ON α-ψ-CONTRACTIVE TYPE MAPPINGS AND THEIR ASYMPTOTIC VERSIONS

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Abstract. In this paper we consider an asymptotically regular α-ψ contractive mappings in complete metric spaces. We also prove fixed point theorems in this spaces.

1. Introduction

In [3], Górnicki proved the following fixed point theorem:

Theorem 1. *Let (X, d) be a complete metric space and $f : X \to X$ an asymptotically regular continuous mapping. Suppose there exist $0 \leq M < 1$ and $0 \leq K < \infty$ satisfying*

$$d(fx, fy) \leq Md(x, y) + Kd(x, fx) + d(y, fy)$$

for all $x, y \in X$. Then f has a unique fixed point $z \in X$ and $f^n x \to z$ for each $x \in X$.

In [1], Bisht give the following common fixed point theorem.

Theorem 2. *Let (X, d) be a complete metric space and $f, g : X \to X$ be R-weakly commuting mappings of type A_g or type A_f. Suppose that f is asymptotically regular with respect to g and there exist $0 \leq M < 1$ and $0 \leq K < \infty$ satisfying*

$$d(fx, fy) \leq Md(gx, gy) + K(d(fx, gx) + d(fy, gy)) \tag{1}$$

for all $x, y \in X$. Then f and g have a unique common fixed point if and only if f and g are (f, g)-orbitally continuous.

And he give a example and prove that Theorem 2 is not a consequence of Theorem 1.

On the other hand in [15] we give a following α-ψ_n contractive mapping which is a generalization of a α-ψ mapping in [14] and also fixed point theorem. Let (X, d) be a metric space. We say that mapping $T : X \to X$ is α-ψ_n contractive if there exist a mapping $\alpha : X \times X \to [0, \infty)$ and a sequence of non-decreasing mappings ψ from $[0, \infty)$ into itself such that the series $\sum_{n=1}^{\infty} \psi_n(t)$ converges for all $t \in [0, \infty)$ and for any $x, y \in X, n \in N$, we have

$$\alpha(x, y)d(T^n x, T^n y) \leq \psi_n(d(x, y)). \tag{2}$$

Theorem 3. *Let (X, d) be an complete metric space. Let $T : X \to X$ be α-ψ_n contractive mappings satisfying the following conditions;*

(i) *there exists $x_0 \in X$ such that $\alpha(x_0, T(x_0)) \geq 1$;*

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(ii) *T is continuous.*

Then there exists a fixed point of mapping T.

In this article we consider the α-ψ_n mapping versions of fixed point theorem which corresponding to Theorem 1. And also for a α-ψ_n mapping we give common fixed point theorem which corresponding to Theorem 2.

2. Preliminaries

In this section we think about the relationship between α-ψ_n (α-ψ) mapping and the asymptotic regularity. Throughout this paper, we refer to the set of all positive integers as N and the set of real numbers as R. We provide several preliminary settings in this section, including α-ψ and α-ψ_n mappings, among others. We give the following.

Definition 4. [5] *Let (X, d) be a metric space and T a mapping from X into itself and $\alpha : X \times X \to (\infty, \infty)$. It is said that T is a triangular α-admissible mapping if*

(i) *$\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$, for $x, y \in X$,*
(ii) *$\alpha(x, y) \geq 1, \alpha(y, z) \geq 1$ imply $\alpha(x, z) \geq 1$ for any $x, y, z \in X$.*

Remark 5. *Let T be a triangular α-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Define sequence $\{x_n\}$ by $x_n = T^n x_0$. Then $\alpha(x_m, x_n) \geq 1$ for all $m, n \in N$ with $m \geq n$, see* [5, Lemma 7]

Definition 6. [6] *Let (X, d) be a metric space and we say that $T : X \to X$ is partial α-ψ contractive (resp. partial α-ψ_n contractive) if there exist $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ (resp. $\psi \in \Psi_n$) such that the following condition hold:*

$$x, y \in X \text{ with } \alpha(x, y) \geq 1 \text{ imples } d(Tx, Ty) \leq \psi(d(x, y)).$$

$$(\text{ resp. } x, y \in X \text{ with } \alpha(x, y) \geq 1 \text{ imples } d(Tx, Ty) \leq \psi_n(d(x, y)), \text{ for any } n.)$$

Definition 7. *A mapping T of a metric space (X, d) into itself is said to be asymptotically regular if*

$$\lim_{n\to\infty} d(T^n x, T^{n+1} x) = 0$$

for all $x \in X$.

