Maximal L^1 regularity for the compressible Stokes equations

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1 Introduction

Let $\mathbb{R}^N_+ := \{x = (x', x_N) \in \mathbb{R}^N \mid x' \in \mathbb{R}^{N-1},\ x_N > 0\}$, $N \geq 2$, be the half space. In this paper, we consider the following linear system of the compressible Stokes system with homogeneous Dirichlet boundary conditions:

$$\begin{cases} \partial_t \rho + \gamma \operatorname{div} \mathbf{u} = 0 & \text{in } \mathbb{R}^N_+, \\ \partial_t \mathbf{u} - \alpha \Delta \mathbf{u} - \beta \nabla \operatorname{div} \mathbf{u} + \gamma \nabla \rho = 0 & \text{in } \mathbb{R}^N_+, \\ \mathbf{u} = 0 & \text{on } \partial\mathbb{R}^N_+, \\ (\rho, \mathbf{u})(0, x) = (\rho_0, \mathbf{u}_0) & \text{in } \mathbb{R}^N_+. \end{cases} \tag{1.1}$$

Here, ρ and $\mathbf{u} = (u_1, \cdots, u_N)$ are respective unkown density and velocity functions, while the initial datum $(\rho_0, \mathbf{u}_0)$ is assumed to be given. Moreover, the coefficients α, β, and γ are assumed to be constants such that $\alpha + \beta > 0$ and $\gamma > 0$. The aim of this paper is to show the generation of a continuous analytic semigroup associated with equations (1.1) and its L_1 in time maximal regularity property in Besov spaces $\mathcal{H}^s_{q,r} = B^{s+1}_{q,r}(\mathbb{R}^N_+) \times B^s_{q,r}(\mathbb{R}^N_+)^N$, where $1 < q < \infty$, $1 \leq r < \infty$, and $-1 + 1/q < s < 1/q$. Our approach is to prove the existence of the solution in $\mathcal{H}^s_{q,1}$ and from real interpolation, combine some new estimates for the resolvent problem by using $B^{s+1}_{q,1}(\mathbb{R}^N_+) \times B^{s\pm\sigma}_{q,1}(\mathbb{R}^N_+)$ norms for some small $\sigma > 0$ satisfying the condition $-1 + 1/q < s - \sigma < s < s + \sigma < 1/q$.

The system (1.1) is the linearized system of the compressible Navier-Stokes equations with homogeneous Dirichlet boundary conditions:

$$\begin{cases} \partial_t \varrho + \operatorname{div}(\varrho \mathbf{v}) = 0 & \text{in } \mathbb{R}^N_+, \\ \varrho(\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}) - \mu \Delta \mathbf{v} - (\mu + \nu) \nabla \operatorname{div} \mathbf{v} + \nabla P(\varrho) = 0 & \text{in } \mathbb{R}^N_+, \\ \mathbf{v} = 0 & \text{on } \partial\mathbb{R}^N_+, \\ (\varrho, \mathbf{v})(0, x) = (\varrho_0, \mathbf{v}_0) & \text{in } \mathbb{R}^N_+, \end{cases}$$

where ϱ and $\mathbf{v}$ describe the unknown density and the velocity field of the compressible viscous field, respectively, while the initial datum $(\varrho_0, \mathbf{v}_0)$ is a pair of given functions. The coefficients μ and ν are assumed to satisfy the ellipticity conditions $\mu > 0$ and $\mu + \nu > 0$. In addition, the

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pressure of the fluid P is a given smooth function with respect to ϱ, which is assumed to satisfy the stability condition $P'(\rho_*) > 0$. Here, ρ_* stands for the reference density that is a postitve constant, and the initial density ϱ_0 is given as a perturbation from ρ_*. As discussed in [4, Sec.8], the coefficients α, β, and γ are defined by $\alpha = \mu/\rho_*$, $\beta = \nu/\rho_*$, and $\gamma = \sqrt{P'(\rho_*)}$, respectively. Clearly, α, β, and γ satisfy the aforementioned given conditions.

Since the 1950s, lots of mathematicians have contributed to the research on the long time behaviors of the global solution of compressible Newtonian fluids. Matsumura and Nishida [6] proved a unique global-in-time solution for the initial value problem of the compressible Navier-Stokes equations with heat-conductive effects in the three-dimensional whole space with the help of a local existence theorem together with the a priori estimates for the solution. More precisely, the a priori estimate were constructed by the linear spectral theory and the L_2-energy method. This result was established in the half space and exterior domains cases by the same authors, see [8]. After that, semigroup approach has been establised. Ströhmer [13] proved the global well-posedness by the semigroup theory. He formulated the system in Lagrange coordinates. By doing so, the convection term $\varrho \cdot \nabla \mathbf{v}$ can be eliminated and the transformed system can be regarded as a pure parabolic type system. Therefore, the derivative loss from the mass conservation equation vanishes. Recently, the Cauchy problem of the compressible Navier-Stokes equations has been studied in terms of the maximal L^p regularity $(1 < p < \infty)$ approach and Enomoto and Shibata [4] obtained the global existence result in a bounded smooth domain with a wider class for the initial data. The strategy in their study is to prove the existence of solution operators of the generalized resolvent problem and to apply the operator-valued Fourier multiplier theorem due to Weis [15]. On the other hand, for the endpoint case $p = 1$, namely the maximal L^1 regularity. Danchin [2] proved the local well-posedness in bounded domains, and Danchin and Tolksdorf [3] for the global well-posedness in bounded domains. Here, they studied the system in the L_1 in time and $B^s_{p,1}$ in space framework, where p and s are taken such that $1 < p < \infty$ and $s = -1 + N/p$.

1.1 Notations

Let us summerize the notations and functional spaces in this paper. Let $\mathbb{R}$, $\mathbb{N}$, and $\mathbb{C}$ be the set of all real, natural, complex numbers, respectively, while let $\mathbb{Z}$ be the set of all integers. Besides, $\mathbb{K}$ stands for either $\mathbb{R}$ or $\mathbb{C}$. Set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

For $N \in \mathbb{N}$ and a Banach space X, let $\mathcal{S}(\mathbb{R}^N; X)$ be the Schwartz class of X-valued rapidly decreasing functions on $\mathbb{R}^N$. We denote $\mathcal{S}'(\mathbb{R}^N; X)$ by the space of X-valued tempered distributions, which means the set of all continuous linear mappings from $\mathcal{S}(\mathbb{R}^N)$ to X. For $N \in \mathbb{N}$, we define the Fourier transform $f \mapsto \mathcal{F}[f]$ from $\mathcal{S}(\mathbb{R}^N; X)$ onto itself and its inverse as

$$\mathcal{F}[f](\xi) := \int_{\mathbb{R}^N} f(x) e^{-ix\cdot\xi} \, dx, \qquad \mathcal{F}^{-1}_\xi[g](x) := \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} g(\xi) e^{ix\cdot\xi} \, d\xi,$$

respectively. In addition, we define the partial Fourier transform $\mathcal{F}'[f(\,\cdot\,, x_N)] = \hat{f}(\xi', x_N)$ and partial inverse Fourier transform $\mathcal{F}^{-1}_{\xi'}$ by

$$\mathcal{F}'[f(\,\cdot\,, x_N)](\xi') := \hat{f}(\xi', x_N) = \int_{\mathbb{R}^{N-1}} f(x', x_N) e^{-ix'\cdot\xi'} \, dx',$$
$$\mathcal{F}^{-1}_{\xi'}[g(\,\cdot\,, x_N)](x') := \frac{1}{(2\pi)^{N-1}} \int_{\mathbb{R}^{N-1}} g(\xi', x_N) e^{ix'\cdot\xi'} \, d\xi',$$

where we have set $x' = (x_1, \cdots, x_{N-1}) \in \mathbb{R}^{N-1}$ and $\xi' = (\xi_1, \cdots, \xi_{N-1}) \in \mathbb{R}^{N-1}$. For $N \geq 2$, we set $(\mathbf{f}, \mathbf{g})_{\mathbb{R}^N_+} = \int_{\mathbb{R}^N_+} \mathbf{f}(x) \cdot \mathbf{g}(x) \, dx$ for N-vector functions $\mathbf{f}$ and $\mathbf{g}$ on $\mathbb{R}^N_+$, where we will write $(\mathbf{f}, \mathbf{g}) = (\mathbf{f}, \mathbf{g})_{\mathbb{R}^N_+}$ for short if there is no confusion. Given function f, define $\bar{\nabla} f = (f, \nabla f)$,

$\bar{\nabla}^2 f = (f, \nabla f, \nabla^2 f)$. By $C > 0$ we will often denote a generic constant that does not depend on the quantities at stake.

Next, we introduce some function spaces on $\mathbb{R}^N$ and $\mathbb{R}^N_+$. In the following, let $s \in \mathbb{N}$ and $p \in (1, \infty)$. Bessel potential spaces $H^s_p(\mathbb{R}^N)$ are defined as the set of all $f \in \mathcal{S}'(\mathbb{R}^N)$ such that $\|f\|_{H^s_p(\mathbb{R}^N)} < \infty$, where the norm $\|\cdot\|_{H^s_p(\mathbb{R}^N)}$ is defined by

$$\|f\|_{H^s_p(\mathbb{R}^N)} := \left\|\mathcal{F}^{-1}_\xi \left[(1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F}[f](\xi)\right]\right\|_{L_p(\mathbb{R}^N)}.$$

It is well-known that, if $s = m \in \mathbb{N}_0$, then $H^s_p(\mathbb{R}^N)$ coincides with the classical Sobolev space $W^m_p(\mathbb{R}^N)$, see, e.g., [1, Theorem 3.7].

