Robustness of feasibility for multi-valued optimization via sublinear characterization

Yuto OGATA
Faculty of Economic Informatics, Kanazawa Gakuin University

Abstract

This paper contains Gordan type theorems of the alternative based on set relations and some robustness of feasibility for multi-valued optimization problems. Gordan's theorem of the alternative standing on the convex separation theorem has shown the duality of linear functions. We investigate generalized ones of Gordan's and they are introduced by a non-linear separation so that they omit the linearity and the convexity from their assumptions.

1 Introduction

Speaking of theorems of the alternative, they are usually referred to as theorems that state conditions for some element x satisfying $f(x)$ meets the negative orthant to exist. In linear cases, we can check Gordan's [4], Farkas' [2], Fredholm's [3], Stimke's [18], Motzkin's [13], etc. From a vectorial point of view, their theorems can be extended with the pointwise ordering (e.g., see [7], [11], [19]). However, it is not easy to state similar assertions for set-valued maps.

At first, we shall discuss how values of a set-valued map meet the negative orthant. Each value of the map are given as a set that consists of many elements. Next, we need to know what a dual form of set-valued maps is. Theorems of the alternative describe some duality to show the alternate statement. In the literature, it is common that some kinds of convexity assumptions are given to vector-valued or set-valued maps and as a result, their values are linearly distinguished from the negative orthant.

We introduce Gordan type theorems of the alternative based on certain binary relations called "set relations" between two sets. The relations are characterized by non-linearly separating functionals introduced in [5, 10], that is, the theorems are independent of the linearity and the convexity. In this paper, we divide the concept of set-to-set ordering into six cases and propose some equivalences, that requires cone-closedness or cone-compactness, between the set relations and the separating functionals in each case. Our results are generalizations or modification of previous studies in [1, 14–16]. As similar works, we refer to ones about the calculability of the separating functionals ([20]), oriented distance functions that perform as the functional in normed spaces ([6, 8]).

Unless otherwise specified, X is a topological vector space, $C \subset X$ is a convex cone satisfying $\mathrm{int}C \neq \emptyset$ throughout the thesis. For two vectors $x, y \in X$, the pointwise ordering (vector ordering) $\leq_C$ denotes $(X, X, \{(x, y) \mid x \in y - C\})$. For two sets $A, B \subset X \setminus \{\emptyset\}$ and

$\alpha \in \mathbb{R}$, $A+B$ and αA denotes $\{a+b \mid a \in A, b \in B\}$ and $\{\alpha a \mid a \in A\}$, respectively. We use the following convex cone properties ([12]) with respect to C:

- A is C-closed if $A + \mathrm{cl}\, C$ is closed;
- A is C-bounded if it holds that $A \subset U + C$ for any open neighborhood U of the origin;
- A is C-compact if any cover of S being like $\{U_\lambda + C \mid U_\lambda \text{ is open}\}$ admits a finite subcover.

We clearly see C-compactness leads to $C-$closedness and C-boundedness.

Moreover, as set orderings, we introduce the six types of set relations originally proposed in [9]. For nonempty sets $A, B \subset X$, $A \preceq_C^{(1)} B$ denotes $B - A \subset C$; $A \preceq_C^{(2)} B$ denotes $B \subset a + C$ for some $a \in A$; $A \preceq_C^{(3)} B$ denotes $B \subset A + C$; $A \preceq_C^{(4)} B$ denotes $A \subset b - C$ for some $b \in B$; $A \preceq_C^{(5)} B$ denotes $A \subset B - C$; $A \preceq_C^{(6)} B$ denotes $B \cap (A + C) \neq \emptyset$ or equivalently $A \cap (B - C) \neq \emptyset$.

It is easily confirmed that $\preceq_C^{(1)}$ implies $\preceq_C^{(2)}$ and $\preceq_C^{(4)}$, which lead to $\preceq_C^{(3)}$ and $\preceq_C^{(5)}$ respectively. The last relation $\preceq_C^{(6)}$ is implied by the others. Note that all the relations $\preceq_C^{(\mathrm{i})}$ coincide with the pointwise ordering $\leq_C$ when two compared sets are both singletons.

2 Characterization functionals

To nonlinearly separate values of a set-valued map, we introduce the following functional $Z_{C,B,d}^{(\mathrm{i})} : 2^{X \times X} \to \mathbb{R} \cup \{\infty\}$ defined by

$$Z_{C,d}^{(\mathrm{i})}(A, B) := \inf\{\gamma \in \mathbb{R} \mid A \preceq_C^{(\mathrm{i})} B + \gamma d\}.$$

for a given set A, B, and a fixed direction d. $Z_{C,d}^{(\mathrm{i})}(\{\cdot\}, \mathbf{0}_X)$ coincides with the linear functional $f \in X^*$ when A, B are singletons, $C := \{x \in X \mid f(x) \geq 0\}$ is a half space, and $k \in \mathrm{int}\, C$. Also, it holds that $A \preceq_C^{(\mathrm{i})} B$ implies $Z_{C,d}^{(\mathrm{i})}(A, B) \leq 0$.