We define the orbit set with respect to operator $f : X \to X$ at t with

$$O(t, f) = \{f^n t \in X, n = 0, 1, \ldots\}.$$

A mapping T of a metric space from (X, d) into itself is contraction, Kannan, then T is asymptotically regular. Asymptotic regularity is also satisfied by other mappings. The asymptotic regularity and non expansiveness (i.e. $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in X$), more generally, continuity, are independent, see [3].

In genearal, α-ψ contaractive mapping does not satisfy asymptotically regularity. However, we give the following lemma.

Lemma 8. *Let (X, d) be a metric space and T a mapping from X into itself. We assume that*

(i) *T is α-admissble;*
(ii) *T is partial α-ψ contractive;*
(iii) *there exists mapping $\alpha : X \times X \to [0, \infty)$ such that $\alpha(x, Tx) \geq 1$ for any $x \in X$.*

Then T is asymptotically regular.

Proof. Let $x_0 \in X$ be any point. We define a sequence $\{x_n\}$ in X as $x_n = T^n x_0$ based on x_0. By assumption (iii), we have $\alpha(x_0, Tx_0) \geq 1$ and assumption (i), we have $\alpha(x_n, x_{n+1}) = \alpha(x_n, Tx_n) \geq 1$. Moreover assumption (ii), we have

$$d(x_n, x_{n+1}) = d(x_n, Tx_n) \leq \psi(d(x_{n-1}, Tx_{n-1})).$$

Repeating this we have

$$d(x_n, x_{n+1}) \leq \psi^n(d(x_0, Tx_0)).$$

Since $\psi \in \Psi$, we have $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. □

For the α-ψ_n mapping, we have the following.

Lemma 9. *Let (X, d) be a metric space and T be a mapping from X into itself. We assume that T is partial α-ψ_n contractive. We assume that*

(i) *T is partial α-ψ contractive;*
(ii) *there exists mapping $\alpha : X \times X \to [0, \infty)$ such that $\alpha(T^n x, T^{n+1} x) \geq 1$ for any $x \in X$,*

then T is asymptotically regular.

Proof. Let $x_0 \in X$ be any point. We define a sequence $\{x_n\}$ in X as $x_n = T^n x_0$. By assumption (ii), we have $\alpha(T^n x_0, T^{n+1} x_0) \geq 1$. Moreover by assumption (i) we have

$$d(T^n x_0, T^{n+1} x_0) \leq \psi_n(d(x_0, Tx_0)),$$

that is,

$$d(x_n, x_{n+1}) = d(T^n x_0, T^{n+1} x_0) \leq \psi_n(d(x_0, Tx_0)).$$

By the definition of $\{\psi_n\}$ and for any $t > 0$ $\psi_n(t) \to 0$ as $n \to \infty$. So we have $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. □

3. Fixed point theorem

In this section we consider the following theorem 12 and 13 which are versions of Theorem 1 for the α-ψ and α-ψ_n mappimgs, repectvely. We begin with considering the extension of α-ψ and α-ψ_n mappings.

Definition 10. *Let (X, d) be a metric space. We say that mapping $T : X \to X$ is α-ψ-L contractive if there exist a mapping $\alpha : X \times X \to [0, \infty)$ and non-decreasing mappings $\psi : [0, \infty) \to [0, \infty)$ such that for any $x, y \in X$, we have*

$$\alpha(x, y) d(Tx, Ty) \leq \psi(d(x, y)) + L\left(d(x, Tx) + d(y, Ty)\right),$$

where $0 \leq L < \infty$.

Definition 11. *Let (X, d) be a metric space. We say that mapping $T : X \to X$ is α-ψ_n-L contractive if there exist a mapping $\alpha : X \times X \to [0, \infty)$ and non-decreasing mappings $\psi : [0, \infty) \to [0, \infty)$ such that for any $x, y \in X, n \in N$, we have*

$$\alpha(x, y) d(T^n x, T^n y) \leq \psi_n(d(x, y)) + L\left(d(T^n x, T^{n+1} x) + d(T^n y, T^{n+1} y)\right),$$

where $0 \leq L < \infty$.

Theorem 12. *Let (X, d) be a complete metric space and $T : X \to X$ a α-ψ-L contractive mappings. We assume that*

(i) T *is triangular* α*-admissible;*
(ii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, T(x_0)) \geq 1$*;*
(iii) $\psi(t) < t$ *for any* $t > 0$*;*
(iv) T *is asymptotically regular continuous.*