To define inhomogeneous Besov spaces, we need to introduce Littlewood-Paley decomposition. Let $\phi \in \mathcal{S}(\mathbb{R}^N)$ with $\operatorname{supp}\phi = \{\xi \in \mathbb{R}^N \mid 1/2 \le |\xi| \le 2\}$ such that $\sum_{k\in\mathbb{Z}} \phi(2^{-k}\xi) = 1$ for all $\xi \in \mathbb{R}^N \setminus \{0\}$. Then, define

$$\phi_k := \mathcal{F}^{-1}_\xi[\phi(2^{-k}\xi)], \quad k \in \mathbb{Z}, \qquad \psi = 1 - \sum_{k\in\mathbb{N}} \phi(2^{-k}\xi).$$

For $1 \le p, q \le \infty$ and $s \in \mathbb{R}$ we denote

$$\|f\|_{B^s_{p,q}(\mathbb{R}^N)} := \begin{cases} \|\psi * f\|_{L_p(\mathbb{R}^N)} + \left(\displaystyle\sum_{k\in\mathbb{N}} \left(2^{sk}\|\phi_k * f\|_{L_p(\mathbb{R}^N)}\right)^q\right)^{1/q} & \text{if } 1 \le q < \infty, \\ \|\psi * f\|_{L_p(\mathbb{R}^N)} + \displaystyle\sup_{k\in\mathbb{N}} \left(2^{sk}\|\phi_k * f\|_{L_p(\mathbb{R}^N)}\right) & \text{if } q = \infty. \end{cases}$$

Here, $f * g$ means the convolution between f and g. Then inhomogeneous Besov spaces $B^s_{p,q}(\mathbb{R}^N)$ are defined as the sets of all $f \in \mathcal{S}'(\mathbb{R}^N)$ such that $\|f\|_{B^s_{p,q}(\mathbb{R}^N)} < \infty$.

Let $\mathcal{D}'(\mathbb{R}^N_+)$ be the collection of all complex-valued distributions on $\mathbb{R}^N_+$. Let $s \in \mathbb{R}$, $p \in (1, \infty)$, and $q \in [1, \infty]$. Then for any $X \in \{H^s_p, B^s_{p,q}\}$, the space $X(\mathbb{R}^N_+)$ is the collection of all $f \in \mathcal{D}'(\mathbb{R}^N_+)$ such that there exists a function $g \in X(\mathbb{R}^N)$ with $g|_{\mathbb{R}^N_+} = f$. Moreover, the norm of $f \in X(\mathbb{R}^N_+)$ is given by

$$\|f\|_{X(\mathbb{R}^N_+)} = \inf \|g\|_{X(\mathbb{R}^N)},$$

where the infimum is taken over all $g \in X(\mathbb{R}^N)$ such that its restriction $g|_{\mathbb{R}^N_+}$ coincides in $\mathcal{D}'(\mathbb{R}^N_+)$ with f. We also define

$$X_0(\mathbb{R}^N_+) := \{f \in X(\mathbb{R}^N) \mid \operatorname{supp} f \subset \overline{\mathbb{R}^N_+}\}.$$

Clearly, we always have $X_0(\mathbb{R}^N_+) \hookrightarrow X(\mathbb{R}^N_+)$.

According to [14, Section 2.9], for $s \in \mathbb{R}$, $p \in (1, \infty)$, and $q \in [1, \infty)$, we have the following density result:

$$X_0(\mathbb{R}^N_+) = \overline{C^\infty_0(\mathbb{R}^N_+)}^{\|\cdot\|_{X(\mathbb{R}^N)}}.$$

Here, $X(\mathbb{R}^N_+)$ and $X_0(\mathbb{R}^N_+)$ may coincide if one restricts s such that $-1 + 1/p < s < 1/p$.

Proposition 1.1. *Let $1 < p < \infty$, $1 \le q \le \infty$, and $-1 + 1/p < s < 1/p$. Then $H^s_p(\mathbb{R}^N_+) = H^s_{p,0}(\mathbb{R}^N_+)$ as well as $B^s_{p,q}(\mathbb{R}^N_+) = B^s_{p,q,0}(\mathbb{R}^N_+)$.*

Proposition 1.2. *Let $p \in (1, \infty)$. Then the following assertions are valid.*

(1) *For $s \in \mathbb{R}$, there holds*

$$(H^s_{p,0}(\mathbb{R}^N_+))' = H^{-s}_{p'}(\mathbb{R}^N_+).$$

(2) *For $-\infty < s \le 1/p$, there holds*

$$(H_p^s(\mathbb{R}_+^N))' = H_{p',0}^{-s}(\mathbb{R}_+^N).$$

(3) *For $-\infty < s_0 < s_1 < \infty$, $1 < p < \infty$, $1 \le q \le \infty$, and $0 < \theta < 1$. there holds*

$$B_{p,q}^{\theta s_0 + (1-\theta)s_1}(\mathbb{R}_+^N) = \big(H_p^{s_0}(\mathbb{R}_+^N), H_p^{s_1}(\mathbb{R}_+^N)\big)_{\theta,q}.$$

1.2 Main theorems

Let $1 < q < \infty$, $-1 + 1/q < s < 1/q$, $1 \le r < \infty$, and $\Omega \in \{\mathbb{R}^N, \mathbb{R}_+^N\}$. Let $B_{q,r}^{\mu}(\Omega)$ denote standard Besov spaces on a domain $\Omega \in \mathbb{R}^N$. Let

$$\begin{aligned}
\mathcal{H}_{q,r}^s(\Omega) &= B_{q,r}^{s+1}(\Omega) \times B_{q,r}^s(\Omega)^N, \\
\mathcal{D}_{q,r}^s(\mathbb{R}^N) &= B_{q,r}^{s+1}(\mathbb{R}^N) \times B_{q,r}^{s+2}(\mathbb{R}^d), \\
\mathcal{D}_{q,r}^s(\mathbb{R}_+^N) &= \{(\rho, \mathbf{v}) \in B_{q,r}^{s+1}(\mathbb{R}_+^N) \times B_{q,r}^{s+2}(\mathbb{R}_+^N)^N \mid \mathbf{v}|_{\partial\mathbb{R}_+^N} = 0\}, \\
\|(f, \mathbf{g})\|_{\mathcal{H}_{q,r}^s(\Omega)} &= \|f\|_{B_{q,r}^{s+1}(\Omega)} + \|\mathbf{g}\|_{B_{q,r}^s(\Omega)}, \\
\|(f, \mathbf{g})\|_{\mathcal{D}_{q,r}^s(\Omega)} &= \|f\|_{B_{q,r}^{s+1}(\Omega)} + \|\mathbf{g}\|_{B_{q,r}^{s+2}(\Omega)}.
\end{aligned} \tag{1.2}$$

In addition, we introduce the operator $\mathcal{A}$ corresponding to equations (1.1) which is defined by setting

$$\mathcal{A}(\rho, \mathbf{v}) = (\gamma \operatorname{div} \mathbf{v}, -\alpha\Delta\mathbf{v} - \beta\nabla \operatorname{div} \mathbf{v} + \gamma\nabla\rho) \quad \text{for } (\rho, \mathbf{v}) \in \mathcal{D}_{q,r}^s(\mathbb{R}_+^N). \tag{1.3}$$

Using $\mathcal{A}$, equations (1.1) are written as

$$\partial_t(\rho, \mathbf{u}) + \mathcal{A}(\rho, \mathbf{u}) = (0, 0) \quad \text{for } t > 0, \quad (\rho, \mathbf{u})|_{t=0} = (\rho_0, \mathbf{u}_0) \in \mathcal{H}_{q,r}^s(\mathbb{R}_+^N) \tag{1.4}$$

for $(\rho, \mathbf{u})$ with

$$(\rho, \mathbf{u}) \in C^0[(0, \infty), \mathcal{H}_{q,r}^s(\mathbb{R}_+^N) \cap C^1((0, \infty), \mathcal{H}_{q,r}^s(\mathbb{R}_+^N)) \cap C^0((0, \infty), \mathcal{D}_{q,r}^s(\mathbb{R}_+^N)).$$

Our main results of this paper read as follows.

Theorem 1.3. *Let $1 < q < \infty$, $-1 + 1/q < s < 1/q$, and $1 \le r < \infty$. Then, the operator $\mathcal{A}$ generates a continuous analytic semigroup $\{T(t)\}_{t\ge0}$ on $\mathcal{H}_{q,r}^s(\mathbb{R}_+^N)$.*

Moreover, there exists a large $\gamma_0 \ge 1$ such that, for any $\gamma \ge \gamma_0$ and $(f, \mathbf{g}) \in \mathcal{H}_{q,1}^s(\mathbb{R}_+^N)$,

$$\int_0^\infty e^{-\gamma t}\|T(t)(f, \mathbf{g})\|_{\mathcal{D}_{q,1}^s(\mathbb{R}_+^N)}dt \le C\|(f, \mathbf{g})\|_{\mathcal{H}_{q,1}^s(\mathbb{R}_+^N)}.$$

To prove Theorem 1.3, we consider the following resolvent problem:

$$\begin{cases} \lambda\rho + \gamma \operatorname{div} \mathbf{u} = f & \text{in } \mathbb{R}_+^N, \\ \lambda\mathbf{u} - \alpha\Delta\mathbf{u} - \beta\nabla \operatorname{div} \mathbf{u} + \gamma\nabla\rho = \mathbf{g} & \text{in } \mathbb{R}_+^N, \\ \mathbf{u} = 0 & \text{on } \partial\mathbb{R}_+^N, \end{cases} \tag{1.5}$$

for $\lambda \in \Lambda_{\epsilon,\nu_0}$. Here, Λ_{ϵ,ν_0} is a subset of $\mathbb{C}$ defined as follows:

$$\begin{aligned}
\Sigma_\epsilon &= \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg\lambda| \le \pi - \epsilon, \}, \\
K_\epsilon &= \Big\{\lambda \in \mathbb{C} \mid \Big(\operatorname{Re}\lambda + \frac{\gamma^2}{\alpha+\beta} + \epsilon\Big)^2 + (\operatorname{Im}\lambda)^2 \ge \Big(\frac{\gamma^2}{\alpha+\beta} + \epsilon\Big)^2\Big\}, \\
\Lambda_{\epsilon,\nu_0} &= K_\epsilon \cap \Sigma_\epsilon \cap \{\lambda \in \mathbb{C} \mid |\lambda| \ge \nu_0\}.
\end{aligned}$$

Theorem 1.3 may be proved by real interpolation theorem with the help of the following theorem.

