Curvatures on curves passing through Whitney umbrella

Hiroyuki Hayashi

Kobe university

We consider curves passing through Whitney umbrella, which is known as a singularity appearing on stable maps from a surface to the three space. By the study of curvature of such curves, we obtain relations between curves and the differential geometric properties of Whitney umbrella.

1 Preliminaries

Definition 1.1. Two map-germs $f, g : (\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$ are said to be $\mathcal{A}$-*equivalent* if there exist diffeomorphism-germs $\phi : (\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$ and $\psi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $\psi \circ f = g \circ \phi$.

Using $\mathcal{A}$-equivalence, we define Whitney umbrella. Generally, Whitney umbrella is a singular point of maps, but we define it for a map-germ.

Definition 1.2. A map-grem $X : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is a *Whitney umbrella* (or a *cross cap*) if X is $\mathcal{A}$-equivalent to the map-germ

$$X_0 : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0), (u, v) \mapsto (u, uv, v^2)$$

at the origin.

For a Whitney umbrella, the following normal form is known.

Theorem 1.3 ([2]). *Let $W : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a Whitney umbrella. Then for any $k \geq 3$, there exist an orientation preserving diffeomorphism $\psi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ and $T \in SO(3)$ such that*

$$T \circ W \circ \psi(u, v) = \begin{pmatrix} u \\ uv + \frac{b_3}{3!}v^3 + \cdots + O(u, v)^{k+1} \\ \frac{a_{20}}{2}u^2 + a_{11}uv + \frac{a_{02}}{2}v^2 + \cdots + O(u, v)^{k+1} \end{pmatrix},$$

where $b_j \in \mathbb{R}(j \geq 3), a_{jk} \in \mathbb{R}(j + k \geq 2)$, and $a_{02} \neq 0$. Here, $O(u, v)^{k+1}$ is the terms whose dgrees are greater than or equal to $k + 1$.

In particular, the coefficients b_j, a_{jk} are all differential geometric invariants. On the other hand, the unit normal vector can not be smoothly extended at the singular point of the Whitney umbrella. A map that can smoothly extend the unit normal vector at the singular point is called frontal, and it is defined as follows.

Definition 1.4. A map-germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is a *frontal* if there exists a C^∞ map-germ $\nu : (\mathbb{R}^2, 0) \to S^2$ such that for any $p \in ((\mathbb{R}^2; u, v), 0)$,

$$\nu(p) \cdot f_u(p) = \nu(p) \cdot f_v(p) = 0$$

holds, where S^2 is the unit sphere in $\mathbb{R}^3$.

The property that a map-germ is a frontal does not depend on the $\mathcal{A}$-equivalence. Since $X_0(u, v) = (u, uv, v^2)$ satisfies

$$\frac{(X_0)_u \times (X_0)_v}{|(X_0)_u \times (X_0)_v|}(u, v) = \frac{(2v^2, -2v, u)}{\sqrt{4v^4 + 4v^2 + u^2}},$$

Whitney umbrella is not a frontal. On the other hand, the following property is known that for Whitney umbrella there is a well-defined unit normal vector after the bowing-up operation as follows ([1]).

We set $\tilde{\pi} : \mathbb{R} \times S^1 \to \mathbb{R}^2; (r, \theta) \to (r\cos\theta, r\sin\theta)$, and set $\mathcal{M}$ which is the quotient set $\mathbb{R} \times S^1/\sim$ of the relation $(r, \theta) \sim (-r, \theta + \pi)$. Considering

$$\pi : \mathcal{M} \to \mathbb{R}^2, [(r, \theta)] \mapsto (r\cos\theta, r\sin\theta),$$

one can see

$$\frac{((X_0)_u \times (X_0)_v) \circ \pi}{|((X_0)_u \times (X_0)_v) \circ \pi|}(0, \theta) = \frac{1}{\sqrt{4\sin^2\theta + \cos^2\theta}} \begin{pmatrix} 0 \\ -2\sin\theta \\ \cos\theta \end{pmatrix}.$$

This implies that by composing π, the unit normal vector field is well-defined near a Whitney umbrella on $\mathcal{M}$. This property strongly suggests that if we fix the angle of incidence of a Whitney umbrella, a smooth unit normal vector is well-defined. We consider a curve passing through $(0, 0)$ which satisfies the following general enough condition.