Then T *has a fixed point.*

Proof. By assumption, there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and define a sequence $\{x_n\}$ in X as $x_n = T^n x_0$. Suppose that $\{x_n\}$ is not Cauchy. Then there exists $\tau > 0$ and for any integer k, there exists $n_k \geq m_k \geq k$, $d(x_{m_k}, x_{n_k}) \geq \tau$ and $d(x_{m_k}, x_{n_k-1}) < \tau$. By triangle inequality $d(x_{m_k}, x_{n_k}) < d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k})$, we have $d(x_{m_k}, x_{n_k}) \to \tau$ as $k \to \infty$, By (i) and (ii), from Remark 5, we have $\alpha(m_k, n_k) \geq 1$, in this case

$$\begin{aligned} d(x_{m_k+1}, x_{n_k+1}) &\leq \alpha(x_{m_k}, x_{n_k}) d(Tx_{m_k}, Tx_{n_k}) \\ &\leq \psi(d(x_{m_k}, x_{n_k})) \\ &\leq \psi(d(x_{m_k}, x_{m_k+1}) + d(x_{m_k+1}, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k})). \end{aligned}$$

Then $k \to \infty$ right side, we have

$$\tau \leq \psi(\tau) < \tau$$

a contradiction. So $\{x_n\}$ is Cauchy. Since X is complete there exists $p \in X$such that $x_n \to p$. Since T is continuous, we have $Tp = p$. Then p is a fixed point of T. □

Theorem 13. *Let* (X, d) *be a complete metric space and* T *is* α-ψ_n-L *contractive mappings from* X *into itself. Assume that*

(i) T *is triangular* α*-admissible;*
(ii) *there exists* x_0 *such that* $\alpha(x_0, Tx_0) \geq 1$*;*
(iii) $\psi_n(t) < t$ *for any* $t > 0$ *and* $n \in N$*;*
(iv) T *is asymptotically regular continuous.*

Then T *has a fixed point.*

Proof. By assumption, there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and define a sequence $\{x_n\}$ in X as $x_n = T^n x_0$. Suppose that $\{x_n\}$ is not Cauchy. By the same argument of α-ψ, there exists $\tau > 0$ and any integer k there exists $n_k \geq m_k \geq k$, $d(x_{m_k}, x_{n_k}) \to \tau$ as $k \to \infty$. By (i) and (ii), from Remark 5, we have $\alpha(m_k, n_k) \geq 1$. Let $l > 0$, then we have

$$\begin{aligned} d(x_{m_k+l}, x_{n_k+l}) &= d(T^l x_{m_k}, T^l x_{n_k}) \\ &\leq \alpha(x_{m_k}, x_{n_k}) d(T^l x_{m_k}, T^l x_{n_k}) \\ &\leq \psi_l(d(x_{m_k}, x_{n_k})) + L(d(T^l x_{m_k}, T^{l+1} x_{m_k}) + d(T^l x_{n_k}, T^{l+1} x_{n_k})) \\ &= \psi_l(d(x_{m_k}, x_{n_k})) + L(d(x_{m_k+l}, x_{m_k+l+1}) + d(x_{n_k+l}, x_{n_k+l+1})). \end{aligned}$$

Since $\psi_l(t) < t$ for any $l > 0$ and T is asymptotic regular, that is, $d(T^n x, T^{n+1} x) \to 0$ for any $x \in X$ and n, and for n_0 there exists $n_k \geq m_k \geq n_0$ such that $d(x_{m_k}, x_{n_k}) \to \tau$ as $k \to \infty$. So we have $\tau \leq \psi_l(\tau) < \tau$, a contradiction. So $\{x_n\}$ is Cauchy. Since X is complete there exists $p \in X$ such that $x_n \to p$. Also T is continuous, we have $Tp = p$. Then p is a fixed point of T. □

Theorem 14. *To the hypotheses of Theorem 12 (or Theorem 13), we assume that*

(i) *for any* $x, y \in X$*, there exists* $z \in X$ *such that* $\alpha(x, z) \geq 1$ *and* $\alpha(y, z) \geq 1$*;*

(ii) $\lim_{n\to\infty}\psi^n(t)=0$ *(resp.* $\lim_{n\to\infty}\psi_n(t)=0$*) for any* $t>0$.

Then the fixed point of T *is unique.*

Proof. Let p be fixed point of T and q another fixed point of T. By assumption (i) we have $z\in X$ such that $\alpha(p,z)\ge 1$ and $\alpha(q,z)\ge 1$. Since T is α-admissible, we get $\alpha(p,T^nz)\ge 1$ and $\alpha(p,T^nz)\ge 1$ for any $n\in N$. Then

$$\begin{aligned} d(p,T^nz) = d(Tp,T(T^{n-1}z)) &\le \alpha(p,T^{n-1}z)d(Tp,T(T^{n-1}z)) \\ &\le \psi(d(p,T^{n-1}z)). \end{aligned}$$