Theorem 1.4. *Let $1 < q < \infty$, $1 \le r < \infty$, $-1+1/q < s < 1/q$, and $\epsilon \in (0, \pi/2)$. Then, there exists a large constant $\gamma > 0$ such that for every $\lambda \in \Lambda_{\epsilon,\gamma}$ and $(f, \mathbf{g}) \in \mathcal{H}^s_{q,r}(\mathbb{R}^N_+)$, there exists a unique solution $(\rho, \mathbf{u}) \in \mathcal{D}^s_{q,r}(\mathbb{R}^N_+)$ to (1.5) satisfying*

$$\|\lambda(\rho, \mathbf{u})\|_{\mathcal{H}^s_{q,r}(\mathbb{R}^N_+)} + \|\mathbf{u}\|_{B^{s+2}_{q,r}(\mathbb{R}^N_+)} \le C\|(f, \mathbf{g})\|_{\mathcal{H}^s_{q,r}(\mathbb{R}^N_+)}.$$

Moreover, let σ be a small positive number such that $-1+1/q < s-\sigma < s < s+\sigma < 1/q$. Then, there exist $\mathbf{u}_1$, $\mathbf{u}_2 \in B^{s+2}_{q,r}(\mathbb{R}^N_+)^N$ such that $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$ and for any $\lambda \in \Lambda_{\epsilon,\gamma}$ there hold

$$\|\mathbf{u}_1\|_{B^{s+2}_{q,r}(\mathbb{R}^N_+)} \le C|\lambda|^{-\frac{\sigma}{2}}\|\mathbf{g}\|_{B^{s+\sigma}_{q,r}(\mathbb{R}^N_+)},$$
$$\|\partial_\lambda \mathbf{u}_1\|_{B^{s+2}_{q,r}(\mathbb{R}^N_+)} \le C|\lambda|^{-(1-\frac{\sigma}{2})}\|\mathbf{g}\|_{B^{s-\sigma}_{q,r}(\mathbb{R}^N_+)}$$

for any $\mathbf{g} \in C^\infty_0(\mathbb{R}^N_+)^N$ as well as

$$\|(\rho, \mathbf{u}_2)\|_{\mathcal{D}^s_{q,r}(\mathbb{R}^N_+)} \le C|\lambda|^{-1}\|(f, \mathbf{g})\|_{\mathcal{H}^s_{q,r}(\mathbb{R}^N_+)},$$
$$\|\partial_\lambda(\rho, \mathbf{u}_2)\|_{\mathcal{D}^s_{q,r}(\mathbb{R}^N_+)} \le C|\lambda|^{-2}\|(f, \mathbf{g})\|_{\mathcal{H}^s_{q,r}(\mathbb{R}^N_+)}$$

for any $(f, \mathbf{g}) \in \mathcal{H}^s_{q,r}(\mathbb{R}^N_+)$.

Remark 1.1. The conditions $1 < q < \infty$, $1 \le r < \infty$ and $-1+1/q < s < 1/q$ assure that $C^\infty_0(\Omega)$ is a dense subset of $B^s_{q,r}(\Omega)$ for $\Omega \in \{\mathbb{R}^N, \mathbb{R}^N_+\}$. This fact is an important point for our analysis in this paper. For a proof of this fact, refer to [9, pp.368–369], [10, p.132], and [14, Theorems 2.9.3 and 2.10.3].

2 Solution formula

In this section, we shall derive solution formulas of equations (1.5). Insert the first equation into the second equation in (1.5), we have the complex Lamé equations

$$\lambda \mathbf{u} - \alpha\Delta\mathbf{u} - \eta_\lambda \nabla \operatorname{div} \mathbf{u} = \mathbf{g} - \gamma\lambda^{-1}\nabla f \quad \text{in } \mathbb{R}^N_+, \quad \mathbf{u}|_{\partial\mathbb{R}^N_+} = 0. \tag{2.1}$$

Here, we have set

$$\eta_\lambda = \beta + \gamma^2\lambda^{-1}.$$

If we find a solution $\mathbf{u}$ of equations (2.1) and if we set $\rho = \lambda^{-1}(f - \gamma \operatorname{div} \mathbf{u})$, then ρ and $\mathbf{u}$ are solutions of equations (1.5). Thus, in this section, we shall drive solution formulas of the complex Lame equations

$$\lambda \mathbf{u} - \alpha\Delta\mathbf{u} - \eta_\lambda \nabla \operatorname{div} \mathbf{u} = \mathbf{g} \quad \text{in } \mathbb{R}^N_+, \quad \mathbf{u}|_{\partial\mathbb{R}^N_+} = 0. \tag{2.2}$$

2.1 Whole space case

For $\epsilon \in (0, \pi/2)$ and $\lambda_0 > 0$ let $\lambda \in \Sigma_{\epsilon,\lambda_0}$ be the resolvent parameter, where λ_0 is assumed to be sufficiently large if necessary. In this subsection, we derive the representation of the solution formula for the following model problem in $\mathbb{R}^N$:

$$\lambda \mathbf{u} - \alpha\Delta\mathbf{u} - \eta_\lambda \nabla \operatorname{div} \mathbf{u} = \mathbf{g} \quad \text{in } \mathbb{R}^N. \tag{2.3}$$

where $\mathbf{g} \in B^s_{q,1}(\mathbb{R}^N)^N$, with $1 < q < \infty$ and $-1+1/q < s < 1/q$. Applying the divergence to equation (2.3) yields

$$\lambda \operatorname{div} \mathbf{u} - (\alpha + \eta_\lambda)\Delta \operatorname{div} \mathbf{u} = \operatorname{div} \mathbf{g} \quad \text{in } \mathbb{R}^N.$$

Applying Fourier transform and the Fourier inverse transform, we can define the solution operator $\mathcal{S}_0(\lambda)$ of the resolvent problem in the whole space as follows:

$$\mathcal{S}^0(\lambda)\mathbf{g} := \mathbf{u} = \mathcal{F}^{-1}\left[\frac{\hat{\mathbf{g}}(\xi)}{\lambda + \alpha|\xi|^2}\right] - \frac{\beta\lambda + \gamma^2}{(\alpha+\beta)\lambda + \gamma^2}\mathcal{F}^{-1}\left[\frac{(\xi \otimes \xi)\hat{\mathbf{g}}(\xi)}{(\lambda + \alpha|\xi|^2)(p(\lambda) + |\xi|^2)}\right]. \tag{2.4}$$

2.2 Half space case

Let $\epsilon \in (0, \pi/2)$ and $\nu_0 > 0$. Let $\gamma_0 > 0$ be a large number such that $\Sigma_\epsilon + \gamma_0 \subset K_\epsilon \cap \Sigma_\epsilon \cap \{\lambda \in \mathbb{C} \mid |\lambda| \geq \nu_0\}$. In this subsection, we derive the solution formula for equations (2.2). To this end, we extend $\mathbf{g} = (g_1, \cdots, g_N)$ by even extension for $j = 1, \cdots, N-1$ and odd extension for $j = N$. We now set $\mathbf{G} := (g_1^e, \cdots, g_{N-1}^e, g_N^o)$, where

$$g_j^e(x) = \begin{cases} g_j(x) & \text{for } x_N > 0, \\ g_j(x', -x_N) & \text{for } x_N < 0, \end{cases} \quad g_N^o(x) = \begin{cases} g_j(x) & \text{for } x_N > 0, \\ -g_j(x', -x_N) & \text{for } x_N < 0. \end{cases}$$

Let $\mathbf{u}$ be a solution of equations (2.2) and let $\mathbf{w} = \mathbf{u} - \mathcal{S}^0(\lambda)\mathbf{G}$, and then $\mathbf{w}$ should satisfy the equations

$$\lambda \mathbf{w} - \alpha \Delta \mathbf{w} - \eta_\lambda \nabla \operatorname{div} \mathbf{w} = 0 \quad \text{in } \mathbb{R}^N_+, \quad \mathbf{w}|_{\partial \mathbb{R}^N_+} = -\mathcal{S}^0(\lambda)\mathbf{G}|_{\partial \mathbb{R}^N_+}. \tag{2.5}$$

In view of (2.4), we may have

$$\mathcal{S}^0(\lambda)\mathbf{G} = \mathcal{F}^{-1}\left[\frac{\hat{\mathbf{G}}(\xi)}{\lambda + \alpha|\xi|^2}\right] - \frac{\beta\lambda + \gamma^2}{(\alpha+\beta)\lambda + \gamma^2}\mathcal{F}^{-1}\left[\frac{(\xi \otimes \xi)\hat{\mathbf{G}}(\xi)}{(\lambda + \alpha|\xi|^2)(p(\lambda) + |\xi|^2)}\right].$$

Let $\mathbf{w} = (w_1, \ldots, w_N)$, and we shall investigate the formula of the partial Fourier transform $\mathcal{F}'[w_j](\xi', x_N)$ of w_j. Applying the partial Fourier transform $\mathcal{F}'$ to equations (2.3), we have the ordinal differential equations in x_N variable, which reads as

$$\begin{cases} (\lambda + \alpha|\xi'|^2)\mathcal{F}[w_j](\xi', x_N) - \alpha\partial_N^2 \mathcal{F}[w_j](\xi', x_N) \\ \quad - \eta_\lambda i\xi_j(i\xi' \cdot \mathcal{F}'[\mathbf{w}'](\xi', x_N) + \partial_N \mathcal{F}'[w_N](\xi', x_N)) = 0, & \text{for } x_N > 0, \\ (\lambda + \alpha|\xi'|^2)\mathcal{F}[w_N](\xi', x_N) - \alpha\partial_N^2 \mathcal{F}[w_N](\xi', x_N) \\ \quad - \eta_\lambda \partial_N(i\xi' \cdot \mathcal{F}'[\mathbf{w}'](\xi', x_N) + \partial_N \mathcal{F}'[w_N](\xi', x_N)) = 0, & \text{for } x_N > 0, \\ \mathcal{F}'[\mathbf{w}](\xi', 0) = -\mathcal{F}'[\mathcal{S}^0(\lambda)\mathbf{G}](\xi', 0). \end{cases}$$

Here, we have set $i\xi' \cdot \mathcal{F}'[\mathbf{w}'](\xi', x_N) = \sum_{j=1}^{N-1} i\xi_j \mathcal{F}'[w_j](\xi', x_N)$.