Definition 1.5. A C^∞ curve-germ $\mathbf{c} : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ is said to be of *finite multiplicity* if there exist a positive integer m and a C^∞-germ $\bar{\mathbf{c}} : (\mathbb{R}, 0) \to \mathbb{R}^2$ such that

$$\mathbf{c}(x) = x^m \bar{\mathbf{c}}(x), \bar{\mathbf{c}}(0) \neq 0.$$

We consider curves of finite multiplicity passing through a Whitney umbrella. Then, we obtain the following lemma.

Lemma 1.6. *Let $\mathbf{c} : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ be a curve of finite multiplicity, and let $W : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a Whitney umbrella. Then the unit normal vector field of W along $W \circ \mathbf{c}$ can be smoothly extended across the Whitney umbrella.*

Similarly, we obtain the following lemma.

Lemma 1.7. *Let* $\mathbf{c} : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ *be a curve of finite multiplicity, and let* $W : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ *be a Whitney umbrella. Consider the composition* $W \circ \mathbf{c}$*. Then the unit tangent vector field of* $W \circ \mathbf{c}$ *can be smoothly extended across the Whitney umbrella.*

Example 1.8. Let us set $Y_0(x) = (x, x)$ and $X_0(u, v) = (u, uv, -v^2)$. We consider the composition $X_0 \circ Y_0 : (\mathbb{R}, 0) \to (\mathbb{R}^3, 0)$. Then we can take a unit normal vector field $\mathbf{n}_0$ and a unit tangent vector field $\mathbf{e}_0$ as

$$\mathbf{n}_0(x) = \frac{1}{\sqrt{4x^2+5}} \begin{pmatrix} -2x \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{e}_0(x) = \frac{1}{\sqrt{8x^2+1}} \begin{pmatrix} 1 \\ 2x \\ -2x \end{pmatrix}$$

along $X_0 \circ Y_0$.

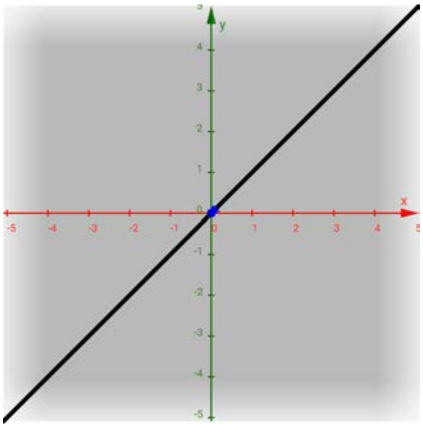

Figure 1.1: $Y_0(x)$

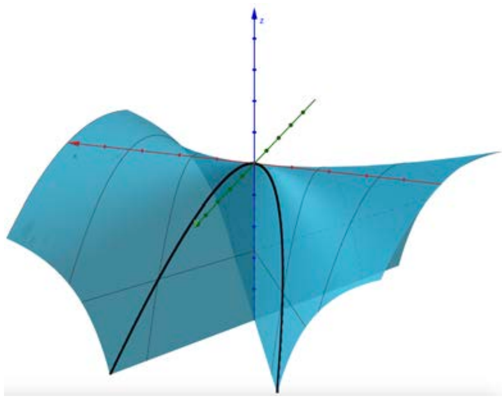

Figure 1.2: $Y_0 \circ X_0(x)$

2 Construction of frame

By the previous section, we have the following lemma.

Lemma 2.1. *The vectors*

$$\frac{(W \circ \mathbf{c})'}{|(W \circ \mathbf{c})'|}(x), \frac{(W_u \times W_v) \circ \mathbf{c}}{|(W_u \times W_v) \circ \mathbf{c}|}(x)$$

are well-defined across the singular point $(x = 0)$ *by taking an appropriate reduction. Setting* $\mathbf{b}(x) = \mathbf{n}(x) \times \mathbf{e}(x)$*, one obtain a Darboux frame* $\{\mathbf{e}(x), \mathbf{b}(x), \mathbf{n}(x)\}$ *along* $W \circ \mathbf{c}$*.*

We assume the condition

$$\operatorname{rank}(\mathbf{c}(0), \mathbf{c}'(0), \mathbf{c}''(0), \cdots) = 2$$

for a curve $\mathbf{c}(x)$. Then we can set

$$\mathbf{c}(x) = \begin{pmatrix} d_1(x)x^{m_1} \\ d_2(x)x^{m_2} \end{pmatrix}, \quad d_1(0) \neq 0,\ d_2(0) \neq 0,$$

where $m_1,\ m_2 \in \mathbb{Z}_{>0}, d_1, d_2 : (\mathbb{R}, 0) \to \mathbb{R}$. The following is an example of $\{\mathbf{e}(x), \mathbf{b}(x), \mathbf{n}(x)\}$.

