Z. The topological interior, topological closure, and complement of a set A are denoted by $\operatorname{int} A$, $\operatorname{cl} A$, and A^c, respectively.

A set $C \in \mathcal{P}(Z)$ is a cone if $tz \in C$ for all $z \in C$ and $t > 0$. The cone C is *convex* if $z + z' \in C$ for all $z, z' \in C$. The transitive relation $\leq_C$ is induced by a convex cone C as follows: for $z, z' \in Z$, $z \leq_C z'$ if $z' - z \in C$.

Definition 2.1 (intuitionistic fuzzy set, [1]). A pair $\widetilde{A} = (\mu_{\widetilde{A}}, \nu_{\widetilde{A}})$ is called an *intuitionistic fuzzy set* or *IFS* on Z, where

$$\mu_{\widetilde{A}} : Z \to [0,1] \text{ and } \nu_{\widetilde{A}} : Z \to [0,1]$$

are the membership and non-membership functions, respectively, if for all $z \in Z$, $0 \leq \mu_{\widetilde{A}}(z) + \nu_{\widetilde{A}}(z) \leq 1$. When $\mu_{\widetilde{A}}(z) + \nu_{\widetilde{A}}(z) = 1$, $\widetilde{A}$ is called a *fuzzy set*.

Let $\ell = \{(\alpha, \beta) \in [0,1]^2 \mid \alpha + \beta \leq 1\}$ be the set which is used to give values for α, β in the (α, β)-*cut* of $\widetilde{A}$ defined as

$$\widetilde{A}_{(\alpha,\beta)} := \begin{cases} \{z \in Z \mid \mu_{\widetilde{A}}(z) \geq \alpha \text{ and } \nu_{\widetilde{A}}(z) \leq \beta\} & \text{if } (\alpha,\beta) \in \ell \setminus \{(0,1)\}; \\ \operatorname{cl}\{z \in Z \mid \mu_{\widetilde{A}}(z) > 0 \text{ and } \nu_{\widetilde{A}}(z) < 1\} & \text{if } (\alpha,\beta) = (0,1). \end{cases}$$

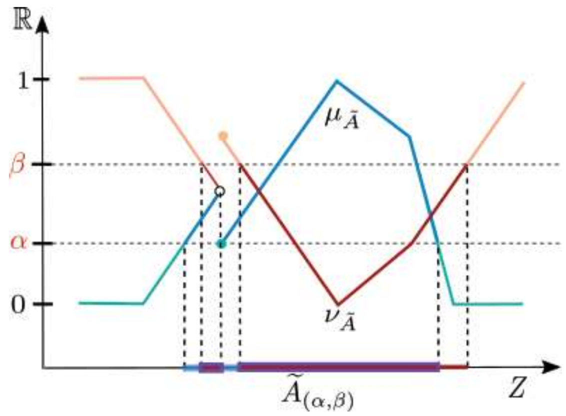


Figure 1: Illustration of an intuitionistic fuzzy set $\widetilde{A}$ with its (α, β)-cut $\widetilde{A}_{(\alpha,\beta)}$.

$\widetilde{A}$ is said to be normal if $\widetilde{A}_{(1,0)} \neq \emptyset$ (or equivalently, for all $(\alpha, \beta) \in \ell$, $\widetilde{A}_{(\alpha,\beta)} \neq \emptyset$). We denote by $\mathcal{F}_N(Z)$ the set of all normal intuitionistic fuzzy sets on Z. For convenience, the set-valued mapping

$$\ell \ni (\alpha, \beta) \mapsto \widetilde{A}_{(\alpha,\beta)} \in \mathcal{P}(Z)$$

is referred to as the *cut mapping* of $\widetilde{A}$.