The calculability of the characterization functionals is given in [20]. If given sets and the ordering cone are convex polyhedral, values of the six functionals can be obtained by solving finitely many linear programming problems.

Let $A, B \in 2^X \setminus \{\emptyset\}$. If A is compact, then $A \preceq_{\mathrm{cl}C}^{(2)} B$ and $A \preceq_{\mathrm{cl}C}^{(3)} B$ follow from $Z_{\mathrm{cl}C,d}^{(2)}(A, B) \leq 0$ and $Z_{\mathrm{cl}C,d}^{(3)}(A, B) \leq 0$ for some $d \in X$, respectively. If B is compact, then $A \preceq_{\mathrm{cl}C}^{(4)} B$ and $A \preceq_{\mathrm{cl}C}^{(5)} B$ follow from $Z_{\mathrm{cl}C,d}^{(4)}(A, B) \leq 0$ and $Z_{\mathrm{cl}C,d}^{(5)}(A, B) \leq 0$ for some $d \in X$, respectively. If both A, B are compact, then $A \preceq_{\mathrm{cl}C}^{(6)} B$ follows from $Z_{\mathrm{cl}C,d}^{(6)}(A, B) \leq 0$ for some $d \in X$.

On the other hand, if both A, B are compact, then $A \preceq_{\mathrm{int}C}^{(1)} B$ follows from $Z_{\mathrm{int}C,d}^{(1)}(A, B) \leq 0$ for some $d \in X$. If B is compact, then $A \preceq_{\mathrm{int}C}^{(2)} B$ and $A \preceq_{\mathrm{int}C}^{(3)} B$ follow from $Z_{\mathrm{int}C,d}^{(2)}(A, B) \leq 0$ and $Z_{\mathrm{int}C,d}^{(3)}(A, B) \leq 0$ for some $d \in X$, respectively. If A is compact, then $A \preceq_{\mathrm{int}C}^{(4)} B$ and $A \preceq_{\mathrm{int}C}^{(5)} B$ follow from $Z_{\mathrm{int}C,d}^{(4)}(A, B) \leq 0$ and $Z_{\mathrm{int}C,d}^{(5)}(A, B) \leq 0$ for some $d \in X$, respectively.

3 Gordan type theorems

The original Gordan's theorem of the alternative states that exactly one of the following two systems is consistent:

[System1] $\exists \mathrm{x} \in \mathbb{R}^n$ such that $A\mathrm{x} < \mathbf{0}$;

[System2] $\exists \mathrm{y} \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$ such that $A^{\mathrm{T}}\mathrm{y} = \mathbf{0}$.

where A is an $m \times n$ matrix. This theorem shows if $A\mathrm{x}$ lies out of the negative orthant, $A\mathrm{x}$ is linearly separated by a nonzero vector y in the positive orthant: $\mathrm{y}^{\mathrm{T}}(A\mathrm{x}) = \mathbf{0}$.

If the matrix A is generally replaced with a vector-valued function $f : \mathbb{R}^n \to \mathbb{R}^m$, it holds by the separation theorem that for all $\mathrm{x} \in \mathbb{R}^n$, there exists $\mathrm{y} \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$ such that $\mathrm{y}^{\mathrm{T}} f(\mathrm{x}) \geq 0$ in the second system. However, it is hardly considered as generalization of Gordan's since the normal vector y depends on x. Addressing this issue in the literature, the image of f satisfies some kinds of convexity (see the left picture in Fig.1). Our results achieve the removal of any convexity assumptions placed on the image of the function. One of the main reasons why this issue occurs is the linearity of the convex set separation. We introduce a sublinear separation (see the pictures in Fig.1 to contrast them) in the second system holding that there exists $\mathrm{y} \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$ such that $\inf\{t \in \mathbb{R} \mid f(\mathrm{x}) \in t\mathrm{y} - \mathbb{R}^m_+\} = \inf\{t \in \mathbb{R} \mid f(\mathrm{x}) \leq_C t\mathrm{y}\} \geq 0$ for all $\mathrm{x} \in \mathbb{R}^n$.