To obtain $\mathcal{F}'[w_j](\xi', x_N)$, first we derive the representation of $\mathcal{F}'[\mathcal{S}^0(\lambda)\mathbf{G}](0)$. Applying residue theorem in theory of one complex variable, we have the representation for h_j for $j = 1, \ldots, N-1$ and $h_N = 0$. According to [4, (4.9)], we have

$$\mathcal{F}'[w_j](\xi', x_N) = h_j e^{-Bx_N} - \frac{i\xi_j \eta_\lambda}{K}\mathcal{M}(x_N) i\xi' \cdot h', \quad \mathcal{F}'[w_N](\xi', x_N) = \frac{A\eta_\lambda}{K}\mathcal{M}(x_N) i\xi' \cdot h'.$$

where

$$K = (\alpha + \eta_\lambda)A + \alpha B, \qquad \mathcal{M}(x_N) = \frac{e^{-Ax_N} - e^{-Bx_N}}{A - B}.$$

and h_j is the j-th component of $\mathcal{F}'[\mathcal{S}^0(\lambda)\mathbf{G}](\xi', 0)$. Thus, we have

$$\begin{aligned} \mathcal{F}'[w_j](\xi', x_N) &= -\int_0^\infty Be^{-(x_N+y_N)B}\frac{1}{\alpha B^2}\mathcal{F}[g_j]\, dy_N \\ &- \int_0^\infty B^2 e^{-Bx_N}\mathcal{M}(y_N)\frac{\beta\lambda + \gamma^2}{\alpha((\alpha+\beta)\lambda + \gamma^2)}\Big(\sum_{k=1}^{N-1}\frac{i\xi_j\xi_k}{(A+B)AB^2}\mathcal{F}'[g_k] - \frac{i\xi_j}{(A+B)B^2}\mathcal{F}'[g_N]\Big)\, dy_N \\ &+ \int_0^\infty Be^{-B(x_N+y_N)}\frac{\beta\lambda + \gamma^2}{\alpha((\alpha+\beta)\lambda + \gamma^2)}\sum_{k=1}^{N-1}\frac{i\xi_j\xi_k}{(A+B)AB^2}\mathcal{F}'[g_k](\xi', y_N)\, dy_N \end{aligned}$$

$$
+ \int_0^\infty B^2 \mathcal{M}(x_N) e^{-By_N} \frac{i\xi_j \eta_\lambda}{K} \frac{1}{\alpha B^3} i\xi' \cdot \mathcal{F}[\mathbf{g}']\, dy_N
$$

$$
- \int_0^\infty B^3 \mathcal{M}(x_N)\mathcal{M}(y_N) \frac{i\xi_j \eta_\lambda}{K} \frac{\beta\lambda + \gamma^2}{\alpha((\alpha+\beta)\lambda + \gamma^2)} (\sum_{k=1}^{N-1} \frac{|\xi'|^2 \xi_k}{(A+B)AB^3} \mathcal{F}'[g_k] - \frac{|\xi'|^2}{(A+B)B^3} \mathcal{F}'[g_N])\, dy_N
$$

$$
+ \int_0^\infty B^2 \mathcal{M}(x_N) e^{-By_N} \frac{i\xi_j \eta_\lambda}{K} \frac{\beta\lambda + \gamma^2}{\alpha((\alpha+\beta)\lambda + \gamma^2)} \sum_{k=1}^{N-1} \frac{|\xi'|^2 \xi_k}{(A+B)AB^3} \mathcal{F}'[g_k](\xi', y_N)\, dy_N,
$$

$$
\mathcal{F}'[w_N](\xi', x_N) = - \int_0^\infty B^2 \mathcal{M}(x_N) e^{-By_N} \frac{A\eta_\lambda}{K} \frac{1}{\alpha B^3} i\xi' \cdot \mathcal{F}[\mathbf{g}']\, dy_N
$$

$$
+ \int_0^\infty B^3 \mathcal{M}(x_N)\mathcal{M}(y_N) \frac{A\eta_\lambda}{K} \frac{\beta\lambda + \gamma^2}{\alpha((\alpha+\beta)\lambda + \gamma^2)} (\sum_{k=1}^{N-1} \frac{|\xi'|^2 \xi_k}{(A+B)AB^3} \mathcal{F}'[g_k] - \frac{|\xi'|^2}{(A+B)B^3} \mathcal{F}'[g_N])\, dy_N
$$

$$
- \int_0^\infty B^2 \mathcal{M}(x_N) e^{-By_N} \frac{A\eta_\lambda}{K} \frac{\beta\lambda + \gamma^2}{\alpha((\alpha+\beta)\lambda + \gamma^2)} \sum_{k=1}^{N-1} \frac{|\xi'|^2 \xi_k}{(A+B)AB^3} \mathcal{F}'[g_k](\xi', y_N)\, dy_N. \tag{2.6}
$$

3 Estimates of solution operators

3.1 Whole space case

In this subsection, we shall estimate the solution operator $\mathcal{S}^0(\lambda)$ defined in (2.4). To this end, we use the Fourier multiplier theorem of Mihlin-Hörmander type [5,7]. Let $m(\xi)$ be a complex-valued function defined on $\mathbb{R}^N \setminus \{0\}$. We say that $m(\xi)$ is a multiplier if it satisfies the multiplier conditions:

$$
|D_\xi^\alpha m(\xi)| \le C_\alpha |\xi|^{-|\alpha|}
$$

for any multi-index $\alpha \in \mathbb{N}_0^N$ with some constant C_α depending on α. Then, the Fourier multiplier operator with kernel function $m(\xi)$ is defined by

$$
T_m f = \mathcal{F}_\xi^{-1}[m(\xi)\mathcal{F}[f](\xi)] = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} m(\xi)\mathcal{F}[f](\xi)\, \mathrm{d}\xi \quad \text{for } f \in \mathcal{S}(\mathbb{R}^N).
$$

Then, we have the following theorem called the Fourier multiplier theorem.

Theorem 3.1. *Let $1 < q < \infty$ and $m(\xi)$ be a multiplier. Then, the Fourier multiplier T_m is an $L_q(\mathbb{R}^N)$ bounded operator, that is T_m is uniquely extended to an operator on $L_q(\mathbb{R}^N)$, which is also written by T_m and there exists a constant depending on q and N such that*

$$
\|T_m f\|_{L_q(\mathbb{R}^N)} \le C(\max_{|\alpha| \le [N/2]+1} C_\alpha) \|f\|_{L_q(\mathbb{R}^N)} \quad \textit{for } f \in L_q(\mathbb{R}^N).
$$

Here, $[N/2]$ denotes the integer part of $N/2$.

The following theorem provide the estimate of the solution operator $\mathcal{S}^0(\lambda)$ of the problem in $\mathbb{R}^N$.

Theorem 3.2. *Let $1 < q < \infty$, $1 \le r \le \infty$, $-1 + 1/q < s < 1/q$, and $\epsilon \in (0, \pi/2)$. Let $\mathcal{S}^0(\lambda)$ be the operator defined in (2.4). Then, there exists a large constant $\gamma_0 > 0$ such that for any $\lambda \in \Sigma_{\epsilon,\gamma_0}$ and $\mathbf{g} \in B_{q,r}^s(\mathbb{R}^N)$, there hold*

$$
\|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\mathcal{S}^0(\lambda)\mathbf{g}\|_{B_{q,r}^s(\mathbb{R}^N)} \le C\|\mathbf{g}\|_{B_{q,r}^s(\mathbb{R}^N)},
$$

$$\|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\partial_\lambda \mathcal{S}^0(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N)} \le C|\lambda|^{-1}\|\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N)}.$$

Moreover, let $\sigma > 0$ be a small number such that $-1+1/q < s-\sigma < s < s+\sigma < 1/q$. Then, there exist a large number $\gamma_1 \ge \gamma_0$ and two operators $T^0_1(\lambda)$ and $T^0_2(\lambda)$ which are holomorphic on $\Lambda_{\epsilon,\gamma_1}$ such that $\mathcal{S}^0(\lambda) = \mathcal{T}^0_1(\lambda) + \mathcal{T}^0_2(\lambda)$ and for any $\mathbf{g} \in C^\infty_0(\mathbb{R}^N)$ and $\lambda \in \Lambda_{\epsilon,\gamma_1}$, there hold

$$\|(\lambda^{1/2}\nabla, \nabla^2)\mathcal{T}^0_1(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N)} \le C|\lambda|^{-\sigma/2}\|\mathbf{g}\|_{B^{s+\sigma}_{q,r}(\mathbb{R}^N)},$$
$$\|(\lambda^{1/2}\nabla, \nabla^2)\partial_\lambda\mathcal{T}^0_1(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N)} \le C|\lambda|^{-(1-\frac{\sigma}{2})}\|\mathbf{g}\|_{B^{s-\sigma}_{q,r}(\mathbb{R}^N)}$$

as well as for any $\lambda \in \Lambda_{\epsilon,\gamma_1}$ and $\mathbf{g} \in B^s_{q,1}(\mathbb{R}^N)$, there hold

$$\|(\lambda^{1/2}\nabla, \nabla^2)\mathcal{T}^0_2(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N)} \le C|\lambda|^{-\sigma/2}\|\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N)},$$
$$\|(\lambda^{1/2}, \nabla^2)\partial_\lambda\mathcal{T}^0_2(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N)} \le C|\lambda|^{-(1-\frac{\sigma}{2})}\|\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N)}.$$

3.2 Half space case

Let $\mathcal{S}^b(\lambda) = (\mathcal{S}^b_1(\lambda), \dots, \mathcal{S}^b_N(\lambda))$ be the operator solution operator corresponding to equations (2.5) defined by

$$\mathcal{S}^b_J(\lambda)\mathbf{g} = w_J$$

for $J = 1, \dots, N$, where the partial Fourier transform $\mathcal{F}'[w_J]$ of w_J are defined by (2.6). In this section, we shall estimate $\mathcal{S}^b_J(\lambda)$. Namely, we shall prove the following theorem.