Example 2.2. We set

$$m_1 = 2m_2,\ d_2 = 1,\ d_1(x) = \sum_{i=0} c_i x^i,$$

then $(W \circ \mathbf{c})'(x)$ and $(W_u \times W_v) \circ \mathbf{c}(x)$ satisfy

$$(W \circ \mathbf{c})'(x) = \begin{pmatrix} \{2m_2c_0 + O(x)\} \\ \{3m_2c_0 + \frac{1}{2}m_2b_3 + O(x)\}x^{m_2} \\ \{m_2a_{02} + O(x)\} \end{pmatrix} x^{2m_2-1} = \mathcal{A}(x)x^{2m_2-1},$$

$$(W_u \times W_v) \circ \mathbf{c}(x) = \begin{pmatrix} \{a_{02} + O(x)\}x^{m_2} \\ \{-a_{02} + O(x)\} \\ \{c_0 + \frac{b_3}{2} + O(x)\}x^{m_2} \end{pmatrix} x^{m_2} = \mathcal{B}(x)x^{m_2},$$

where $|\mathcal{A}(0)| \neq 0, |\mathcal{B}(0)| \neq 0$. Therefore, we have

$$\mathbf{e}(x) = \frac{\mathcal{A}}{|\mathcal{A}|} = \frac{1}{|\mathcal{A}|} \begin{pmatrix} e_1(x) \\ e_2(x)x^{m_2} \\ e_3(x) \end{pmatrix}, \begin{cases} e_1(0) = 2m_2c_0, \\ e_2(0) = m_2(3c_0 + \frac{1}{2}b_3), \\ e_3(0) = m_2a_{02}, \end{cases}$$

$$\mathbf{b}(x) = \frac{1}{|\mathcal{A}||\mathcal{B}|} \begin{pmatrix} b_1(x) \\ b_2(x)x^{m_2} \\ b_3(x) \end{pmatrix}, \begin{cases} b_1(0) = -m_2{a_{02}}^2, \\ b_2(0) = m_2(2{c_0}^2 + b_3c_0 - {a_{02}}^2), \\ b_3(0) = 2m_2a_{02}c_0, \end{cases}$$

$$\mathbf{n}(x) = \frac{\mathcal{B}}{|\mathcal{B}|} = \frac{1}{|\mathcal{B}|} \begin{pmatrix} n_1(x)x^{m_2} \\ n_2(x) \\ n_3(x)x^{m_2} \end{pmatrix}, \begin{cases} n_1(0) = a_{02}, \\ n_2(0) = -a_{02}, \\ n_3(0) = c_0 + \frac{1}{2}b_3. \end{cases}$$

3 Curvature

We define functions $\kappa_1, \kappa_2, \kappa_3$ by

$$\frac{d}{dx}\begin{pmatrix} \mathbf{e}(x) \\ \mathbf{b}(x) \\ \mathbf{n}(x) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1(x) & \kappa_2(x) \\ -\kappa_1(x) & 0 & \kappa_3(x) \\ -\kappa_2(x) & -\kappa_3(x) & 0 \end{pmatrix}\begin{pmatrix} \mathbf{e}(x) \\ \mathbf{b}(x) \\ \mathbf{n}(x) \end{pmatrix}.$$

We see the relations between $\kappa_1, \kappa_2, \kappa_3$ and the geodesic curvature κ_g, the normal curvature κ_ν, and the geodesic torsion κ_t which are defined on a set of regular points.