By assumption (ii), this implies that

$$d(p,T^nz)\le \psi^n(d(p,z))\to 0,$$

as $n\to\infty$. Then $T^nz\to p$ as $n\to\infty$, similary, we have $T^nz\to q$ as $n\to\infty$. The uniqueness of the limit gives us $p=q$. If the T is α-ψ_n-L contractive,

$$\begin{aligned} d(p,T^nz) = d(T^np,T^nz)) &\le \alpha(p,z)d(T^np,T^nz)) \\ &\le \psi_n(d(p,z)). \end{aligned}$$

Then $T^nz\to p$ as $n\to\infty$, similary, we have $T^nz\to q$ as $n\to\infty$. □

Note that if ψ is nondecreasing, then for each $t>0$, $\lim_{n\to\infty}\psi^n(t)=0$ implies $\psi(t)<t$.

Proof. Suppose that there exists $t_0>0$ such that $\lim_{n\to\infty}\psi^n(t_0)=0$ and $\psi(t_0)\ge t_0$. Since ψ is a non-decreasing function, from $\psi(t_0)\ge t_0$ we have $\psi^2(t_0)=\psi(\psi(t_0))\ge\psi(t_0)\ge t_0$. It implies that $\psi^n(t_0)\ge t_0$ for all $n=1,2,\ldots$. Hence $\lim_{n\to\infty}\psi^n(t_0)\ge t_0>0$. This is a contradiction. □

If ψ_n is nondecreasing for each $n\in N$, then for each $t>0$, $\lim_{n\to\infty}\psi_n(t)=0$ implies $\psi_n(t)<t$ for each n.

Proof. Suppose that there exists $t_0>0$ such that $\lim_{n\to\infty}\psi_n(t_0)=0$ and there exists n_0 such that $\psi_{n_0}(t_0)\ge t_0$. Since ψ_n is non-decreasing with respect to each n, We have $\psi_{n_0+1}(t_0)\ge\psi_{n_0}(t_0)\ge t_0$. It implies that $\psi_n(t_0)\ge t_0$ for all $n\ge n_0$. Hence $\lim_{n\to\infty}\psi_n(t_0)\ge t_0>0$. This is a contradiction. □

4. Common fixed point theory

In this section we consider the common fixed point theory with respect to two α-ψ-Lcontaractive mappings f and g, and also two α-ψ_n-L contaractive mapppings f and g. We give the following definitions.

Definition 15. *Let* f *and* g *be two self mappings of a metric space* (X,d)*. Then*

(i) f *is called asymptotically regular with respect to* g *at* $x_0\in X$ [2, 7, 13] *if there exists a sequence* $\{x_n\}$ *in* X *such that* $gx_{n+1}=fx_n$, $n=0,1,2,\ldots$, *and* $d(gx_{n+1},gx_{n+2})\to 0$ *as* $n\to\infty$.

(ii) f *is called iterative asymptotically regular with respect to* g *at* $x_0\in X$ *if there exists a iterative sequence* $\{x_n\}$ *in* X *with respect to* f *such that* $g(x_{n+1})=f(x_n)$, $g(x_0)=x_0$, $n=0,1,2,\ldots$, *and* $d(g(x_n),g(x_{n+1}))\to 0$ *as* $n\to\infty$.

(iii) f *and* g *are called* R*-weakly commuting mappings* [8] *if there exists some real number* $R>0$ *such that* $d(fgx,gfx)\le Rd(fx,gx)$ *for all* $x\in X$.

(iv) *f and g are called R-weakly commuting mappings of type A_g (resp. type A_f)* [11] *if there exists some real number $R > 0$ such that*

$$d(ffx, gfx) \leq Rd(fx, gx)$$
$$(resp.\, d(fgx, ggx) \leq Rd(fx, gx))$$

for all $x \in X$.

(v) *f and g are called non trivially weakly compatible* [4] *if f and g commute on the set of coincidence points whenever the set of their coincidences is non-empty.*

(vi) *f and g are called non-compatible* [9] *if there exists a sequence $\{x_n\}$ in X such that $fx_n, gx_n \to t$ as $n \to \infty$ for some $t \in X$ but the limit of $d(fgx_n, gfx_n)$ is either non-zero or non-existent.*

(vii) *f and g are called g-reciprocally continuous* [10] *if $ffx_n \to ft$ and $gfx_n \to gt$, whenever $\{x_n\}$ is a sequence in X such that fx_n, $gx_n \to t$ as $n \to \infty$ for some $t \in X$.*

(viii) *Let f and g be two self-mappings of a metric space (X, d) and let $\{x_n\}$ be a sequence in X such that $fx_n = gx_{n+1}$. Then the set $O(x_0, f, g) = \{fx_n : n = 0, 1, 2, \ldots\}$ is called the (f, g)-orbit at x_0.*