Theorem 3.3. *Let $1 < q < \infty$, $1 \le r \le \infty$, $-1+1/q < s < 1/q$, $\epsilon \in (0, \pi/2)$ and $\lambda_0 > 0$. Then, for any $\lambda \in \Lambda_{\epsilon,\lambda_0}$ and $\mathbf{g} \in B^s_{q,r}(\mathbb{R}^N_+)$, there hold*

$$\|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\mathcal{S}^b(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N_+)} \le C\|\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N_+)},$$
$$\|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\partial_\lambda\mathcal{S}^b(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N_+)} \le C|\lambda|^{-1}\|\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N_+)}. \tag{3.1}$$

Moreover, let $\sigma > 0$ be a small number such that $-1+1/q < s-\sigma < s < s+\sigma < 1/q$. Then, there exist a large number $\lambda_0 > 0$ and two operators $\mathcal{T}^b_1(\lambda)$ and $\mathcal{T}^b_2(\lambda)$ which are holomorphic on $\Lambda_{\epsilon,\lambda_0}$ such that $\mathcal{S}^b(\lambda) = \mathcal{T}^b_1(\lambda) + \mathcal{T}^b_2(\lambda)$ and for any $\mathbf{g} \in C^\infty_0(\mathbb{R}^N_+)^N$, there hold

$$\|(\lambda, \lambda^{1/2}\bar{\nabla}, \bar{\nabla}^2)\mathcal{T}^b_1(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N_+)} \le C|\lambda|^{-\sigma/2}\|\mathbf{g}\|_{B^{s+\sigma}_{q,r}(\mathbb{R}^N_+)},$$
$$\|(\lambda, \lambda^{1/2}\bar{\nabla}, \nabla^2)\partial_\lambda\mathcal{T}^b_1(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N_+)} \le C|\lambda|^{-(1-\frac{\sigma}{2})}\|\mathbf{g}\|_{B^{s-\sigma}_{q,r}(\mathbb{R}^N_+)},$$
$$\|(\lambda, \lambda^{1/2}\bar{\nabla}, \bar{\nabla}^2)\mathcal{T}^b_2(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N_+)} \le C|\lambda|^{-1}\|\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N_+)},$$
$$\|(\lambda, \lambda^{1/2}\bar{\nabla}, \bar{\nabla}^2)\partial_\lambda\mathcal{T}^b_2(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N_+)} \le C|\lambda|^{-2}\|\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N_+)}. \tag{3.2}$$

Proof. To prove Theorem 3.3, the argument based on interpolation theory plays an important role, see Shibata [11]. Rewrite $\mathcal{S}^b(\lambda)$ as

$$\mathcal{S}^b(\lambda)\mathbf{g} = \sum_{J=1}^N \mathcal{T}^b_{1J}(\lambda)\mathbf{g} + \mathcal{T}^b_{2J}(\lambda)\mathbf{g},$$

where $\mathcal{T}^b_{1J}(\lambda)\mathbf{g}$ are enssential parts of the solution formula. On the other hand, $\mathcal{T}^b_{2J}(\lambda)\mathbf{g}$ are the terms of the forms which contain λ^{-1}, which decay fast enough. In this paper, we focus on one term of $\mathcal{T}^b_{1J}(\lambda)\mathbf{g}$, which is

$$\mathcal{T}^b_{11}(\lambda)g_j = \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[Be^{-(x_N+y_N)B}\frac{1}{\alpha B^2}\mathcal{F}[g_j](\xi', y_N)\Big](x')\,dy_N$$

and derive the follwing estimate:

$$\|\lambda \mathcal{T}_{11}^b(\lambda)g_j\|_{B^s_{q,r}(\mathbb{R}^N_+)} \leq C\|g_j\|_{B^s_{q,r}(\mathbb{R}^N_+)} \tag{3.3}$$

with the aid of the real interpolation. To do this, we first prove

$$\|\lambda \mathcal{T}_{11}^b(\lambda)g_j\|_{H^1_q(\mathbb{R}^N_+)} \leq C\|g_j\|_{H^1_q(\mathbb{R}^N_+)} \tag{3.4}$$

and, in this paper, we especially estimate $\|\lambda\partial_N \mathcal{T}_{11}^b(\lambda)g_j\|_{L_q(\mathbb{R}^N_+)}$. Since we can assume $\mathbf{g} \in C_0^\infty(\mathbb{R}^N_+)$ by Remark 1.1, from the integration by parts, we have

$$\begin{aligned}
\partial_N\lambda\mathcal{T}_{11}^b(\lambda)g_j &= \mathcal{F}^{-1}_{\xi'}\left[\frac{\lambda}{\alpha B}\int_0^\infty \hat{g}_j(\xi', y_N)e^{-Bx_N}(-B)e^{-By_N}\,dy_N\right](x') \\
&= \mathcal{F}^{-1}_{\xi'}\left[\frac{\lambda}{\alpha B}\int_0^\infty \hat{g}_j(\xi', y_N)e^{-Bx_N}\partial_N(e^{-By_N})\,dy_N\right](x') \\
&= -\mathcal{F}^{-1}_{\xi'}\left[\frac{\lambda}{\alpha B}\int_0^\infty \widehat{\partial_N g_j}(\xi', y_N)e^{-Bx_N}e^{-By_N}\,dy_N\right](x') \\
&= -\mathcal{F}^{-1}_{\xi'}\left[\frac{\lambda}{\alpha B^2}\int_0^\infty \widehat{\partial_N g_j}(\xi', y_N)Be^{-B(x_N+y_N)}\,dy_N\right](x')
\end{aligned}$$

To show the estimate (3.3), we need the following propostition.

Proposition 3.4. *Let $1 < q < \infty$, $\epsilon \in (0, \pi/2)$, $\lambda_0 > 0$, and $\lambda \in \Lambda_{\epsilon,\lambda_0}$. Suppose that $m_0 \in \mathbb{M}_0$. Define the integral operators L_i, $i = 1, \cdots, 6$, by the formula:*

$$\begin{aligned}
L_1(\lambda)f &= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[m_0(\lambda,\xi')B^3\mathcal{M}(x_N)\mathcal{M}(y_N)\hat{f}(\xi', y_N)\right](x')\,dy_N, \\
L_2(\lambda)f &= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[m_0(\lambda,\xi')B^2\mathcal{M}(x_N)e^{-Ay_N}\hat{f}(\xi', y_N)\right](x')\,dy_N, \\
L_3(\lambda)f &= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[m_0(\lambda,\xi')B^2\mathcal{M}(x_N)e^{-By_N}\hat{f}(\xi', y_N)\right](x')\,dy_N, \\
L_4(\lambda)f &= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[m_0(\lambda,\xi')B^2e^{-Ax_N}\mathcal{M}(y_N)\hat{f}(\xi', y_N)\right](x')\,dy_N, \\
L_5(\lambda)f &= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[m_0(\lambda,\xi')B^2e^{-Bx_N}\mathcal{M}(y_N)\hat{f}(\xi', y_N)\right](x')\,dy_N, \\
L_6(\lambda)f &= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[m_0(\lambda,\xi')Be^{-Jx_N}e^{-Qy_N}\hat{f}(\xi', y_N)\right](x')\,dy_N,
\end{aligned}$$

respectively, where (J, Q) stands for an element of $\{(A, A), (A, B), (B, A), (B, B)\}$ in the formula of L_6. Then for every $f \in L_q(\mathbb{R}^N_+)$, it holds

$$\|L_i(\lambda)f\|_{L_q(\mathbb{R}^N_+)} \leq C_q\|f\|_{L_q(\mathbb{R}^N_+)} \quad (i = 1, 2, 3, 4, 5, 6).$$

Proposition 3.5. [12, A.3 p.271]. *Let $1 < q < \infty$. Define the integral operator G by the formula:*

$$Gf(x_N) = \int_0^\infty \frac{f(y_N)}{x_N + y_N}\,dy_N.$$

Then, for every $f \in L_q(0, \infty)$ there exists a constant A_q such that

$$\|Gf\|_{L_q((0,\infty))} \leq A_q\|f\|_{L_q((0,\infty))}.$$

By Theorem 3.1 and Propostiion 3.4, we deduce

$$\begin{aligned}\|\partial_N\lambda\mathcal{T}^b_{11}(\lambda)g_j\|_{L_q(\mathbb{R}^N_+)} &\le C\left\|\int_0^\infty \frac{\|\partial_N g_j(\cdot,y_N)\|_{L_q(\mathbb{R}^{N-1})}}{x_N+y_N}\,dy_N\right\|_{L_q(0,\infty)}\\ &\le C\|\partial_N g_j\|_{L_q(\mathbb{R}^N_+)}.\end{aligned}$$