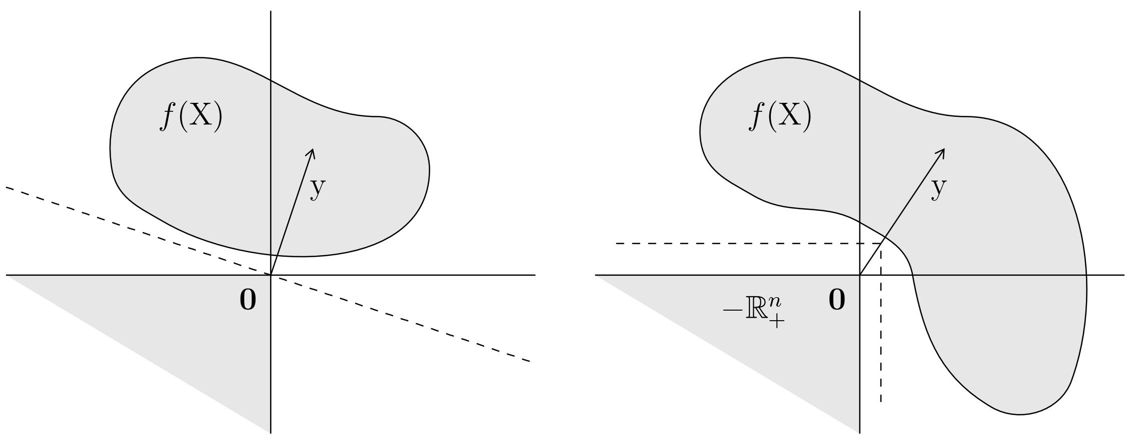

Figure 1: linear and sublinear separations

Theorem 3.1 ([17]). Let S be a set, X a topological vector space, $C \subset X$ a convex cone having the nonempty interior, $W \subset X$ a nonempty set, $F : S \to 2^X \setminus \{\mathbf{0}\}$. Then, exactly one of the following two systems is consistent:

[System1] $\exists s \in S$ such that $F(s) \preceq^{(\mathrm{i})}_{\mathrm{cl}\, C} W$;

[System2] $\exists k \in \mathrm{int}\, C$ such that $Z^{(\mathrm{i})}_{C,k}(F(s), W) > 0$ for all $s \in S$.

where F is C-compact valued for case $i = 2$; S_1 is C-closed valued for case $i = 3$; W is $(-C)$-compact for case $i = 4$; W is $(-C)$-closed for case $i = 5$; F is C-closed valued and W is $(-C)$-compact or F is C-compact valued and W is $(-C)$-closed for case $i = 6$.

Theorem 3.1 can describe the sublinear separation depicted in Fig.1 by treating X as $\mathbb{R}^m$, C as $\mathbb{R}^m_+$, a one-to-one mapping F as $f : S \to X$, and a singleton W as $\mathbf{0}$.

Example 3.1. Let $S := [-\pi, \pi]$, $W := [0, \pi] \times [0, \pi]$. We define a set-valued map $F : S \to \mathbb{R}^2$ by $F(s) := \left\{(x, y) \mid (x - s)^2 + (y - \cos s)^2 \leq \frac{1}{4}(s + \pi)\right\}$ (see Fig.3.1). Fig.3 depicts values of the characterization functionals $Z^{(\mathrm{i})}_{\mathbb{R}^2_+,k}$ for $k = (1, 1) \in \operatorname{int} \mathbb{R}^2_+$ where $\alpha \in S$ satisfies $\alpha = -\cos \alpha$. The cases $i = 4, 5$ and $i = 2, 3$ coincide, respectively. Moreover, one can check that for all $s \in S$, it holds that

$$Z^{(1)}_{\mathbb{R}^2_+,k}(F(s), W) \geq Z^{(2,3)}_{\mathbb{R}^2_+,k}(F(s), W) \geq Z^{(6)}_{\mathbb{R}^2_+,k}(F(s), W),$$
$$Z^{(1)}_{\mathbb{R}^2_+,k}(F(s), W) \geq Z^{(4,5)}_{\mathbb{R}^2_+,k}(F(s), W) \geq Z^{(6)}_{\mathbb{R}^2_+,k}(F(s), W).$$