Lemma 3.1. *We set*

$$\gamma'(x) = \mathcal{A}(x)x^{\alpha}, \mathcal{A}(0) \neq 0,$$
$$(W_u \times W_v) \circ \mathbf{c}(x) = \mathcal{B}(x)x^{\beta}, \mathcal{B}(0) \neq 0.$$

Then it holds that

$$\begin{aligned}
\kappa_1(x) &= \operatorname{sgn}(x^{\alpha+\beta})|\mathcal{A}(x)|x^{\alpha}\kappa_g(x),\\
\kappa_2(x) &= \operatorname{sgn}(x^{\beta})|\mathcal{A}(x)|x^{\alpha}\kappa_{\nu}(x),\\
\kappa_3(x) &= |\mathcal{A}(x)|x^{\alpha}\kappa_t(x).
\end{aligned}$$

By lemma 3.1, we can see a boundedness of $\kappa_g, \kappa_\nu, \kappa_t$ if we give degrees and top terms of $\kappa_1, \kappa_2, \kappa_3$. We consider a curve $\mathbf{c}(x)$ such that

$$\mathbf{c}'(0) \in \operatorname{Ker} dW.$$

Then $\mathbf{c}(x)$ satisfies

$$\mathbf{c}(x) = \begin{pmatrix} c(x)x^{m_1} \\ x^{m_2} \end{pmatrix}, c(0) \neq 0, m_1 > m_2.$$

Therefore, we set

$$\mathbf{c}(x) = \begin{pmatrix} c(x)x^{mp+q} \\ x^{m} \end{pmatrix}, 1 \leq p, 1 \leq q < m, \tag{3.1}$$

or

$$\mathbf{c}(x) = \begin{pmatrix} c(x)x^{mp} \\ x^{m} \end{pmatrix}, 2 \leq p \tag{3.2}$$

where $c(x) = c_0 + c_1 x + c_2 x^2 + \cdots,\ c_0 \neq 0$.

Theorem 3.2. *We assume that* $\mathbf{c}(x)$ *satisfies* (3.1)*. Then it holds that*

$$\kappa_1(x) = \begin{cases} \tilde{\kappa}_1(x)x^{m-q-1} & (p = 1),\\ \tilde{\kappa}_1(x)x^{m(p-2)+q-1} & (2 \leq p < 4),\\ \tilde{\kappa}_1(x)x^{2m-1} & (4 \leq p), \end{cases}$$

$$\kappa_2(x) = \tilde{\kappa}_2(x)x^{m-1}$$

$$\kappa_3(x) = \begin{cases} \tilde{\kappa}_3(x)x^{q-1} & (p = 1),\\ \tilde{\kappa}_3(x)x^{m-1} & (2 \leq p). \end{cases}$$

Theorem 3.3. *We assume that* $\mathbf{c}(x)$ *satisfies* (3.2). *Then it holds that*

$$\kappa_1(x) = \begin{cases} \tilde{\kappa}_1(x)x^0 & (p=2), \\ \tilde{\kappa}_1(x)x^{m-1} & (p=3), \\ \tilde{\kappa}_1(x)x^{2m-1} & (4 \le p), \end{cases}$$

$$\kappa_2(x) = \tilde{\kappa}_2(x)x^{m-1},$$
$$\kappa_3(x) = \tilde{\kappa}_3(x)x^{m-1}.$$

In particular, if

$$\mathbf{c}(x) = \begin{pmatrix} c(x)x^{2m} \\ x^m \end{pmatrix},$$

the top terms of $\tilde{\kappa}_1, \tilde{\kappa}_2, \tilde{\kappa}_3$ satisfy

$$\tilde{\kappa}_1(x) = \begin{cases} \frac{1}{|\mathcal{A}|^2|\mathcal{B}|}a_{02}(6a_{11}c_0^2 + a_{03}c_0 - 3a_{02}c_1) + O(x)(m_2 = 1), \\ -\frac{1}{|\mathcal{A}|^2|\mathcal{B}|}m_2(2m_2+1)a_{02}^2c_1 + O(x)(2 \le m_2), \end{cases} \tag{3.3}$$

$$\tilde{\kappa}_2(x) = -\frac{1}{|\mathcal{A}||\mathcal{B}|}m_2^2 a_{02}\left(3c_0 + \frac{b_3}{2}\right) + O(x), \tag{3.4}$$

$$\tilde{\kappa}_3(x) = -\frac{1}{|\mathcal{A}||\mathcal{B}|^2}m_2^2 a_{02}(2c_0^2 + b_3c_0 - a_{02}^2) + O(x). \tag{3.5}$$