(viiii) *f (or g) is called (f, g)-orbitally continuous [8, 23] if $\lim_{n\to\infty} fx_n = z$ implies $\lim_{n\to\infty} ffx_n = fz$ or ($\lim_{n\to\infty} fx_n = z$ implies $\lim_{n\to\infty} gfx_n = gz$).*

(x) *f and g are called orbitally continuous if f is (f, g)-orbitally continuous and g is (f, g)-orbitally continuous.*

Definition 16. *Let f and g are mappinga from X into itself. f and g are called mutual α-ψ-L contractive if there exist a mapping $\alpha : X \times X \to [0, \infty)$ and a sequence of non-decreasing mappings ψ of $[0, \infty)$ into itself such that $\psi(t) < t$ for all $t \in [0, \infty)$ and for any $x, y \in X$, we have*

$$\alpha(x, y)d(fx, fy) \leq \psi(d(gx, gy)) + L\left(d(fx, gx) + d(fy, gy)\right). \tag{3}$$

where $0 \leq L < \infty$.

Definition 17. *Let f and g are mappinga from X into itself. f and g are called mutual α-ψ_n-L contractive if there exists a mapping $\alpha : X \times X \to [0, \infty)$ and a sequence of non-decreasing mappings ψ_n of $[0, \infty)$ into itself such that for all $t \in [0, \infty)$, any $n \in N$, and for any $x, y \in X$, we have*

$$\alpha(x, y)d(f^n x, f^n y) \leq \psi_n(d(gx, gy)) + L\left(d(f^n x, g^n x) + d(f^n y, g^n y)\right), \tag{4}$$

where $0 \leq L < \infty$.

For the α-ψ-L mapping, the following holds.

Theorem 18. *Let (X, d) be a complete metric space and f and g mutual α-ψ-L contractive mappings from X into itself. We assume that*

(i) *f is triangular α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;*

(iii) *either $\alpha(fx, fy) > 1$ or $\alpha(fy, fx) > 1$ whenever $fx = gx$ and $fy = gy$, see* [12]*;*

(iv) *$\psi(t) < t$ for any $t > 0$;*

(v) *f is asymptotically regular with respect to g;*

(vi) *f is R-weakly commuting mappings of type A_g and f and g are g-reciprocally continuous.*

Then f and g have a unique common fixed point.

Proof. Since f is asymptotically regular with respect to g at $x_0 \in X$, there exists a sequence $\{y_n\} \subset X$ such that $y_n = fx_n = gx_{n+1}$ for all $n \in N$ and

$$\lim_{n\to\infty} d(gx_{n+1}, gx_{n+2}) = \lim_{n\to\infty} d(y_n, y_{n+1}) = 0.$$

Suppose that $\{y_n\}$ is not a Cauchy sequence. There exists $\tau > 0$ and for all n_0 there exists $m_k \geq n_k \geq n_0$ such that $d(y_{m_k}, y_{n_k}) \geq \tau$ and $d(y_{m_k-1}, y_{n_k}) < \tau$. In light of triangle inequality,

$$d(y_{m_k}, y_{n_k}) \leq d(y_{m_k}, y_{m_k-1}) + d(y_{m_k-1}, y_{n_k})$$

Making $k \to \infty$, we have

$$d(y_{m_k}, y_{n_k}) < \tau \text{ and } d(y_{m_k}, y_{n_k}) \to \tau.$$

We also $d(y_{m_k+l}, y_{n_k+l}) \to \tau$ as $k \to \infty$. By (i) and (ii), we have $\alpha(y_{m_k+1}, y_{n_k+1}) \geq 1$. Then by (3) we have

$$\begin{aligned} d(y_{m_k+1}, y_{n_k+1}) &\leq \alpha(y_{m_k+1}, y_{n_k+1}) d(fx_{m_k+1}, fx_{n_k+1}) \\ &\leq \psi(d(gx_{m_k+1}, gx_{n_k+1})) + L(d(fx_{m_k}, gx_{m_k}) + d(fx_{n_k}, gx_{n_k})) \\ &= \psi(d(y_{m_k}, y_{n_k})) + L(d(y_{m_k}, y_{m_k+1})) + L(d(y_{n_k}, y_{n_k+1})). \end{aligned}$$

Making $k \to \infty$, $\tau \leq \psi(\tau) < \tau$ a contradiction. Then $\{y_n\}$ is a Cauchy sequence. Since X is complete, there exists $p \in X$ such that y_n converges to $p \in X$. Since f is g-reciprocally continuous, $\lim_{n\to\infty} fx_n = p$ implies $\lim_{n\to\infty} gfx_n = gp$ (or $\lim_{n\to\infty} fx_n = p$ implies $\lim_{n\to\infty} ffx_n = fp$). Moreover $fx_n \to p$ implies $ffx_n \to fp$, and $gfx_n \to gp$. Then R-weakly commuting mappings of type A_g of f yields

$$d(ffx_n, gfx_n) \leq d(fx_n, gx_n).$$

On letting $n \to \infty$, we get $fp = gp$. Also R-weakly commuting mappings of type A_g of f implies commutativity at p, that is, $fgp = gfp$. Hence $gfp = fgp = ffp = ggp$.