We define $\preceq$ as the partial order where $(\alpha_1, \beta_1) \preceq (\alpha_2, \beta_2)$ means $\alpha_1 \leq \alpha_2$ and $\beta_1 \geq \beta_2$. Let $K := \{(x, y) \in \mathbb{R}^2 \mid x \geq 0 \text{ and } y \leq 0\}$. Then K is pointed and $(\alpha_2, \beta_2) \in (\alpha_1, \beta_1) + K$ if and only if $(\alpha_1, \beta_1) \preceq (\alpha_2, \beta_2)$.
It can be seen that $\preceq$ is a partial order on ℓ and for any $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \ell$,

$$\widetilde{A}_{(\alpha_2,\beta_2)} \subset \widetilde{A}_{(\alpha_1,\beta_1)} \text{ whenever } (\alpha_1, \beta_1) \preceq (\alpha_2, \beta_2). \tag{1}$$

For $\Delta \subset \ell$, the sets of *Pareto minimal points* and *Pareto maximal points* of Δ with respect to K are defined as

$$\operatorname{Min} \Delta := \{(\alpha, \beta) \in \Delta \mid \Delta \cap ((\alpha, \beta) - K) = \{(\alpha, \beta)\}\}$$

and

$$\operatorname{Max} \Delta := \{(\alpha, \beta) \in \Delta \mid \Delta \cap ((\alpha, \beta) + K) = \{(\alpha, \beta)\}\},$$

respectively. Every point of Δ is said to be *dominated* by a minimal (resp., maximal) point of Δ if $\Delta \subset \operatorname{Min} \Delta + K$ (resp., $\Delta \subset \operatorname{Max} \Delta - K$), and this is called the *domination property* with respect to K (resp., $-K$).

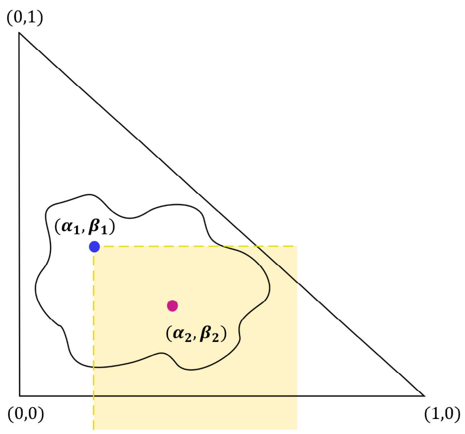


Figure 2: Illustration of a point (α_2, β_2) dominated by a minimal point (α_1, β_1).

For a topological space X and $x_0 \in X$, we denote by $\mathcal{N}_X(x_0)$ the set of all neighborhoods of x_0 in X.

Remark 2.1. The topological structure of a topological vector space Z about any point is determined by a base of neighborhoods of θ_Z. If $\mathcal{U}$ is a base of neighborhoods of θ_Z, then the sets $z + U$ constitute a base of neighborhoods of z for some $U \in \mathcal{U}$ and then $\mathcal{U}$ is called the *local base* in Z. Also, every $U \in \mathcal{U}$ is absorbing, and for $U \in \mathcal{U}$, there exists a balanced neighborhood $V \in \mathcal{U}$ such that $V + V \subset U$; V is called a balanced set when $tV \subset V$ for $|t| \leq 1$. V is balanced if and only if V is symmetric ($-V = V$) and $tV \subset V$ for $0 \leq t < 1$.

Let Z be a topological vector space and X a topological space. A set-valued map $F : X \to \mathcal{P}(Z)$ is said to be

(i) *upper continuous at* $x_0 \in X$ if

$$\forall W \in \mathcal{P}(Z), W \text{ open}, F(x_0) \subset W, \exists U \in \mathcal{N}_X(x_0), \forall x \in U, F(x) \subset W;$$

(ii) *lower continuous* at $x_0 \in X$ if

$$\forall W \in \mathcal{P}(Z), W \text{open}, F(x_0) \cap W \neq \emptyset, \exists U \in \mathcal{N}_X(x_0), \forall x \in U, F(x) \cap W \neq \emptyset;$$

(iii) *Hausdorff upper continuous* at $x_0 \in X$ if

$$\forall W \in \mathcal{U}, \exists U \in \mathcal{N}_X(x_0), \forall x \in U, F(x) \subset F(x_0) + W;$$

(iv) *Hausdorff lower continuous* at $x_0 \in X$ if

$$\forall W \in \mathcal{U}, \exists U \in \mathcal{N}_X(x_0), \forall x \in U, F(x_0) \subset F(x) + W;$$

(v) *upper continuous (resp., lower continuous)* if F is so at every $x \in X$;

(vi) *continuous* if F is both upper and lower continuous.