Hence, we have (3.4). By duality argument, we have

$$\|\lambda\mathcal{T}^b_{11}(\lambda)g_j\|_{H^{-1}_{q,0}(\mathbb{R}^N_+)} \le C\|g_j\|_{H^{-1}_{q,0}(\mathbb{R}^N_+)}.$$

By interpolation, for $-1+1/q<s<1/q$, we deduce that

$$\|\lambda\mathcal{T}^b_{11}(\lambda)g_j\|_{B^s_{q,1}(\mathbb{R}^N_+)} \le C\|g_j\|_{B^s_{q,1}(\mathbb{R}^N_+)}.$$

Other terms of $\mathcal{T}^b_{1J}(\lambda)\mathbf{g}$ and $\mathcal{T}^b_{2J}(\lambda)\mathbf{g}$ can be estimated in the similiar way. Moreover, we can derive (3.1)- (3.2) by applying the the same method as above. This completes the proof of Theorem 3.3. □

4 Proof of Main Results

In this section, we first construct solution operators of equations:

$$\begin{cases} \lambda\rho+\gamma\operatorname{div}\mathbf{u}=f & \text{in } \mathbb{R}^N_+,\\ \lambda\mathbf{u}-\alpha\Delta\mathbf{u}-\beta\nabla\operatorname{div}\mathbf{u}+\gamma\nabla\rho=\mathbf{g} & \text{in } \mathbb{R}^N_+,\\ \mathbf{u}=0 & \text{on } \partial\mathbb{R}^N_+.\end{cases} \tag{4.1}$$

First, from the first equation in (4.1), we set $\rho=\lambda^{-1}(f-\gamma\operatorname{div}\mathbf{u})$. Inserting this formula into the second equation in (4.1), we have the complex Lamé equation:

$$\lambda\mathbf{u}-\alpha\Delta\mathbf{u}-\eta_\lambda\nabla\operatorname{div}\mathbf{u}=\mathbf{g}-\gamma\lambda^{-1}\nabla f \quad \text{in } \mathbb{R}^N_+, \quad \mathbf{u}|_{\partial\mathbb{R}^N_+}=0. \tag{4.2}$$

From Theorems 3.2 and 3.3, we have

$$\mathbf{u}=\mathcal{S}^0(\lambda)(\mathbf{g}-\gamma\lambda^{-1}\nabla f)-\mathcal{S}^b(\lambda)(\mathbf{g}-\gamma\lambda^{-1}\nabla f).$$

Thus, defining ρ by

$$\rho=\lambda^{-1}(f-\gamma\operatorname{div}\mathbf{u})=\lambda^{-1}f-\gamma\lambda^{-1}\operatorname{div}(\mathcal{S}^0(\lambda)(\mathbf{g}-\gamma\lambda^{-1}\nabla f)-\mathcal{S}^b(\lambda)(\mathbf{g}-\gamma\lambda^{-1}\nabla f)),$$

we see that $\mathbf{u}$ and ρ are solutions of equations (4.1). In view of Theorems 3.2 and 3.3, we decompose $\mathbf{u}$ as

$$\mathbf{u}=\mathcal{T}^0_1(\lambda)\mathbf{g}-\mathcal{T}^b_1(\lambda)\mathbf{g}+\mathcal{T}^0_2(\lambda)\mathbf{g}-\mathcal{T}^b_2(\lambda)\mathbf{g}-\gamma\lambda^{-1}\mathcal{S}^0(\lambda)\nabla f+\gamma\lambda^{-1}\mathcal{S}^b(\lambda)\nabla f.$$

Summing up, there exist solution operators $\mathcal{S}(\lambda)$, $\mathcal{S}^1(\lambda)$, $\mathcal{S}^2(\lambda)$ such that $\mathbf{u}=\mathcal{S}(\lambda)(f,\mathbf{g})$, $\rho=\mathcal{R}(\lambda)(f,\nabla\mathbf{g})$, and

$$\begin{aligned}\mathcal{S}^1(\lambda)\mathbf{g}&=\mathcal{T}^0_1(\lambda)\mathbf{g}-\mathcal{T}^b_1(\lambda)\mathbf{g},\\ \mathcal{S}^2(\lambda)(f,\mathbf{g})&=\mathcal{T}^0_2(\lambda)\mathbf{g}-\mathcal{T}^b_2(\lambda)\mathbf{g}-\gamma\lambda^{-1}\mathcal{S}^0(\lambda)\nabla f+\gamma\lambda^{-1}\mathcal{S}^b(\lambda)\nabla f,\\ \mathcal{S}(\lambda)(f,\mathbf{g})&=\mathcal{S}^1(\lambda)\mathbf{g}+\mathcal{S}^2(\lambda)(f,\mathbf{g}),\\ \mathcal{R}(\lambda)(f,\mathbf{g})&=\lambda^{-1}f-\gamma\lambda^{-1}\operatorname{div}\mathcal{S}^0(\lambda)\mathbf{g}+\gamma^2\lambda^{-2}\operatorname{div}\mathcal{S}^0(\lambda)\nabla f\\ &\quad-\gamma\lambda^{-1}\operatorname{div}\mathcal{S}^b(\lambda)\mathbf{g}+\gamma^2\lambda^{-2}\operatorname{div}\mathcal{S}^b(\lambda)\nabla f.\end{aligned} \tag{4.3}$$

We see easily that

$$\begin{aligned}
\mathcal{S}(\lambda) &\in \mathrm{Hol}\,(\Lambda_{\epsilon,\lambda_0}, \mathcal{L}(B^{s+1}_{q,r}(\mathbb{R}^N_+) \times B^s_{q,r}(\mathbb{R}^N_+), B^{s+2}_{q,r}(\mathbb{R}^N_+))),\\
\mathcal{S}^1(\lambda) &\in \mathrm{Hol}\,(\Lambda_{\epsilon,\lambda_0}, \mathcal{L}(B^{s}_{q,r}(\mathbb{R}^N_+), B^{s+2}_{q,r}(\mathbb{R}^N_+))),\\
\mathcal{S}^2(\lambda) &\in \mathrm{Hol}\,(\Lambda_{\epsilon,\lambda_0}, \mathcal{L}(B^{s+1}_{q,r}(\mathbb{R}^N_+) \times B^s_{q,r}(\mathbb{R}^N_+), B^{s+1}_{q,r}(\mathbb{R}^N_+) \times B^{s+2}_{q,r}(\mathbb{R}^N_+))),\\
\mathcal{R}(\lambda) &\in \mathrm{Hol}\,(\Lambda_{\epsilon,\lambda_0}, \mathcal{L}(B^{s+1}_{q,r}(\mathbb{R}^N_+) \times B^s_{q,r}(\mathbb{R}^N_+), B^{s+1}_{q,r}(\mathbb{R}^N_+))).
\end{aligned}$$

Moreover, by Theorems 3.2 and 3.3, we see the following theorem.

Theorem 4.1. *Let $1 < q < \infty$, $1 \le r < \infty$, $-1 + 1/q < s < 1/q$, and $\epsilon \in (0, \pi/2)$. Then, ther exists a large number $\lambda_0 > 0$ such that for any $\lambda \in \Lambda_{\epsilon,\lambda_0}$, $f \in B^{s+1}_{q,r}(\mathbb{R}^N_+)$, and $\mathbf{g} \in B^s_{q,r}(\mathbb{R}^N_+)^N$, there hold*

$$\begin{aligned}
\|(\lambda, \lambda^{1/2}\nabla, \nabla^2)\mathcal{S}(\lambda)(f, \mathbf{g})\|_{\mathcal{H}^s_{q,r}(\mathbb{R}^N_+)} &\le C\|(f, \mathbf{g})\|_{B^{s+1}_{q,r}(\mathbb{R}^N_+)\times B^s_{q,r}(\mathbb{R}^N_+)},\\
\|(\lambda^{1/2}\nabla, \nabla^2)\mathcal{S}^1(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N_+)} &\le C|\lambda|^{-\frac{\sigma}{2}}\|\mathbf{g}\|_{B^{s+\sigma}_{q,r}(\mathbb{R}^N_+)},\\
\|(\lambda^{1/2}\nabla, \nabla^2)\partial_\lambda\mathcal{S}^1(\lambda)\mathbf{g}\|_{B^s_{q,r}(\mathbb{R}^N_+)} &\le C|\lambda|^{-(1-\frac{\sigma}{2})}\|\mathbf{g}\|_{B^{s-\sigma}_{q,r}(\mathbb{R}^N_+)},\\
\|(\lambda^{1/2}\nabla, \nabla^2)\mathcal{S}^2(\lambda)(f, \mathbf{g})\|_{B^s_{q,r}(\mathbb{R}^N_+)\times B^s_{q,r}(\mathbb{R}^N_+)} &\le C|\lambda|^{-1}\|(f, \mathbf{g})\|_{B^{s+1}_{q,r}(\mathbb{R}^N_+)\times B^s_{q,r}(\mathbb{R}^N_+)},\\
\|(\lambda^{1/2}\nabla, \nabla^2)\partial_\lambda\mathcal{S}^2(\lambda)(f, \mathbf{g})\|_{B^s_{q,r}(\mathbb{R}^N_+)\times B^s_{q,r}(\mathbb{R}^N_+)} &\le C|\lambda|^{-2}\|(f, \mathbf{g})\|_{B^{s+1}_{q,r}(\mathbb{R}^N_+)\times B^s_{q,r}(\mathbb{R}^N_+)},\\
\|\mathcal{R}(\lambda)f\|_{B^{s+1}_{q,r}(\mathbb{R}^N_+)} &\le C|\lambda|^{-1}\|(f, \mathbf{g})\|_{B^{s+1}_{q,r}(\mathbb{R}^N_+)\times B^s_{q,r}(\mathbb{R}^N_+)},\\
\|\partial_\lambda\mathcal{R}(\lambda)f\|_{B^{s+1}_{q,r}(\mathbb{R}^N_+)} &\le C|\lambda|^{-2}\|(f, \mathbf{g})\|_{B^{s+1}_{q,r}(\mathbb{R}^N_+)\times B^s_{q,r}(\mathbb{R}^N_+)},
\end{aligned}$$

Theorem 1.4 follows from Theorem 4.1 immediately.