Let $fp \neq ffp$. By (iii) $fp = gp$ and $ffp = gfp$ imples $\alpha(fp, ffp) \geq 1$ and using (3) we have

$$\begin{aligned} d(fp, ffp) &\leq \alpha(fp, ffp) d(fp, ff) \\ &\leq \psi(d(gp, gfp) + L(d(ffp, gfp) + d(fffp, gffp))) \end{aligned}$$

Since f is asymptotic regular, we have $d(fp, ffp) < d(fp, ffp)$. It is a contradiction. Hence $fp = ffp = gfp$ and fp is a common fixed point of f and g.

□

Also for the α-ψ_n-L mapping, the following holds.

Theorem 19. *Let (X, d) be a complete metric space and f and g mutual α-ψ_n-L contractive mappings from X into itself. We assume that*

(i) *f is triangular α-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geq 1$;*
(iii) *either $\alpha(fx, fy) > 1$ or $\alpha(fy, fx) > 1$ whenever $fx = gx$ and $fy = gy$, see* [12]*;*
(iv) *$\psi_n(t) < t$ for any $t > 0$ and $n \in N$;*
(v) *f is iterative asymptotically regular with respect to g at $x \in X$;*

(vi) *f is R-weakly commuting mappings of type A_g and f and g are g-reciprocally continuous.*

Then f and g have a common fixed point.

Proof. By (v) there exists sequence $\{y_n\}$ such that $y_n = f^n x = fx_n = gx_{n+1}$ for all $n \in N$ and

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = \lim_{n\to\infty} d(gx_{n+1}, gx_{n+2}) = 0.$$

We also prove that $\{y_n\}$ is Cauchy sequnces. Suppose that $\{y_n\}$ is not a Cauchy sequence. Then there exists $\tau > 0$ and for all k there exists $m_k > n_k \geq k$ such that $d(y_{m_k}, y_{n_k}) \geq \tau$ and $d(y_{m_k-1}, y_{n_k}) < \tau$. In light of triangle inequality,

$$d(y_{m_k}, y_{n_k}) \leq d(y_{m_k}, y_{m_k-1}) + d(z_{m_k-1}, z_{n_k})$$

On letting $k \to \infty$ we have $d(y_{m_k}, y_{n_k}) < \tau$ and $d(y_{m_k}, y_{n_k}) \to \tau$. Let $l > 0$ and we also $d(y_{m_k+l}, y_{n_k+l}) \to \tau$ as $k \to \infty$. By (i) and (ii) we have $\alpha(y_{m_k}, y_{n_k}) \geq 1$. Then we have

$$\begin{aligned} d(y_{m_k+l}, y_{n_k+l}) &\leq \alpha(y_{m_k}, y_{n_k}) d(f^l x_{m_k}, f^l x_{n_k})) \\ &\leq \psi_l(gx_{m_k}, gx_{n_k}) + L(d(f^l x_{m_k}, f^{l+1} x_{m_k})) \\ &+ L(d(f^l x_{n_k}, f^{l+1} x_{n_k}) \\ &= \psi_l(d(y_{m_k}, y_{n_k})) + L(d(y_{m_k+l}, y_{m_k+l+1})) \\ &+ L(d(y_{n_k+l}, y_{n_k+l+1})). \end{aligned}$$

Since f is asymptotically regular with respect to g at $x_0 \in X$, on letting $k \to \infty$, $\tau \leq \psi_l(\tau) < \tau$ a contradiction. Then $\{y_n\} = \{f^n x_0)\}$ is a Cauchy sequence. Since X is complete, there exists $p \in X$ such that z_n converges to $p \in X$. Since f is g-reciprocally continuous, $\lim_{n\to\infty} fx_n = p$ implies $\lim_{n\to\infty} gfx_n = gp$ or ($\lim_{n\to\infty} fx_n = p$ implies $\lim_{n\to\infty} ffx_n = fp$). Then $fx_n \to p$, implies $ffx_n \to fp$, and $gfx_n \to gp$. Then R-weakly commuting mappings of type A_g of f and g yields