Remark 2.2. If a set-valued map $F : X \to \mathcal{P}(Z)$ is upper continuous at $x_0 \in X$, then F is Hausdorff upper continuous at x_0; the converse is true when $F(x_0)$ is compact. If F is Hausdorff lower continuous at $x_0 \in X$, then F is lower continuous at x_0; the converse is true when $F(x_0)$ is compact.

Lemma 2.3 ([3, Lemma 3.1])**.** *Let $\emptyset \neq K \subset Z$ be compact and $\emptyset \neq O \subset Z$ be open. Then $\bigcap_{v \in K} (v + O)$ is open.*

For a topological space X, a function $f : X \to \mathbb{R} \cup \{\pm\infty\}$ is said to be lower semicontinuous at $x_0 \in X$ if for any $s < f(x_0)$, there exists $U \in \mathcal{N}_X(x_0)$ such that $s \leq f(x)$ for all $x \in U$. f is lower semicontinuous, denoted by l.s.c., if it is so at every $x \in X$. As is well known, a lower semicontinuous function defined on a compact space always has a minimum.

Definition 2.2 (set relations, [4]). Let $C \subset Z$ be a convex cone. The eight types of *set relations* are defined by

$$
\begin{aligned}
A \leq_C^{(1)} B &\overset{\text{def}}{\iff} \forall a \in A, \forall b \in B, a \leq_C b \iff A \subset \bigcap_{b \in B} (b - C); \\
A \leq_C^{(2L)} B &\overset{\text{def}}{\iff} \exists a \in A \text{ s.t. } \forall b \in B, a \leq_C b \iff A \cap \left(\bigcap_{b \in B} (b - C) \right) \neq \emptyset; \\
A \leq_C^{(2U)} B &\overset{\text{def}}{\iff} \exists b \in B \text{ s.t. } \forall a \in A, a \leq_C b \iff \left(\bigcap_{a \in A} (a + C) \right) \cap B \neq \emptyset; \\
A \leq_C^{(2)} B &\overset{\text{def}}{\iff} A \leq_C^{(2L)} B \text{ and } A \leq_C^{(2U)} B; \\
A \leq_C^{(3L)} B &\overset{\text{def}}{\iff} \forall b \in B, \exists a \in A, a \leq_C b \iff B \subset A + C; \\
A \leq_C^{(3U)} B &\overset{\text{def}}{\iff} \forall a \in A, \exists b \in B, a \leq_C b \iff A \subset B - C; \\
A \leq_C^{(3)} B &\overset{\text{def}}{\iff} A \leq_C^{(3L)} B \text{ and } A \leq_C^{(3U)} B; \\
A \leq_C^{(4)} B &\overset{\text{def}}{\iff} \exists a \in A, \exists b \in B, a \leq_C b \iff A \cap (B - C) \neq \emptyset.
\end{aligned}
$$

for $A, B \in \mathcal{P}(Z)$.

Definition 2.3 (intuitionistic fuzzy-set relations, [7]). Let $C \subset Z$ be a convex cone and $\emptyset \neq \Delta \subset \ell$. For each $j = 1, 2L, 2U, 2, 3L, 3U, 3, 4$, the *intuitionistic fuzzy-set relation* $\leq_C^{\Delta(j)}$, IFS relation shortly, is defined by

$$
\widetilde{A} \leq_C^{\Delta(j)} \widetilde{B} \iff \forall (\alpha, \beta) \in \Delta, \widetilde{A}_{(\alpha,\beta)} \leq_C^{(j)} \widetilde{B}_{(\alpha,\beta)}
$$

for $\widetilde{A}, \widetilde{B} \in \mathcal{F}_N(Z)$.

The values α and β in the (α, β)-cut of $\widetilde{A}$ indicate the suitable or preferable degrees for membership and nonmembership functions of the fuzzy set $\widetilde{A}$. The set Δ can be viewed as a collection of such values in comparing intuitionistic fuzzy sets. The IFS relations seem to be a multi-criteria comparison of intuitionistic fuzzy sets.