Now, we consider an initial value problem:

$$\begin{cases}
\partial_t\Pi + \gamma\,\mathrm{div}\,\mathbf{U} = 0 & \text{in } \mathbb{R}^N_+ \times \mathbb{R}_+,\\
\partial_t\mathbf{U} - \alpha\Delta\mathbf{U} - \beta\nabla\,\mathrm{div}\,\mathbf{U} + \gamma\nabla\Pi = 0 & \text{in } \mathbb{R}^N_+ \times \mathbb{R}_+,\\
\mathbf{U} = 0 & \text{on } \partial\mathbb{R}^N_+ \times \mathbb{R}_+,\\
(\Pi, \mathbf{U})|_{t=0} = (\Pi_0, \mathbf{U}_0) & \text{in } \mathbb{R}^N_+.
\end{cases} \tag{4.4}$$

To formulate problem (4.4) in the semigroup setting, we introduce spaces $\mathcal{H}^s_{q,r}(\mathbb{R}^N_+)$, $\mathcal{D}^s_{q,r}(\mathbb{R}^N_+)$ and an operator $\mathcal{A}$ defined in (1.2) and (1.3), respectively. Then, as was seen in (1.4), equations (4.4) are written as

$$\partial_t(\Pi, \mathbf{U}) + \mathcal{A}(\Pi, \mathbf{U}) = (0, 0) \quad \text{for } t > 0, \quad (\Pi, \mathbf{U})|_{t=0} = (\Pi_0, \mathbf{U}_0) \in \mathcal{H}^s_{q,r}.$$

And, the corresponding resolvent problem (4.1) is written as

$$\lambda(\rho, \mathbf{u}) + \mathcal{A}(\rho, \mathbf{u}) = (f, \mathbf{g})$$

for $(f, \mathbf{g}) \in \mathcal{H}^s_{q,r}(\mathbb{R}^N_+)$ and $(\rho, \mathbf{u}) \in \mathcal{D}^s_{q,r}(\mathbb{R}^N_+)$. From Theorem 4.1 it follows that the resolvent operator $(\lambda + \mathcal{A})^{-1}$ exists for any $\lambda \in \Lambda_{\epsilon,\lambda_0}$ for sufficient large $\lambda_0 > 0$. In fact, $(\lambda + \mathcal{A})^{-1}(f, \mathbf{g}) = (\mathcal{R}(\lambda), \mathcal{S}(\lambda))(f, \mathbf{g})$ for $(f, \mathbf{g}) \in \mathcal{H}^s_{q,r}$. Thus, the resolvent estimate: $\|\lambda(\lambda + \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H}^s_{q,r})} \le C$ holds for any $\lambda \in \Lambda_{\epsilon\lambda_0}$.

From these observations, by theory of C_0 analytic semigroup ([16]), there exists a C_0 analytic semigroup $\{T(t)\}_{t\ge 0}$ associated with (4.4) and $(\Pi, \mathbf{U}) = T(t)(\Pi_0, \mathbf{U}_0)$ is a unique solution of (4.4), which satisfies the regularity condition:

$$(\Pi, \mathbf{U}) \in C^0([0, \infty), \mathcal{H}^s_{q,r}) \cap C^0((0, \infty), \mathcal{D}^s_{q,r}) \cap C^1((0, \infty), \mathcal{H}^s_{q,r})$$

as well as

$$\lim_{t\to 0}\|(\Pi(\cdot,t),\mathbf{U}(\cdot,t))-(\Pi_0,\mathbf{U}_0)\|_{\mathcal{H}^s_{q,r}}=(0,0).$$

Finally, we shall show the following theorem about the maximal L_1 regularity of $\{T(t)\}_{t\geq 0}$. Obvisouly, combining the results about continuous analytic property mentioned above and the following theorem completes the proof of Theorem 1.3.

Theorem 4.2. *Let $1<q<\infty$ and $-1+1/q<s<1/q$. Then, there exists $\gamma>0$ such that for any $(\Pi_0,\mathbf{U}_0)\in\mathcal{H}^s_{q,1}$, there holds*

$$\int_0^\infty e^{-\gamma t}(\|\partial_t T(t)(\Pi_0,\mathbf{U}_0)\|_{\mathcal{H}^s_{q,1}}+\|T(t)(\Pi_0,\mathbf{U}_0)\|_{\mathcal{D}^s_{q,1}})\ \mathrm{d}t\leq\|(\Pi_0,\mathbf{U}_0)\|_{\mathcal{H}^s_{q,1}}.$$

In the sequel, we shall prove Theorem 4.2. We start with the following lemma.

Proposition 4.3. *Let X_0 and X_1 be Banach spaces which arean interpolation couple, and Y be another Banach space. Assume that $0<\sigma_0,\sigma_1,\theta<1$ satisfy $1=(1-\theta)(1-\sigma_0)+\theta(1+\sigma_1)$. Let $\gamma\geq 0$. For $t>0$ let $T(t)\colon Y\to X_0+X_1$ be a bounded linear operator such that*

$$\begin{aligned}\|T(t)f\|_Y&\leq Ce^{\gamma t}t^{-1+\sigma_0}\|f\|_{X_0}, && f\in X_0,\\ \|T(t)f\|_Y&\leq Ce^{\gamma t}t^{-1-\sigma_1}\|f\|_{X_1}, && f\in X_1.\end{aligned}$$

Then, there holds

$$\int_0^\infty e^{-\gamma t}\|T(t)f\|_Y\ \mathrm{d}t\leq C\|f\|_{(X_0,X_1)_{\theta,1}}$$

with a constant $C>0$ independent of γ.

A Proof of Theorem 4.2. Let $\gamma>0$ be a large number such that $\Sigma_\epsilon+\gamma\subset\Lambda_{\epsilon,\lambda_0}$. Let Γ be a contour in $\mathbb{C}$ defined by $\Gamma=\Gamma_+\cup\Gamma_-$ with

$$\Gamma_\pm=\{\lambda=re^{\pm(\pi-\epsilon)}\mid r\in(0,\infty)\}.$$

As was well-known in theory of C_0 analytic semigroup (cf. [16]), we have

$$T(t)(\Pi_0,\mathbf{U}_0)=\frac{1}{2\pi i}\int_{\Gamma+\gamma}(\mathcal{S}(\lambda),\mathcal{R}(\lambda))(\Pi_0,\mathbf{U}_0)\,d\lambda\quad\text{for } t>0.$$

In order to show the L_1 integrability of $T(t)$, we use Theorem 4.1. According to the formulas in (4.3), we divide $T(t)$ into three parts as follows.

$$T_1(t)\mathbf{U}_0=\frac{1}{2\pi i}\int_{\Gamma+\gamma}\mathcal{S}^1(\lambda)\mathbf{U}_0\ \mathrm{d}\lambda, \tag{4.5}$$

$$T_2(t)(\Pi_0,\mathbf{U}_0)=\frac{1}{2\pi i}\int_{\Gamma+\gamma}\mathcal{S}^2(\lambda)(\Pi_0,\mathbf{U}_0)\ \mathrm{d}\lambda, \tag{4.6}$$

$$T_3(t)(\Pi_0,\mathbf{U}_0)=\frac{1}{2\pi i}\int_{\Gamma+\gamma}\mathcal{R}(\lambda)(\Pi_0,\mathbf{U}_0)\ \mathrm{d}\lambda. \tag{4.7}$$

We have $T(t)(\Pi_0,\mathbf{U}_0)=(T_3(t)(\Pi_0,\mathbf{U}_0),T_1(t)\mathbf{U}_0+T_2(t)(\Pi_0,\mathbf{U}_0))$.