$$d(ffx_n, gfx_n) \leq d(fx_n, gx_n).$$

On letting $n \to \infty$, we get $fp = gp$. Also R-weakly commuting mappings of type A_g of f implies commutativity at p, that is, $fgp = gfp$. Hence $ggp = gfp = fgp = ffp$. Repeating this we have $g^{n+1}p = gf^n p = f^n gp = f^{n+1}p$. By (iii) we have $\alpha(f^n p, f^n fp) \geq 1$. In fact $f^n p = g^n p$ and $f^n fp = g^n fp$, we have $\alpha(f^n p, f^n fp) \geq 1$. By (4) we have

$$\begin{aligned} d(f^n p, f^n fp) &\leq \alpha(f^{n-1}p, f^{n-1}fp) d(f^n p, f^n fp) \\ &\leq \psi_1(d(gf^{n-1}p, gf^{n-1}fp) + L[d(ff^n p, gf^n p) + d(ff^n fp, gf^n fp)]) \\ &< d(f^n p, f^n fp), \end{aligned}$$

It is a contradiction. Hence $fp = f^n fp = g^n fp$ for each $n \in N$ and fp is a common fixed point of f^n and g^n. □

Remark 20. *If we assume that f and g are commute and take $z_n = g^n x_n$, then $z_n = f^n x_0$, and it is iterative with respect to f. In fact by induction, $gx_1 = fx_0$, and $g^{n+1}x_{n+1} = g^n fx_n = fg^n x_n = ff^n x_0 = f^{n+1}x_0$.*

Theorem 21. *Adding the conditions of Theorem 18(or 19), suppose that the mapping f and g satisfying the follwing assumptions, then the coincidence point is unique and $\{f^n p\}$, $p \in X$ converges to unique common fixed point.*

(i) *for any* $x, y \in X$*, there exists* $z \in X$ *such that* $\alpha(x,z) \geq 1$ *and* $\alpha(y,z) \geq 1$*,*
(ii) $\lim_{n\to\infty} \psi^n(t) = 0$ *(resp.* $\lim_{n\to\infty} \psi_n(t) = 0$*).*

Proof. Let fp is common fixed point of f and g. Suppose fq is another common fixed point of f and g. By assumption we have $z \in X$ such that $\alpha(fp,z) \geq 1$ and $\alpha(fq,z) \geq 1$. Since f is α-admissible, we get $\alpha(fp, f^n z) \geq 1$ and $\alpha(fp, f^n z) \geq 1$, for all $n \in N$. Then

$$\begin{aligned}
d(fp, f^n z) &= d(ffp, ff^{n-1}z) \\
&\leq \alpha(fp,z)d(gfp, gf^{n-1}z)) + L(d(fp, ffp) + d(f^{n-1}z, f^n z)) \\
&\leq \psi(d(gfp, gf^{n-1}z)) + L(d(fp, ffp) + d(f^{n-1}z, f^n z)).
\end{aligned}$$

Repeaing this we have

$$d(fp, f^n z) \leq \psi^n(d(gfp, gz)),$$

By (ii), this implies that

$$d(fp, f^n z) \leq \psi^n(d(gfp, gz)) \to 0,$$

as $n \to \infty$. Then $f^n z \to fp$ as $n \to \infty$, similary, we have $f^n z \to fq$ as $n \to \infty$. The uniqueness of the limit gives us $fp = fq$.

Next if f is α-ψ_n-L contractive, we have

$$\begin{aligned}
d(fp, f^n z) &= d(f^n fp, f^n z)) \\
&\leq \alpha(gfp, gz)d(f^n fp, f^n z)) + L(d(f^n fp, g^n fp) + d(f^n z, g^n z)) \\
&\leq \psi_n(d(gfp, gz)) + L(d(f^n fp, g^n fp) + d(f^n z, g^n z)).
\end{aligned}$$

Then $f^n z \to fp$ as $n \to \infty$, similary, we have $f^n z \to fq$ as $n \to \infty$. □

Next we consider the example. First, we take the mappings f and g which are given in [1].