The following implications follow directly from the definition above.

$$
\begin{gathered}
\widetilde{A} \leq_C^{\Delta(1)} \widetilde{B} \Longrightarrow \widetilde{A} \leq_C^{\Delta(2L)} \widetilde{B} \Longrightarrow \widetilde{A} \leq_C^{\Delta(3L)} \widetilde{B} \Longrightarrow \widetilde{A} \leq_C^{\Delta(4)} \widetilde{B}; \\
\widetilde{A} \leq_C^{\Delta(1)} \widetilde{B} \Longrightarrow \widetilde{A} \leq_C^{\Delta(2U)} \widetilde{B} \Longrightarrow \widetilde{A} \leq_C^{\Delta(3U)} \widetilde{B} \Longrightarrow \widetilde{A} \leq_C^{\Delta(4)} \widetilde{B}; \\
\widetilde{A} \leq_C^{\Delta(1)} \widetilde{B} \Longrightarrow \widetilde{A} \leq_C^{\Delta(2)} \widetilde{B} \Longrightarrow \widetilde{A} \leq_C^{\Delta(3)} \widetilde{B} \Longrightarrow \widetilde{A} \leq_C^{\Delta(4)} \widetilde{B}
\end{gathered} \tag{2}
$$

for any intuitionistic fuzzy sets $\widetilde{A}, \widetilde{B}$.

An evaluation measure for the difference between two IFS is useful in comparing two intuitionistic fuzzy sets with respect to IFS relation.

Definition 2.4 (difference evaluation function for IFS, [7])**.** Let C be a convex cone in Z, $k \in \operatorname{int} C$, and $\emptyset \neq \Delta \subset \ell$. For each $j = 1, 2L, 2U, 2, 3L, 3U, 3, 4$, the *difference evaluation function* $D^{\Delta(j)}_{C,k} : \mathcal{F}_N(Z) \times \mathcal{F}_N(Z) \to \mathbb{R} \cup \{\pm\infty\}$ is defined by

$$D^{\Delta(j)}_{C,k}(\widetilde{A}, \widetilde{B}) := \sup \left\{ t \in \mathbb{R} \,\middle|\, \widetilde{A} + tk \leq^{\Delta(j)}_{C} \widetilde{B} \right\}.$$

3 Main results

Let Δ be a nonempty subset of ℓ. An IFS $\widetilde{A}$ on a topological vector space Z is said to be Δ-*compact* if $\widetilde{A}_{(\alpha,\beta)}$ is compact for all $(\alpha, \beta) \in \Delta$. When $\widetilde{A}$ is $\{(\alpha, \beta)\}$-compact for some $(\alpha, \beta) \in \ell$, we simply say $\widetilde{A}$ is (α, β)-compact.

Theorem 3.1. *Let $C \subset Z$ be a convex cone, $k \in \operatorname{int} C$, $\emptyset \neq \Delta \subset \ell$, and $\widetilde{A}, \widetilde{B} \in \mathcal{F}_N(Z)$. Then the following statements hold:*

(i) if $\widetilde{A} \leq^{\Delta(j)}_{\operatorname{cl} C} \widetilde{B}$, then $D^{\Delta(j)}_{C,k}(\widetilde{A}, \widetilde{B}) \geq 0$ for $j = 1, 2, 3, 4$;

(ii) if $D^{\Delta(1)}_{C,k}(\widetilde{A}, \widetilde{B}) \geq 0$, then $\widetilde{A} \leq^{\Delta(1)}_{\operatorname{cl} C} \widetilde{B}$;

(iii) if $\Delta \subset \operatorname{Min} \Delta + K$, $\operatorname{Min} \Delta$ is closed, the cut mappings of $\widetilde{A}$ and $\widetilde{B}$ are lower and upper continuous, respectively, and $\widetilde{A}$, $\widetilde{B}$ are $(\operatorname{Min} \Delta)$-compact, then $\widetilde{A} \leq^{\Delta(1)}_{\operatorname{int} C} \widetilde{B}$ implies $D^{\Delta(1)}_{C,k}(\widetilde{A}, \widetilde{B}) > 0$;

(iv) if $\widetilde{A}$ is Δ-compact and $D^{\Delta(2L)}_{C,k}(\widetilde{A}, \widetilde{B}) \geq 0$, then $\widetilde{A} \leq^{\Delta(2L)}_{\operatorname{cl} C} \widetilde{B}$;

(v) if Δ is closed, the cut mappings of $\widetilde{A}$ and $\widetilde{B}$ are continuous, and $\widetilde{A}, \widetilde{B}$ are Δ-compact, then $\widetilde{A} \leq^{\Delta(2)}_{\operatorname{int} C} \widetilde{B}$ implies $D^{\Delta(2)}_{C,k}(\widetilde{A}, \widetilde{B}) > 0$;