We first show that

$$\int_0^\infty e^{-\gamma t}\|T_1(t)\mathbf{U}_0\|_{B^{s+2}_{q,1}(\mathbb{R}^N_+)}\ \mathrm{d}t\leq C\|\mathbf{U}_0\|_{B^s_{q,1}(\mathbb{R}^N_+)}. \tag{4.8}$$

To this end, in view of Proposition 4.3, we first prove that for every $t > 0$ and $\mathbf{U}_0 \in C_0^\infty(\mathbb{R}^N_+)^N$ there hold

$$\|T_1(t)\mathbf{U}_0\|_{B^{s+2}_{q,1}(\mathbb{R}^N_+)} \le Ce^{\gamma t}t^{-1+\frac{\sigma}{2}}\|\mathbf{U}_0\|_{B^{s+\sigma}_{q,1}(\mathbb{R}^N_+)}, \tag{4.9}$$

$$\|T_1(t)\mathbf{U}_0\|_{B^{s+2}_{q,1}(\mathbb{R}^N_+)} \le Ce^{\gamma t}t^{-1-\frac{\sigma}{2}}\|\mathbf{U}_0\|_{B^{s-\sigma}_{q,1}(\mathbb{R}^N_+)}. \tag{4.10}$$

Notice that for $\lambda \in \Gamma_\pm + \gamma$, $\lambda = \gamma + re^{\pm i(\pi-\epsilon)}$, and thus $|e^{\lambda t}| = e^{\gamma t}e^{\cos(\pi-\epsilon)rt} = e^{\gamma t}e^{-rt\cos\epsilon}$. Since $\|\mathcal{S}^1(\lambda)\mathbf{U}\|_{B^{s+2}_{q,1}(\mathbb{R}^N_+)} \le C|\lambda|^{-\frac{\sigma}{2}}\|\mathbf{U}\|_{B^{s+\sigma}_{q,1}(\mathbb{R}^N_+)}$ as follows from Theorem 4.1, using (4.5), for $t > 0$ we have

$$\begin{aligned}
\|T_1(t)\mathbf{U}_0\|_{B^{s+2}_{q,1}(\mathbb{R}^N_+)} &\le Ce^{\gamma t}\int_0^\infty e^{-rt\cos\epsilon}r^{-\sigma/2}\,\mathrm{d}r\|\mathbf{U}_0\|_{B^{s+\sigma}_{q,1}(\mathbb{R}^N_+)} \\
&= Ce^{\gamma t}t^{-1+\frac{\sigma}{2}}\int_0^\infty e^{-\ell\cos\epsilon}\ell^{-\sigma/2}\,\mathrm{d}\ell\,\|\mathbf{U}_0\|_{B^{s+\sigma}_{q,1}(\mathbb{R}^N_+)},
\end{aligned}$$

which yields (4.9). To prove (4.10), by integration by parts we write

$$T_1(t)\mathbf{U}_0 = -\frac{1}{2\pi it}\int_{\Gamma+\gamma} e^{\lambda t}\partial_\lambda\mathcal{S}^1(\lambda)\mathbf{U}_0\,\mathrm{d}\lambda.$$

Since $\|\partial_\lambda\mathcal{S}^1(\lambda)\mathbf{U}_0\|_{B^{s+2}_{q,1}(\mathbb{R}^N_+)} \le C|\lambda|^{-(1-\frac{\sigma}{2})}\|\mathbf{U}_0\|_{B^{s-\sigma}_{q,1}(\mathbb{R}^N_+)}$ as follows from Theorem 4.1, we have

$$\begin{aligned}
\|T_1(t)\mathbf{U}_0\|_{B^{s+2}_{q,1}(\mathbb{R}^N_+)} &\le Ct^{-1}e^{\gamma t}\int_0^\infty e^{-rt\cos\epsilon}r^{-(1-\frac{\sigma}{2})}\,\mathrm{d}r\,\|\mathbf{U}_0\|_{B^{s-\sigma}_{q,1}(\mathbb{R}^N_+)} \\
&= Ce^{\gamma t}t^{-1-\frac{\sigma}{2}}\int_0^\infty e^{-\ell\cos\epsilon}\ell^{-1+\frac{\sigma}{2}}\,\mathrm{d}\ell\,\|\mathbf{U}_0\|_{B^{s-\sigma}_{q,1}(\mathbb{R}^N_+)},
\end{aligned}$$

which yields (4.10). Choosing $\theta = 1/2$ in Proposition 4.3 and using the fact that

$$(B^{s+\sigma}_{q,1}(\mathbb{R}^N_+), B^{s-\sigma}_{q,1}(\mathbb{R}^N_+))_{1/2,1} = B^s_{q,1}(\mathbb{R}^N_+),$$

by Proposition 4.3, we have (4.8) for $\mathbf{U}_0 \in C_0^\infty(\mathbb{R}^N_+)^N$. But, since $C_0^\infty(\mathbb{R}^N_+)^N$ is dense in $B^s_{q,r}(\mathbb{R}^N_+)^N$, the estimate (4.8) holds for any $\mathbf{U}_0 \in B^s_{q,1}(\mathbb{R}^N_+)^N$.

We now show that

$$\begin{aligned}
\int_0^\infty e^{-\gamma t}\|T_2(t)(\Pi_0,\mathbf{U}_0)\|_{B^{s+2}_{q,1}}\,\mathrm{d}t &\le C\|(\Pi_0,\mathbf{U}_0)\|_{B^{s+1}_{q,1}(\mathbb{R}^N_+)\times B^s_{q,1}(\mathbb{R}^N_+)}, \\
\int_0^\infty e^{-\gamma t}\|T_3(t)(\Pi_0,\mathbf{U}_0)\|_{B^{s+1}_{q,1}}\,\mathrm{d}t &\le C\|(\Pi_0,\mathbf{U}_0)\|_{B^{s+1}_{q,1}(\mathbb{R}^N_+)\times B^s_{q,1}(\mathbb{R}^N_+)}.
\end{aligned} \tag{4.11}$$

In fact, using Theorem 4.1 and $|\lambda| \ge \lambda_0$, we have

$$\begin{aligned}
\|(\lambda^{1/2}\nabla,\nabla^2)\mathcal{S}^2(\lambda)(f,\mathbf{g})\|_{B^s_{q,r}(\mathbb{R}^N_+)} &\le C|\lambda|^{-1}\|(f,\mathbf{g})\|_{B^{s+1}_{q,r}(\mathbb{R}^N_+)\times B^s_{q,r}(\mathbb{R}^N_+)} \\
&\le C\lambda_0^{-(1-\frac{\sigma}{2})}|\lambda|^{-\frac{\sigma}{2}}\|(f,\mathbf{g})\|_{B^{s+1}_{q,r}(\mathbb{R}^N_+)\times B^s_{q,r}(\mathbb{R}^N_+)}, \\
\|(\lambda^{1/2}\nabla,\nabla^2)\partial_\lambda\mathcal{S}^2(\lambda)(f,\mathbf{g})\|_{B^s_{q,r}(\mathbb{R}^N_+)} &\le C|\lambda|^{-2}\|(f,\mathbf{g})\|_{B^{s+1}_{q,r}(\mathbb{R}^N_+)\times B^s_{q,r}(\mathbb{R}^N_+)} \\
&\le C\lambda_0^{-(1+\frac{\sigma}{2})}|\lambda|^{-(1-\frac{\sigma}{2})}\|(f,\mathbf{g})\|_{B^{s+1}_{q,r}(\mathbb{R}^N_+)\times B^s_{q,r}(\mathbb{R}^N_+)}, \\
\|\mathcal{R}(\lambda)f\|_{B^{s+1}_{q,r}(\mathbb{R}^N_+)} &\le C|\lambda|^{-1}\|(f,\mathbf{g})\|_{B^{s+1}_{q,r}(\mathbb{R}^N_+)\times B^s_{q,r}(\mathbb{R}^N_+)} \\
&\le C\lambda_0^{-(1-\frac{\sigma}{2})}|\lambda|^{-\frac{\sigma}{2}}\|(f,\mathbf{g})\|_{B^{s+1}_{q,r}(\mathbb{R}^N_+)\times B^s_{q,r}(\mathbb{R}^N_+)},
\end{aligned}$$

$$\begin{aligned}\|\partial_\lambda \mathcal{R}(\lambda) f\|_{B^{s+1}_{q,r}(\mathbb{R}^N_+)} &\le C|\lambda|^{-2}\|(f,\mathbf{g})\|_{B^{s+1}_{q,r}(\mathbb{R}^N_+)\times B^s_{q,r}(\mathbb{R}^N_+)} \\ &\le C\lambda_0^{-(1+\frac{\sigma}{2})}|\lambda|^{-(1-\frac{\sigma}{2})}\|(f,\mathbf{g})\|_{B^{s+1}_{q,r}(\mathbb{R}^N_+)\times B^s_{q,r}(\mathbb{R}^N_+)}\end{aligned}$$

for any $\lambda \in \Sigma_\epsilon + \gamma$ and $(f, \mathbf{g}) \in \mathcal{H}^s_{q,1}$. In view of (4.6) and (4.7), employing the same argument as in the proof of (4.9) and (4.10), we have

$$\begin{aligned}\|T_2(t)(\Pi_0,\mathbf{U}_0)\|_{B^{s+2}_{q,1}(\mathbb{R}^N_+)} &\le C\lambda_0^{-\frac{\sigma}{2}}e^{\gamma t}t^{-1+\frac{\sigma}{2}}\|(\Pi_0,\mathbf{U}_0)\|_{\mathcal{H}^s_{q,1}}, \\ \|T_2(t)(\Pi_0,\mathbf{U}_0)\|_{B^{s+2}_{q,1}(\mathbb{R}^N_+)} &\le C\lambda_0^{-(1+\frac{\sigma}{2})}e^{\gamma t}t^{-1-\frac{\sigma}{2}}\|(\Pi_0,\mathbf{U}_0)\|_{\mathcal{H}^s_{q,1}}, \\ \|T_3(t)(\Pi_0,\mathbf{U}_0)\|_{B^{s+1}_{q,1}(\mathbb{R}^N_+)} &\le C\lambda_0^{-\frac{\sigma}{2}}e^{\gamma t}t^{-1+\frac{\sigma}{2}}\|(\Pi_0,\mathbf{U}_0)\|_{\mathcal{H}^s_{q,1}}, \\ \|T_3(t)(\Pi_0,\mathbf{U}_0)\|_{B^{s+1}_{q,1}(\mathbb{R}^N_+)} &\le C\lambda_0^{-(1+\frac{\sigma}{2})}e^{\gamma t}t^{-1-\frac{\sigma}{2}}\|(\Pi_0,\mathbf{U}_0)\|_{\mathcal{H}^s_{q,1}}.\end{aligned}$$

Thus, using Proposition 4.3 and noting that $(\mathcal{H}^s_{q,1}, \mathcal{H}^s_{q,1})_{1/2,1} = \mathcal{H}^s_{q,1} = B^{s+1}_{q,1}(\mathbb{R}^N_+) \times B^s_{q,1}(\mathbb{R}^N_+)$, we have (4.11). This completes the proof of Theorem 4.2. □

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