Example 1. *Let* $X = [2, 20]$ *and let* d *be the usual metric on* X*. Define self mappings* f *and* g *on* X *as follows;*

$$\begin{cases}
f(x) = 2 \ \textit{if}\ x = 2 \ \textit{or}\ x > 5, \\
f(x) = 6 \ \textit{if}\ 2 < x \leq 5, \\
g(2) = 2, \\
g(x) = 11 \ \textit{if}\ 2 < x \leq 5, \\
g(x) = \frac{x+1}{3} \ \textit{if}\ x > 5.
\end{cases} \tag{5}$$

In (5), f *and* g *satisfy the following conditions of Theorem 18, 19 and have a fixed point* $x = 2$ *at which* f *and* g *are discontinuous.*

(i) f *and* g *satisfy the condition* $d(fx, fy) \leq \frac{4}{5}(d(fx, gx) + d(fy, gy))$ *for* $\psi(t) = 0$ *or* $\psi_n(t) = 0$ *any* $t > 0$ *and* $x, y \in X$*;*
(ii) f *is asymptotically regular with respect to* g*,*
(iii) f *and* g *are R-weakly commuting mappings of type* A_g*, i.e.,* $d(gfx, ffx) \leq d(fx, gx)$ *for* $R = 1$*.*
(iv) f *and* g *are g-reciprocally continuous.*

And define $\alpha : X \times X \to [0, \infty)$ *as*

$$\alpha(x,y) = \begin{cases} 1, & \textit{if}\ x, y \in [2,5] \\ 0, & \textit{otherwise,} \end{cases}$$

for any $x, y \in X$.

In (5), f and g satisfy the following conditions of Theorem 8 and have a fixed point $x = 2$ at which f and g are discontinuous.

(i) f and g satisfy the condition $d(f^n x, f^n y) \leq \frac{4}{5}(d(f^n x, g^n x) + d(f^n y, g^n y))$ for $\psi_n(t) = 0$ for any $t > 0$, $n \in N$, and $x, y \in X$;
(ii) f is asymptotically regular with respect to g,
(iii) f and g are R-weakly commuting mappings of type A_g, i.e., $d(gfx, ffx) \leq d(fx, gx)$ for $R = 1$.
(iv) f and g are g-reciprocally continuous.

Remark 22. *In view of the above example one can check that in Theorem 18 (resp, Theorem 19), f and g are discontinuous at the fixed point $x = 2$ but Theorem 12 (resp. Theorem 13) demands continuity of the mapping f. Hence, Theorem 18 (resp, Theorem 19) is not a consequence of Theorem 12 (resp. Theorem 13). Therefore, coincidence point or common fixed point theorems are indeed real generalizations of their corresponding fixed point theorems.*

References

[1] R. K. Bisht, A Remark on Asymptotic Regularity and Fixed Point Property, Filomat 33:14 (2019), 4665-4671.

[2] F. E. Browder, W. V. Petryshyn, The solution by iteration of nonlinear functional equations in Banach spaces, Bulletin of the American Mathematical Society 72 (1966) 571-575.

[3] J. Górnicki, Remarks on asymptotic regularity and xed points, Journal of Fixed Point Theory and Applications 21–29 (2019) https://doi.org/10.1007/s11784-019-0668-0.

[4] G. Jungck, Common xed points for noncontinuous nonself maps on non-metric spaces, Far East J. Math. Sci. (FJMS) 4 (1996) 199215.

[5] Karapinar, E., Kumam, P., Salimi, P.: On α-ϕ-MeirKeeler contractive mappings. Fixed Point Theory Appl. 2013, 94 (2013)

[6] A. Ninsri and W. Sintunavarat, Fixed point theorems for partial - contractive mappings in generalized metric spaces, J. Nonlinear Sci. Appl. 9 (2016), 83-91.

[7] A. Pant, R. P. Pant, Orbital continuity and xed points, Filomat 31:11 (2017), 3495-3499.

[8] R. P. Pant, Common xed points of noncommuting mappings, Journal of Mathematical Analysis and Applications 188(1994) 436-440.

[9] R. P. Pant, Discontinuity and xed points, Journal of Mathematical Analysis and Applications 240 (1999) 284-289.

[10] V. Pant, R. K. Bisht, A new continuity condition and xed point theorems with applications, Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas 108 (2) (2014) 653-668.

[11] H. K. Pathak, Y. J. Cho, S. M. Kang, Remarks on R-weakly commuting mappings and common xed point theorems, Bulletin Korean Mathematical Society 34 (1997) 247-257.

[12] V. La Rosa, P. Vetro Common fixed points for α-ψ-φ-contractions in generalized metric spaces, Nonlinear Analysis: Modelling and Control, 2014, Vol. 19, No. 1, 43–54.

[13] K. P. R. Sastry, S. V. R. Naidu, I. H. N. Rao, K. P. R. Rao, Common xed points for asymptotically regular mappings, Indian Journal of Pure and Applied Mathematics 15-8(1984) 849-854.

[14] B. Samet, C. Vetro and P. Vetro, *Fixed point theorems for α-ψ contractive type mappings.* Nonlinear Anal., 75 (2012), 2154–2165.

[15] M. Toyoda and T. Watanabe, *Fixed point theorems for asymptotic mappinga of a generalized contractive type in complete metric spaces*, nihonkai math 29 (2018), 19–26.

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