(vi) if $\widetilde{B}$ is Δ-compact and $D^{\Delta(3U)}_{C,k}(\widetilde{A}, \widetilde{B}) \geq 0$, then $\widetilde{A} \leq^{\Delta(3U)}_{\operatorname{cl} C} \widetilde{B}$;

(vii) if Δ is closed, the cut mappings of $\widetilde{A}$ and $\widetilde{B}$ are continuous, and $\widetilde{A}, \widetilde{B}$ are Δ-compact, then $\widetilde{A} \leq^{\Delta(3)}_{\operatorname{int} C} \widetilde{B}$ implies $D^{\Delta(3)}_{C,k}(\widetilde{A}, \widetilde{B}) > 0$;

(viii) if $\widetilde{A}$, $\widetilde{B}$ are Δ-compact and $D^{\Delta(4)}_{C,k}(\widetilde{A}, \widetilde{B}) \geq 0$, then $\widetilde{A} \leq^{\Delta(4)}_{\operatorname{cl} C} \widetilde{B}$;

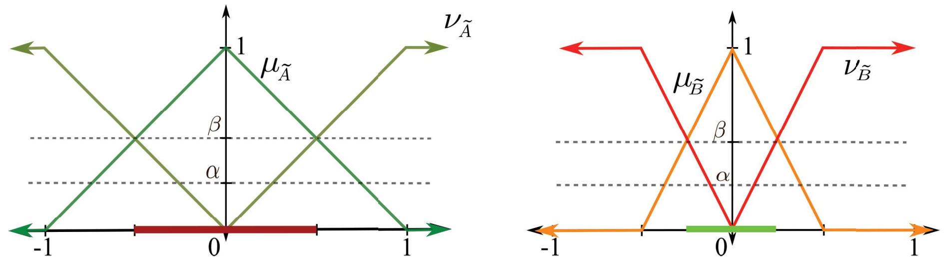
(ix) if Δ is closed, $\widetilde{A}, \widetilde{B}$ are Δ-compact and have lower continuous cut mappings, then $\widetilde{A} \leq^{\Delta(4)}_{\operatorname{int} C} \widetilde{B}$ implies $D^{\Delta(4)}_{C,k}(\widetilde{A}, \widetilde{B}) > 0$;

(x) if $D^{\Delta(j)}_{C,k}(\widetilde{A}, \widetilde{B}) > 0$, then $\widetilde{A} \leq^{\Delta(j)}_{\operatorname{int} C} \widetilde{B}$, for $j = 1, 2, 3, 4$.

Example 3.1. Let $Z = \mathbb{R}$, $C = \mathbb{R}_+$ and define two IFS $\widetilde{A}$ and $\widetilde{B}$ by

$$\mu_{\widetilde{A}}(x) := \begin{cases} 0, & x < -1 \\ x+1, & -1 \le x < 0 \\ 1-x, & 0 \le x < 1 \\ 0, & x > 1 \end{cases}, \quad \mu_{\widetilde{B}}(x) := \begin{cases} 0, & x < -0.5 \\ 2x+1, & -0.5 \le x < 0.5 \\ 1-2x, & 0 \le x < 0.5 \\ 0, & x > 1 \end{cases},$$

$\nu_{\widetilde{A}}(x) := 1 - \mu_{\widetilde{A}}(x)$ and $\nu_{\widetilde{B}}(x) := 1 - \mu_{\widetilde{B}}(x)$ with the following illustration.



When $\Delta = \{(\frac{1}{4}, \frac{1}{2})\} \subset \ell$, we have $\widetilde{A}_{(\alpha,\beta)} = [-\frac{1}{2}, \frac{1}{2}]$ and $\widetilde{B}_{(\alpha,\beta)} = [-\frac{1}{4}, \frac{1}{4}]$. Hence, both are Δ-compact. Moreover, since $D^{\Delta(2L)}_{C,k}(\widetilde{A}, \widetilde{B}) = \frac{1}{4} > 0$, it follows that $\widetilde{A} \le^{\Delta(2L)}_{C} \widetilde{B}$. ◀

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