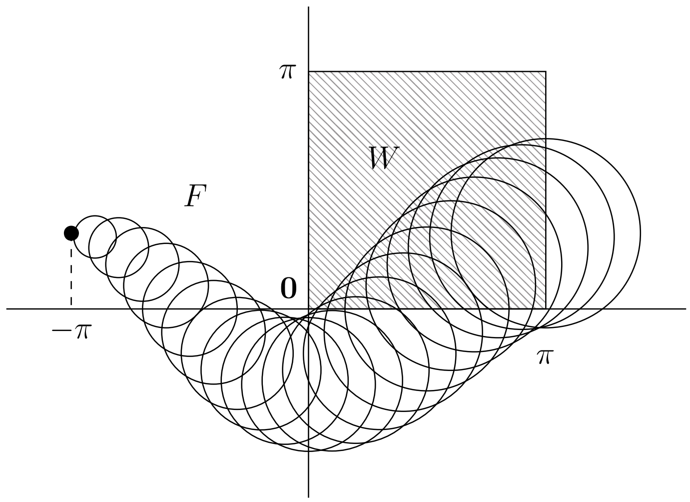

Figure 2: Illustration of Example 3.1

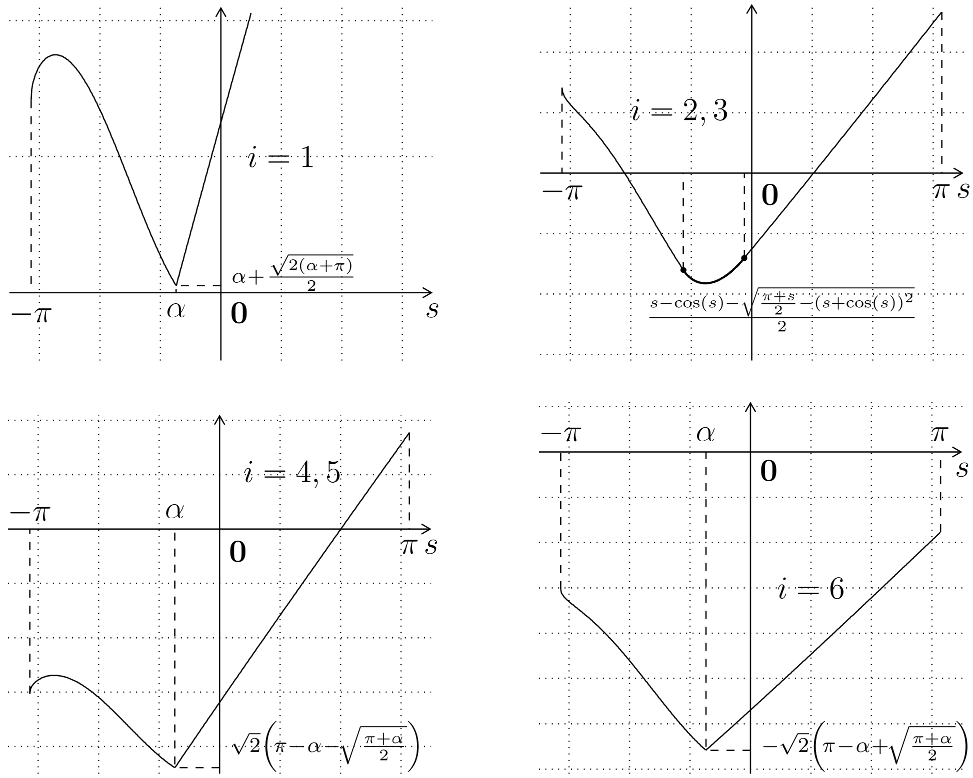

Figure 3: Values of $Z^{(\mathrm{i})}_{\mathbb{R}^2_+,k}(F(s),W)$ in Example 3.1

4 Robustness of the feasibility

For a nonempty set S, consider the following optimization problem:

$$\text{(P) Minimize } f(x) \text{ subject to } g(x) \leq_C r$$

where $f : S \to \mathbb{R}^n$, $g : S \to \mathbb{R}^m$, $r \in \mathbb{R}^m$. In the constraint condision of (P), the function g and the vector r are to be perturbed in the sets G and R, respectively.

Theorem 4.1 ([17])**.** We assume (P) is feasible. Then, the following statements hold on $G(x) := \{g(x) \mid g \in G\}$ and R,

(i) (P) is still feasible for all $g \in G$ and $r \in R$ if and only if $Z^{(1)}_{C,k}(G(x),R) \leq 0$ for some $k \in \mathrm{int}C$.

(ii) There exists $g \in G$ such that (P) is feasible for all $r \in R$ if and only if $Z^{(2)}_{C,k}(G(x),R) \leq 0$ for some $k \in \mathrm{int}C$;

(iii) For all $r \in R$, we can find $g \in G$ to make (P) remain feasible if and only if $Z^{(3)}_{C,k}(G(x),R) \leq 0$ for some $k \in \mathrm{int}C$.

(iv) There exists $r \in R$ such that (P) is feasible for all $g \in G$ if and only if $Z^{(4)}_{C,k}(G(x),R) \leq 0$ for some $k \in \mathrm{int}C$.

(v) For all $g \in G$, we can find $r \in R$ to make (P) remain feasible if and only if $Z^{(5)}_{C,k}(G(x), R) \leq 0$ for some $k \in \text{int}C$.

(vi) (P) is feasible for some $g \in G$ and some $r \in R$ if and only if $Z^{(6)}_{C,k}(G(x), R) \leq 0$ for some $k \in \text{int}C$.

The above theorem implies the values of characterization functionals indicate the robustness of feasibility for a multi-valued optimization problem. Remark that each value is calculated by solving linear programming problems under some convexity assumptions.

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