If we assume these top terms vanish, then we have the following relations in (3.3), (3.4), and (3.5),

$$6a_{11}c_0^2 + a_{03}c_0 - 3a_{02}c_1 = 0, \tag{3.6}$$

$$3c_0 + \frac{b_3}{2} = 0, \tag{3.7}$$

$$2c_0^2 + b_3c_0 - a_{02}^2 = 0, \tag{3.8}$$

where c_0, c_1 are differential geometric invariants of curve $\mathbf{c}$, $a_{02}, a_{11}, a_{03}, b_3$ are differential geometric invariants of Whitney umbrella W. When a Whitney umbrella and a curve satisfy these relations, the degrees of $\kappa_1, \kappa_2, \kappa_3$ are at least greater than 1 for the non-zero cases. Thus we can consider that these relations have a geometric meaning.

Example 3.4. Let us set

$$\mathbf{c}^1(x) = \begin{pmatrix} x^2 \\ x \end{pmatrix}, \tag{3.9}$$

$$W_1(u,v) = \begin{pmatrix} u \\ uv + \frac{b_3}{6}v^3 \\ a_{11}uv + \frac{a_{02}}{2}v^2 + \frac{a_{03}}{6}v^3 \end{pmatrix}. \tag{3.10}$$

We consider the composition $W_1 \circ \mathbf{c}^1$. We give examples of the cases

(1) $\tilde{\kappa}_1(0) \neq 0,\ \tilde{\kappa}_2(0) \neq 0,\ \tilde{\kappa}_3(0) \neq 0,$

(2) $\tilde{\kappa}_1(0) = 0, \ \tilde{\kappa}_2(0) \neq 0, \ \tilde{\kappa}_3(0) \neq 0,$

(3) $\tilde{\kappa}_1(0) \neq 0, \ \tilde{\kappa}_2(0) = 0, \ \tilde{\kappa}_3(0) \neq 0,$

(4) $\tilde{\kappa}_1(0) \neq 0, \ \tilde{\kappa}_2(0) \neq 0, \ \tilde{\kappa}_3(0) = 0,$

by

(1) $(b_3, a_{11}, a_{02}, a_{03}) = (1, -1, -2, -1),$

(2) $(b_3, a_{11}, a_{02}, a_{03}) = (1, 1/6, -2, -1),$

(3) $(b_3, a_{11}, a_{02}, a_{03}) = (-6, -1, -4, -1),$

(4) $(b_3, a_{11}, a_{02}, a_{03}) = (2, -1, -2, -1).$

The figures of W_1 with $W_1 \circ \mathbf{c}^1$ of the cases (1) - (4) are given in Figure 3.1 - 3.4.

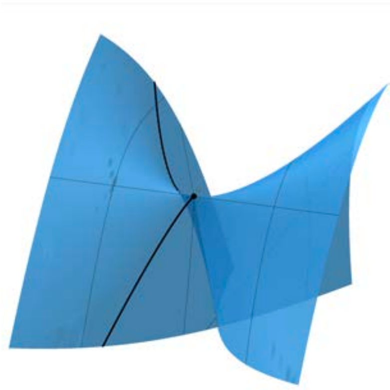

Figure 3.1: $\tilde{\kappa}_1(0) \neq 0, \tilde{\kappa}_2(0) \neq 0, \tilde{\kappa}_3(0) \neq 0$

Figure 3.2: $\tilde{\kappa}_1(0) = 0, \tilde{\kappa}_2(0) \neq 0, \tilde{\kappa}_3(0) \neq 0$

Figure 3.3: $\tilde{\kappa}_1(0) \neq 0, \tilde{\kappa}_2(0) = 0, \tilde{\kappa}_3(0) \neq 0$

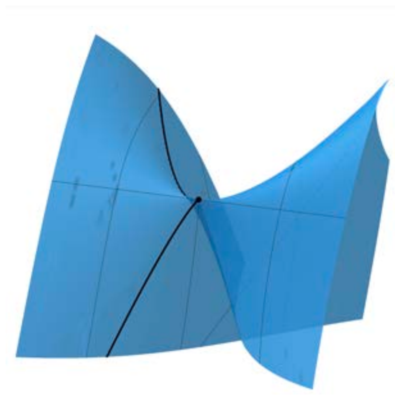

Figure 3.4: $\tilde{\kappa}_1(0) \neq 0, \tilde{\kappa}_2(0) \neq 0, \tilde{\kappa}_3(0) = 0$

Example 3.5. Let us set

$$\mathbf{c}^2(x) = \begin{pmatrix} x^4 + x^5 \\ x^2 \end{pmatrix}, \tag{3.11}$$

We consider the composition $W_1 \circ \mathbf{c}^2$. We give examples of the cases

(1) $\tilde{\kappa}_1(0) \neq 0,\ \tilde{\kappa}_2(0) \neq 0,\ \tilde{\kappa}_3(0) \neq 0,$

(2) $\tilde{\kappa}_1(0) = 0,\ \tilde{\kappa}_2(0) \neq 0,\ \tilde{\kappa}_3(0) \neq 0,$

(3) $\tilde{\kappa}_1(0) \neq 0,\ \tilde{\kappa}_2(0) = 0,\ \tilde{\kappa}_3(0) \neq 0,$

(4) $\tilde{\kappa}_1(0) \neq 0,\ \tilde{\kappa}_2(0) \neq 0,\ \tilde{\kappa}_3(0) = 0,$

by

(1) $(b_3, a_{11}, a_{02}, a_{03}) = (1, -1, -2, -1),$

(2) $(b_3, a_{11}, a_{02}, a_{03}) = (1, -1, -2, -0),$

(3) $(b_3, a_{11}, a_{02}, a_{03}) = (-6, -1, -4, -1),$

(4) $(b_3, a_{11}, a_{02}, a_{03}) = (2, -1, -2, -1).$

The figures of W_1 with $W_1 \circ \mathbf{c}^2$ of the cases (1) - (4) are given in Figure 3.5 - 3.8.

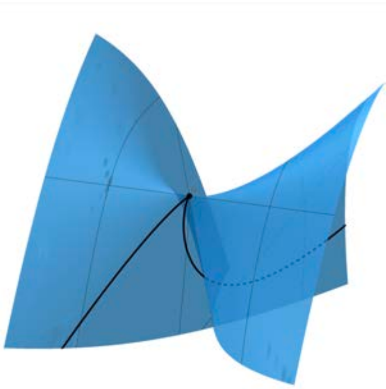

Figure 3.5: $\tilde{\kappa}_1(0) \neq 0, \tilde{\kappa}_2(0) \neq 0, \tilde{\kappa}_3(0) \neq 0$

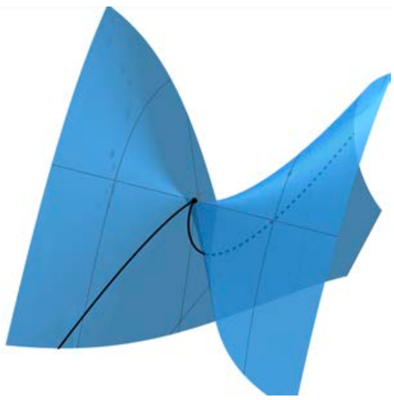

Figure 3.6: $\tilde{\kappa}_1(0) = 0, \tilde{\kappa}_2(0) \neq 0, \tilde{\kappa}_3(0) \neq 0$

Figure 3.7: $\tilde{\kappa}_1(0) \neq 0, \tilde{\kappa}_2(0) = 0, \tilde{\kappa}_3(0) \neq 0$

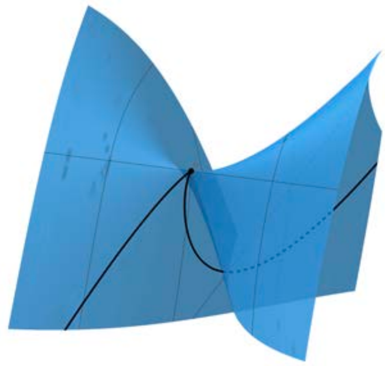

Figure 3.8: $\tilde{\kappa}_1(0) \neq 0, \tilde{\kappa}_2(0) \neq 0, \tilde{\kappa}_3(0) = 0$

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Department of Mathematics Graduate school of science
Kobe University
Kobe 657-8501
JAPAN
E-mail address: 224s017s@cloud.kobe-u.jp